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INTEGRATING FACTORS AND FIRST INTEGRALS OF A CLASS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The principle of finding an integrating factor for a none exact differential equations is extended to a class of third order differential equations. If the third order equation is not exact, under certain conditions, an integrating factor exists which transforms it to an exact one. Hence, it can be reduced into a second order differential equation. In this paper, we give explicit forms for certain integrating factors of a class of the third order differential equations.

Key words and phrases: Third order differential equation; Exact differential equations; None exact differential equations; Integrating factor; First integrals.

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1. INTRODUCTION

Third order nonlinear differential equations play a major role in Applied Mathematics, Physics, and Engineering [4, 8, 10, 14, 15]. To find the general solution of a third order nonlinear differential equation is not an easy problem in the general case. In fact, a very specific class of nonlinear third order differential equations can be solved by using special transformations. Other technique is to reduce the order of the differential equation into the second order, by finding a proper integrating factor. Recently, many studies appear to deal with the problem of the existence of an integrating factor for certain differential equations. For example, in [1, 3, 5, 9], the authors investigated the existence of an integrating factor of some classes of second order differential equations. In [3], the authors investigated the existence of an integrating factor of $n-th$ order differential equations which has known symmetries of certain type. In [7], the authors improve some symbolic algorithms to compute the integrating factor for certain class of third order nonlinear differential equation. In this paper, we investigate the existence of an integrating factor of the following class of third order nonlinear differential equations:

$$F_3(t,y,y',y'')y''' + F_2(t,y,y',y'')y'' + F_1(t,y,y',y'')y' + F_0(t,y,y',y'') = 0, \tag{1.1}$$

where F_0, F_1, F_2 and F_3 are continuous with their first partial derivatives with respect to t, y, y', and y'', respectively, on some simply connected domain $\Omega \subset \mathbb{R}^4$. In fact, we present some theoretical results related to the existence of certain forms of the integrating factor for (1.1). We also present some illustrative examples.

2. INTEGRATING FACTOR AND FIRST INTEGRAL OF A CLASS OF THIRD ORDER DIFFERENTIAL EQUATIONS

In this section, we investigate the existence of some special forms of integrating factors of (1.1) in case that it is not exact differential equation. In general, the n-th order differential equation

$$f(t,y,y',\cdots,y^{(n-1)},y^{(n)}) = 0$$

is called exact if there exists a differentiable function $\Psi(t,y,y',\cdots,y^{(n-1)}) = c$, such that

$$\frac{d}{dt}\Psi(t,y,y',\cdots,y^{(n-1)}) = f(t,y,y',\cdots,y^{(n-1)},y^{(n)}) = 0.$$

In this case, $\Psi(t,y,y',\cdots,y^{(n-1)}) = c$ is called the first integral of $f(t,y,y',\cdots,y^{(n-1)},y^{(n)}) = 0,$ e.g., see, [11, 13]. In [2], the author gave the conditions so that the $n-$th order differential equation

$$\begin{aligned} &F_n\left(t,y,y',y'',\ldots,y^{(n-1)}\right)y^{(n)} + F_{n-1}\left(t,y,y',y'',\ldots,y^{(n-1)}\right)y^{(n-1)} + \cdots + \\ &+ F_1\left(t,y,y',y'',\ldots,y^{(n-1)}\right)y' + F_0\left(t,y,y',y''\ldots,y^{(n-1)}\right) = 0. \end{aligned} \tag{2.1}$$

is exact. Also, he gave an explicit formula for its first integral, $\Psi\left(t,y,y',\cdots,y^{(n)}\right) = c$. In particular, the class of third order differential equations (1.1) is exact if the following conditions:

$$\partial_{y''}F_0 = \partial_t F_3,\ \partial_{y''}F_1 = \partial_y F_3,\ \partial_{y''}F_2 = \partial_{y'}F_3, \partial_{y'}F_0 = \partial_t F_2,\ \partial_{y'}F_1 = \partial_y F_2,\ \text{and } \partial_y F_0 = \partial_t F_1 \tag{2.2}$$

hold. Moreover, the first integral of (1.1) is given by

$$\begin{aligned}\Psi\left(t,y,y',y''\right) &= \int_{t_0}^{t} F_0\left(\xi,y,y',y''\right)d\xi + \int_{y_0}^{y} F_1\left(t_0,\xi,y',y''\right)d\xi \\ &+ \int_{y_0'}^{y'} F_2\left(t_0,y_0,\xi,y''\right)d\xi + \int_{y_0''}^{y''} F_3\left(t_0,y_0,y_0',\xi\right)d\xi \\ &= c \end{aligned} \tag{2.3}$$

where c is the integration constant. Assume that (1.1) is not exact differential equation. Then, according to conditions (2.2), an integrating factor $\mu(t,y,y',y'')$ of (1.1) exists if it solves the following system of partial differential equations:

$$\begin{cases} \mu(\mathbf{y})F_{3t}(\mathbf{y})+\mu_t(\mathbf{y})F_3(\mathbf{y})=\mu(\mathbf{y})F_{0y''}(\mathbf{y})+\mu_{y''}(\mathbf{y})F_0(\mathbf{y}), \\ \mu(\mathbf{y})F_{2t}(\mathbf{y})+\mu_t(\mathbf{y})F_2(\mathbf{y})=\mu(\mathbf{y})F_{0y'}(\mathbf{y})+\mu_{y'}(\mathbf{y})F_0(\mathbf{y}), \\ \mu(\mathbf{y})F_{1t}(\mathbf{y})+\mu_t(\mathbf{y})F_1(\mathbf{y})=\mu(\mathbf{y})F_{0y}(\mathbf{y})+\mu_{y}(\mathbf{y})F_0(\mathbf{y}), \\ \mu(\mathbf{y})F_{1y'}(\mathbf{y})+\mu_{y'}(\mathbf{y})F_1(\mathbf{y})=\mu(\mathbf{y})F_{2y}(\mathbf{y})+\mu_{y}(\mathbf{y})F_2(\mathbf{y}), \\ \mu(\mathbf{y})F_{1y''}(\mathbf{y})+\mu_{y''}(\mathbf{y})F_1(\mathbf{y})=\mu(\mathbf{y})F_{3y}(\mathbf{y})+\mu_{y}(\mathbf{y})F_3(\mathbf{y}), \\ \mu(\mathbf{y})F_{2y''}(\mathbf{y})+\mu_{y''}(\mathbf{y})F_2(\mathbf{y})=\mu(\mathbf{y})F_{3y'}(\mathbf{y})+\mu_{y'}(\mathbf{y})F_3(\mathbf{y}), \end{cases} \tag{2.4}$$

where $\mathbf{y}=(t,y,y',y'')$. In general, to solve such system of first order partial differential equations is not easy. Thus, we consider some special cases of $\mu(t,y,y',y'')$. Particularly, we are looking for an integrating factor of the form $\mu(\xi)$, where $\xi := \xi(t,y,y',y'') = \alpha(t)\beta(y)\gamma(y')\delta(y'')$. Here, we assume that $\alpha(t),\ \beta(y),\ \gamma(y'),$ and $\delta(y'')$ to be differentiable functions with respect to t, y, y', and y'', respectively. By substituting $\mu(\xi)=\mu(\alpha(t)\beta(y)\gamma(y')\delta(y''))$ in (2.4), we get

$$\begin{cases} \mu(\xi)F_{3t}(\mathbf{y})+\mu'(\xi)\xi_tF_3(\mathbf{y})=\mu(\xi)F_{0y''}(\mathbf{y})+\mu'(\xi)\xi_{y''}F_0(\mathbf{y}), \\ \mu(\xi)F_{2t}(\mathbf{y})+\mu'(\xi)\xi_tF_2(\mathbf{y})=\mu(\xi)F_{0y'}(\mathbf{y})+\mu'(\xi)\xi_{y'}F_0(\mathbf{y}), \\ \mu(\xi)F_{1t}(\mathbf{y})+\mu'(\xi)\xi_tF_1(\mathbf{y})=\mu(\xi)F_{0y}(\mathbf{y})+\mu'(\xi)\xi_{y}F_0(\mathbf{y}), \\ \mu(\xi)F_{1y'}(\mathbf{y})+\mu'(\xi)\xi_{y'}F_1(\mathbf{y})=\mu(\xi)F_{2y}(\mathbf{y})+\mu'(\xi)\xi_{y}F_2(\mathbf{y}), \\ \mu(\xi)F_{1y''}(\mathbf{y})+\mu'(\xi)\xi_{y''}F_1(\mathbf{y})=\mu(\xi)F_{3y}(\mathbf{y})+\mu'(\xi)\xi_{y}F_3(\mathbf{y}), \\ \mu(\xi)F_{2y''}(\mathbf{y})+\mu'(\xi)\xi_{y''}F_2(\mathbf{y})=\mu(\xi)F_{3y'}(\mathbf{y})+\mu'(\xi)\xi_{y'}F_3(\mathbf{y}), \end{cases} \tag{2.5}$$

where $\mu'(\xi)=\frac{d\mu}{d\xi}$. Equivalently, we have

$$\begin{cases} \dfrac{\mu'(\xi)}{\mu(\xi)}=\dfrac{F_{3t}(\mathbf{y})-F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})-\xi_tF_3(\mathbf{y})}, \quad \dfrac{\mu'(\xi)}{\mu(\xi)}=\dfrac{F_{2t}(\mathbf{y})-F_{0y'}(\mathbf{y})}{\xi_{y'}F_0(\mathbf{y})-\xi_tF_2(\mathbf{y})}, \\ \\ \dfrac{\mu'(\xi)}{\mu(\xi)}=\dfrac{F_{1t}(\mathbf{y})-F_{0y}(\mathbf{y})}{\xi_{y}F_0(\mathbf{y})-\xi_tF_1(\mathbf{y})}, \quad \dfrac{\mu'(\xi)}{\mu(\xi)}=\dfrac{F_{1y'}(\mathbf{y})-F_{2y}(\mathbf{y})}{\xi_{y}F_2(\mathbf{y})-\xi_{y'}F_1(\mathbf{y})}, \\ \\ \dfrac{\mu'(\xi)}{\mu(\xi)}=\dfrac{F_{1y''}(\mathbf{y})-F_{3y}(\mathbf{y})}{\xi_{y}F_3(\mathbf{y})-\xi_{y''}F_1(\mathbf{y})}, \quad \dfrac{\mu'(\xi)}{\mu(\xi)}=\dfrac{F_{2y''}(\mathbf{y})-F_{3y'}(\mathbf{y})}{\xi_{y'}F_3(\mathbf{y})-\xi_{y''}F_2(\mathbf{y})}. \end{cases} \tag{2.6}$$

Hence, an integrating factor $\mu(\xi)$ of (1.1) exists if

a) $\frac{F_{3t}(\mathbf{y})-F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})-\xi_tF_3(\mathbf{y})}$, $\frac{F_{2t}(\mathbf{y})-F_{0y'}(\mathbf{y})}{\xi_{y'}F_0(\mathbf{y})-\xi_tF_2(\mathbf{y})}$, $\frac{F_{1t}(\mathbf{y})-F_{0y}(\mathbf{y})}{\xi_{y}F_0(\mathbf{y})-\xi_tF_1(\mathbf{y})}$, $\frac{F_{1y'}(\mathbf{y})-F_{2y}(\mathbf{y})}{\xi_{y}F_2(\mathbf{y})-\xi_{y'}F_1(\mathbf{y})}$, $\frac{F_{1y''}(\mathbf{y})-F_{3y}(\mathbf{y})}{\xi_{y}F_3(\mathbf{y})-\xi_{y''}F_1(\mathbf{y})}$, and $\frac{F_{2y''}(\mathbf{y})-F_{3y'}(\mathbf{y})}{\xi_{y'}F_3(\mathbf{y})-\xi_{y''}F_2(\mathbf{y})}$ are functions of ξ,
and

b)
$$\begin{aligned}\frac{F_{3t}(\mathbf{y})-F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})-\xi_tF_3(\mathbf{y})} &= \frac{F_{2t}(\mathbf{y})-F_{0y'}(\mathbf{y})}{\xi_{y'}F_0(\mathbf{y})-\xi_tF_2(\mathbf{y})}=\frac{F_{1t}(\mathbf{y})-F_{0y}(\mathbf{y})}{\xi_yF_0(\mathbf{y})-\xi_tF_1(\mathbf{y})}=\\ \frac{F_{1y'}(\mathbf{y})-F_{2y}(\mathbf{y})}{\xi_yF_2(\mathbf{y})-\xi_{y'}F_1(\mathbf{y})} &= \frac{F_{1y''}(\mathbf{y})-F_{3y}(\mathbf{y})}{\xi_yF_3(\mathbf{y})-\xi_{y''}F_1(\mathbf{y})}=\frac{F_{2y''}(\mathbf{y})-F_{3y'}(\mathbf{y})}{\xi_{y'}F_3(\mathbf{y})-\xi_{y''}F_2(\mathbf{y})}\end{aligned}$$
hold. Therefore, we have the following theorem:

Theorem 2.1. *Let $\mathbf{y}=(t,y,y',y'')$ and $\xi=\alpha(t)\beta(y)\gamma(y')\delta(y'')$ where $\alpha(t),\beta(y),\gamma(y')$, and $\delta(y'')$ are differentiable functions. Assume that Equation* (1.1) *is a none exact differential equation. Then it admits a none constant integrating factor $\mu(\xi)$; if*
$$\begin{aligned}\frac{F_{3t}(\mathbf{y})-F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})-\xi_tF_3(\mathbf{y})} &= \frac{F_{2t}(\mathbf{y})-F_{0y'}(\mathbf{y})}{\xi_{y'}F_0(\mathbf{y})-\xi_tF_2(\mathbf{y})}=\frac{F_{1t}(\mathbf{y})-F_{0y}(\mathbf{y})}{\xi_yF_0(\mathbf{y})-\xi_tF_1(\mathbf{y})}=\\ \frac{F_{1y'}(\mathbf{y})-F_{2y}(\mathbf{y})}{\xi_yF_2(\mathbf{y})-\xi_{y'}F_1(\mathbf{y})} &= \frac{F_{1y''}(\mathbf{y})-F_{3y}(\mathbf{y})}{\xi_yF_3(\mathbf{y})-\xi_{y''}F_1(\mathbf{y})}=\frac{F_{2y''}(\mathbf{y})-F_{3y'}(\mathbf{y})}{\xi_{y'}F_3(\mathbf{y})-\xi_{y''}F_2(\mathbf{y})},\end{aligned}$$
and they are functions in ξ. Moreover, the integrating factor is given explicitly by
$$\mu(\xi)=\exp\left\{\int\frac{F_{3t}(\mathbf{y})-F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})-\xi_tF_3(\mathbf{y})}d\xi\right\}.$$

In the following sections, we presents some special cases of the above theorem. Moreover, we present some illustrative examples.

3. INTEGRATING FACTORS OF THE FORMS $\mu(\alpha(t))$, $\mu(\beta(y))$, $\mu(\gamma(y'))$ AND $\mu(\delta(y''))$

In this section, we give conditions so that an integrating factor of one of the forms $\mu(\alpha(t))$, $\mu(\beta(y))$, $\mu(\gamma(y'))$ and $\mu(\delta(y''))$ for equation (1.1) exists. As a result of Theorem 2.1, we have the following corollaries:

Corollary 3.1. *Let $\mathbf{y}=(t,y,y',y'')$, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor $\mu(\xi)=\mu(\alpha(t))$, where $\alpha(t)$ is differentiable function; if the following two conditions hold:*

a) $F_{1y'}(\mathbf{y})=F_{2y}(\mathbf{y})$, $F_{1y''}(\mathbf{y})=F_{3y}(\mathbf{y})$, *and* $F_{2y''}(\mathbf{y})=F_{3y'}(\mathbf{y})$,
and
b) $\frac{F_{0y''}(\mathbf{y})-F_{3t}(\mathbf{y})}{\xi_tF_3(\mathbf{y})}=\frac{F_{0y'}(\mathbf{y})-F_{2t}(\mathbf{y})}{\xi_tF_2(\mathbf{y})}=\frac{F_{0y}(\mathbf{y})-F_{1t}(\mathbf{y})}{\xi_tF_1(\mathbf{y})}$, *and they are functions in $\xi:=\xi(\alpha(t))$.*

Moreover, the integrating factor is given by
$$\mu(\xi)=\exp\left\{\int\frac{F_{0y''}(\mathbf{y})-F_{3t}(\mathbf{y})}{\xi_tF_3(\mathbf{y})}d\xi\right\}.$$

Corollary 3.2. *Let $\mathbf{y}=(t,y,y',y'')$, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor $\mu(\xi)=\mu(\beta(y))$, where $\beta(y)$ is differentiable function; if the following two conditions hold:*

a) $F_{3t}(\mathbf{y})=F_{0y''}(\mathbf{y})$, $F_{2t}(\mathbf{y})=F_{0y'}(\mathbf{y})$, *and* $F_{2y''}(\mathbf{y})=F_{3y'}(\mathbf{y})$,
and
b) $\frac{F_{1t}(\mathbf{y})-F_{0y}(\mathbf{y})}{\xi_yF_0(\mathbf{y})}=\frac{F_{1y'}(\mathbf{y})-F_{2y}(\mathbf{y})}{\xi_yF_2(\mathbf{y})}=\frac{F_{1y''}(\mathbf{y})-F_{3y}(\mathbf{y})}{\xi_yF_3(\mathbf{y})}$, *and they are functions in $\xi:=\xi(\beta(y))$.*

Moreover, the integrating factor is given by
$$\mu(\xi)=\exp\left\{\int\frac{F_{1t}(\mathbf{y})-F_{0y}(\mathbf{y})}{\xi_yF_0(\mathbf{y})}d\xi\right\}.$$

Corollary 3.3. *Let* $\mathbf{y} = (t, y, y', y'')$*, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor* $\mu(\xi) = \mu(\gamma(y'))$ *for some differentiable functions* $\gamma(y')$*; if the two conditions hold:*

a) $F_{3t}(\mathbf{y}) = F_{0y''}(\mathbf{y})$, $F_{1t}(\mathbf{y}) = F_{0y}(\mathbf{y})$, *and* $F_{1y''}(\mathbf{y}) = F_{3y}(\mathbf{y})$, *and*

b) $\frac{F_{2t}(\mathbf{y})-F_{0y'}(\mathbf{y})}{\xi_{y'}F_0(\mathbf{y})} = \frac{F_{2y}(\mathbf{y})-F_{1y'}(\mathbf{y})}{\xi_{y'}F_1(\mathbf{y})} = \frac{F_{2y''}(\mathbf{y})-F_{3y'}(\mathbf{y})}{\xi_{y'}F_3(\mathbf{y})}$, *and they are functions in* $\xi := \gamma(y')$.

Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{\int \frac{F_{2t}(\mathbf{y}) - F_{0y'}(\mathbf{y})}{\xi_{y'}F_0(\mathbf{y})} d\xi\right\}.$$

Corollary 3.4. *Let* $\mathbf{y} = (t, y, y', y'')$*, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor* $\mu(\xi) = \mu(\delta(y''))$ *for some differentiable function* $\delta(y'')$*; if the following two conditions hold:*

a) $F_{2t}(\mathbf{y}) = F_{0y'}(\mathbf{y})$, $F_{1t}(\mathbf{y}) = F_{0y}(\mathbf{y})$, *and* $F_{1y'}(\mathbf{y}) = F_{2y}(\mathbf{y})$, *and*

b) $\frac{F_{3t}(\mathbf{y})-F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})} = \frac{F_{3y}(\mathbf{y})-F_{1y''}(\mathbf{y})}{\xi_{y''}F_1(\mathbf{y})} = \frac{F_{3y'}(\mathbf{y})-F_{2y''}(\mathbf{y})}{\xi_{y''}F_2(\mathbf{y})}$, *and they are functions in* $\xi := \delta(y'')$.

Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{\int \frac{F_{3t}(\mathbf{y}) - F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})} d\xi\right\}.$$

Example 3.1. *Consider the differential equation*

$$(y')^3 y''' + 2yy'' - (y')^2 + (y')^3 = 0. \tag{3.1}$$

Clearly,

$$F_3(t,y,y',y'') = (y')^3, F_2(t,y,y',y'') = 2y, F_1(t,y,y',y'') = -y', \text{and} F_0(t,y,y',y'') = (y')^3.$$

Moreover,

$$F_{3t}(t,y,y',y'') = F_{0y''}(t,y,y',y'') = 0, F_{1t}(t,y,y',y'') = F_{0y}(t,y,y',y'') = 0,$$

and

$$F_{1y''}(t,y,y',y'') = F_{3y}(t,y,y',y'') = 0,$$

In addition, we have

$$\frac{F_{2t}(\mathbf{y}) - F_{0y'}(\mathbf{y})}{\xi_{y'}F_0(\mathbf{y})} = \frac{F_{2y}(\mathbf{y}) - F_{1y'}(\mathbf{y})}{\xi_{y'}F_1(\mathbf{y})} = \frac{F_{3y'}(\mathbf{y}) - F_{2y''}(\mathbf{y})}{\xi_{y'}F_3(\mathbf{y})} = \frac{-3}{y'}.$$

Therefor, the conditions given in Corollary 3.3 hold. Hence, an integrating factor in terms of y' *of* (3.1) *exists and is given by* $\mu(y') = (y')^{-3}$*. By multiplying* (3.1) *by* $\mu(y') = (y')^{-3}$*, we get*

$$y''' + 2y\,(y')^{-3}\,y'' - (y')^{-2}y' + 1 = 0. \tag{3.2}$$

For this equation, $F_3(t,y,y',y'') = 1$, $F_2(t,y,y',y'') = 2y\,(y')^{-3}$, $F_1(t,y,y',y'') = -(y')^{-2}$, *and* $F_0(t,y,y',y'') = 1$*. Clearly,* $\partial_{y''}F_0 = \partial_t F_3 = 0,\ \partial_{y''}F_1 = \partial_y F_3 = 0,\ \partial_{y''}F_2 = \partial_{y'}F_3 = 0, \partial_{y'}F_0 = \partial_t F_2 = 0,\ \partial_{y'}F_1 = \partial_y F_2 = 2(y')^{-3},$ *and* $\partial_y F_0 = \partial_t F_1 = 0$*. Hence, it is exact differential equation. Its first integral is given by*

$$\Psi\,(t,y,y',y'') = \int_{t_0}^{t} d\xi - (y')^{-2}\int_{y_0}^{y} d\xi + 2y_0\int_{y'_0}^{y'} \xi^{-3}d\xi + \int_{y''_0}^{y''} d\xi = c, \tag{3.3}$$

More precisely,

$$\Psi\,(t,y,y',y'') = y'' - (y')^{-2}y + t = c \tag{3.4}$$

Example 3.2. *Consider the third order linear differential equation*

$$p_2(t)y''' + \alpha p_2(t)y'' + p_1(t)y' + p_0(t)y = h(t),\ p_2(t) \neq 0, \tag{3.5}$$

where $P_1(t)$ and $p_2(t)$ are differentiable functions, and $p_0(t)$ and $h(t)$ are continuous function on some open interval $I \subset \mathbb{R}$. Assume that $p_2(t)p_1'(t) - p_2'(t)p_1(t) = p_0(t)p_2(t)$. Then the above equation satisfies the conditions in Corollary 3.1. Hence, it admits an integrating factor $\mu(t) = \frac{1}{p_2(t)}$. In fact, if we multiply Eq. (3.5) *by $\mu(t) = \frac{1}{p_2(t)}$, then we get*

$$y''' + \alpha y'' + \frac{p_1(t)}{p_2(t)}y' + \frac{p_0(t)}{p_2(t)}y = \frac{h(t)}{p_2(t)}.$$

From the condition $p_2(t)p_1'(t) - p_2'(t)p_1(t) = p_0(t)p_2(t)$, we have $\left(\frac{p_1(t)}{p_2(t)}\right)' = \frac{p_0(t)}{p_2(t)}$. Hence, the above equation becomes

$$y''' + \alpha y'' + \left(\frac{p_1(t)}{p_2(t)}\right)y' + \left(\frac{p_1(t)}{p_2(t)}\right)' y = \frac{h(t)}{p_2(t)}.$$

This equation can be written as

$$\frac{d}{dt}\left[y'' + \alpha y' + \left(\frac{p_1(t)}{p_2(t)}\right)y\right] = \frac{h(t)}{p_2(t)}.$$

Hence, the first integral of Eq. (3.5) *is given by*

$$y'' + \alpha y' + \left(\frac{p_1(t)}{p_2(t)}\right)y = \int^t \frac{h(s)}{p_2(s)ds} + c.$$

4. INTEGRATING FACTORS OF THE FORMS $\mu(\alpha(t)\beta(y))$, $\mu(\alpha(t)\gamma(y'))$, $\mu(\alpha(t)\delta(y''))$, $\mu(\beta(y)\gamma(y'))$, $\mu(\beta(y)\delta(y''))$ AND $\mu(\gamma(y')\delta(y''))$

In this section, we give conditions so that an integrating factor of the forms $\mu(\alpha(t)\beta(y))$, $\mu(\alpha(t)\gamma(y'))$, $\mu(\alpha(t)\delta(y''))$, $\mu(\beta(y)\gamma(y'))$, $\mu(\beta(y)\delta(y''))$ and $\mu(\gamma(y')\delta(y''))$ for equation (1.1) exists. As a result of Theorem 2.1, we have the following corollaries:

Corollary 4.1. *Let $\mathbf{y} = (t, y, y', y'')$, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor $\mu(\xi) = \mu(\alpha(t)\beta(y))$, where $\alpha(t)$ and $\beta(y))$ are differentiable functions; if the following two conditions hold:*

a) $F_{2y''}(\mathbf{y}) = F_{3y'}(\mathbf{y})$, *and*
b) $\frac{F_{0y''}(\mathbf{y})-F_{3t}(\mathbf{y})}{\xi_t F_3(\mathbf{y})} = \frac{F_{0y'}(\mathbf{y})-F_{2t}(\mathbf{y})}{\xi_t F_2(\mathbf{y})} = \frac{F_{1y'}(\mathbf{y})-F_{2y}(\mathbf{y})}{\xi_y F_2(\mathbf{y})} = \frac{F_{1y''}(\mathbf{y})-F_{3y}(\mathbf{y})}{\xi_y F_3(\mathbf{y})} = \frac{F_{1t}(\mathbf{y})-F_{0y}(\mathbf{y})}{\xi_y F_0(\mathbf{y})-\xi_t F_1(\mathbf{y})}$, *and they are functions in $\xi := \xi(\alpha(t)\beta(y))$.*

Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{\int \frac{F_{0y''}(\mathbf{y}) - F_{3t}(\mathbf{y})}{\xi_t F_3(\mathbf{y})} d\xi\right\}.$$

Corollary 4.2. *Let $\mathbf{y} = (t, y, y', y'')$, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor $\mu(\xi) = \mu(\alpha(t)\gamma(y'))$ for some differentiable functions $\alpha(t)$ and $\gamma(y')$; if the two conditions hold:*

a) $F_{1y''}(\mathbf{y}) = F_{3y}(\mathbf{y})$,
and
b) $\frac{F_{0y''}(\mathbf{y})-F_{3t}(\mathbf{y})}{\xi_t F_3(\mathbf{y})} = \frac{F_{0y}(\mathbf{y})-F_{1t}(\mathbf{y})}{\xi_t F_1(\mathbf{y})} = \frac{F_{2y}(\mathbf{y})-F_{1y'}(\mathbf{y})}{\xi_{y'} F_1(\mathbf{y})} = \frac{F_{2y''}(\mathbf{y})}{\xi_{y'} F_3(\mathbf{y})-F_{3y'}(\mathbf{y})} = \frac{F_{2t}(\mathbf{y})-F_{0y'}(\mathbf{y})}{\xi_{y'} F_0(\mathbf{y})-\xi_t F_2(\mathbf{y})}$. *and they are functions in $\xi := \alpha(t)\gamma(y')$.*

Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{ \int \frac{F_{0y''}(\mathbf{y}) - F_{3t}(\mathbf{y})}{\xi_t F_3(\mathbf{y})} d\xi \right\}.$$

Corollary 4.3. *Let* $\mathbf{y} = (t, y, y', y'')$, *and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor* $\mu(\xi) = \mu(\alpha(t)\delta(y''))$ *for some differentiable function* $\delta(y'')$*; if the following two conditions hold:*

a) $F_{1y'}(\mathbf{y}) = F_{2y}(\mathbf{y})$,
and
b) $\frac{F_{0y'}(\mathbf{y}) - F_{2t}(\mathbf{y})}{\xi_t F_2(\mathbf{y})} = \frac{F_{0y}(\mathbf{y}) - F_{1t}(\mathbf{y})}{\xi_t F_1(\mathbf{y})} = \frac{F_{2y}(\mathbf{y}) - F_{1y''}(\mathbf{y})}{\xi_{y''} F_1(\mathbf{y})} = \frac{F_{2y''}(\mathbf{y}) - F_{3y'}(\mathbf{y})}{\xi_{y''} F_2(\mathbf{y})} = \frac{F_{3t}(\mathbf{y}) - F_{0y''}(\mathbf{y})}{\xi_{y''} F_0(\mathbf{y}) - \xi_t F_3}$. *and they are functions in* $\xi := \alpha(t)\delta(y'')$.

Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{ \int \frac{F_{0y'}(\mathbf{y}) - F_{2t}(\mathbf{y})}{\xi_t F_2(\mathbf{y})} d\xi \right\}.$$

Corollary 4.4. *Let* $\mathbf{y} = (t, y, y', y'')$, *and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor* $\mu(\xi) = \mu(\beta(y)\gamma(y'))$, *where* $\beta(y)$ *and* $\gamma(y')$ *are differentiable functions; if the following two conditions hold:*

a) $F_{3t}(\mathbf{y}) = F_{0y''}$ *and*
b) $\frac{F_{2t}(\mathbf{y}) - F_{0y'}(\mathbf{y})}{\xi_{y'} F_0(\mathbf{y})} = \frac{F_{1t}(\mathbf{y}) - F_{0y}(\mathbf{y})}{\xi_y F_0(\mathbf{y})} = \frac{F_{1y''}(\mathbf{y}) - F_{3y}(\mathbf{y})}{\xi_y F_3(\mathbf{y})} = \frac{F_{2y''}(\mathbf{y}) - F_{3y'}(\mathbf{y})}{\xi_{y'} F_3(\mathbf{y})} = \frac{F_{1y'}(\mathbf{y}) - F_{2y}(\mathbf{y})}{\xi_y F_2(\mathbf{y}) - \xi_{y'} F_1(\mathbf{y})}$ *and they are functions in* $\xi := \beta(y)\gamma(y')$.

Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{ \int \frac{F_{2t}(\mathbf{y}) - F_{0y'}(\mathbf{y})}{\xi_{y'} F_0(\mathbf{y})} d\xi \right\}.$$

Corollary 4.5. *Let* $\mathbf{y} = (t, y, y', y'')$, *and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor* $\mu(\xi) = \mu(\beta(y)\delta(y''))$, *where* $\beta(y)$ *and* $\delta(y'')$ *are differentiable functions; if the following two conditions hold:*

a) $F_{2t}(\mathbf{y}) = F_{0y'}(\mathbf{y})$, *and*
b) $\frac{F_{3t}(\mathbf{y}) - F_{0y''}(\mathbf{y})}{\xi_{y''} F_0(\mathbf{y})} = \frac{F_{1t}(\mathbf{y}) - F_{0y}(\mathbf{y})}{\xi_y F_0(\mathbf{y})} = \frac{F_{1y'}(\mathbf{y}) - F_{2y}(\mathbf{y})}{\xi_y F_2(\mathbf{y})} = \frac{F_{3y'}(\mathbf{y}) - F_{2y''}(\mathbf{y})}{\xi_{y''} F_2(\mathbf{y})} = \frac{F_{1y''}(\mathbf{y}) - F_{3y}(\mathbf{y})}{\xi_y F_3(\mathbf{y}) - \xi_{y''} F_1(\mathbf{y})}$ *and they are functions in* $\xi := \beta(y)\delta(y'')$.

Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{ \int \frac{F_{3t}(\mathbf{y}) - F_{0y''}(\mathbf{y})}{\xi_{y''} F_0(\mathbf{y})} d\xi \right\}.$$

Corollary 4.6. *Let* $\mathbf{y} = (t, y, y', y'')$, *and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor* $\mu(\xi) = \mu(\gamma(y')\delta(y''))$, *where* $\gamma(y')$ *and* $\delta(y'')$ *are differentiable functions; if the following two conditions hold:*

a) $F_{1t}(\mathbf{y}) = F_{0y}(\mathbf{y})$, *and*
b) $\frac{F_{3t}(\mathbf{y}) - F_{0y''}(\mathbf{y})}{\xi_{y''} F_0(\mathbf{y})} = \frac{F_{2t}(\mathbf{y}) - F_{0y'}(\mathbf{y})}{\xi_{y'} F_0(\mathbf{y})} = \frac{F_{2y}(\mathbf{y}) - F_{1y'}(\mathbf{y})}{\xi_{y'} F_1(\mathbf{y})} = \frac{F_{3y}(\mathbf{y}) - F_{1y''}(\mathbf{y})}{\xi_{y''} F_1(\mathbf{y})} = \frac{F_{2y''}(\mathbf{y}) - F_{3y'}(\mathbf{y})}{\xi_{y'} F_3(\mathbf{y}) - \xi_{y''} F_2(\mathbf{y})}$ *and they are functions in* $\xi = \gamma(y')\delta(y'')$.

Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{ \int \frac{F_{3t}(\mathbf{y}) - F_{0y''}(\mathbf{y})}{\xi_{y''} F_0(\mathbf{y})} d\xi \right\}.$$

5. INTEGRATING FACTORS OF THE FORMS $\mu(\alpha(t)\beta(y)\gamma(y'))$, $\mu(\alpha(t)\beta(y)\delta(y''))$, $\mu(\alpha(t)\gamma(y')\delta(y''))$ AND $\mu(\beta(y)\gamma(y')\delta(y''))$

In this section, we give conditions so that an integrating factor of the forms $\mu(\alpha(t)\beta(y)\gamma(y'))$, $\mu(\alpha(t)\beta(y)\delta(y''))$, $\mu(\alpha(t)\gamma(y')\delta(y''))$ and $\mu(\beta(y)\gamma(y')\delta(y''))$ for equation (1.1) exists. As a result of Theorem 2.1, we have the following corollaries:

Corollary 5.1. *Let $\mathbf{y} = (t, y, y', y'')$, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor $\mu(\xi) = \mu(\xi(t, y, y')) = \mu(\alpha(t)\beta(y)\gamma(y'))$ where $\alpha(t), \beta(y)$, and $\gamma(y')$ are differentiable functions; if*

$$\frac{F_{0y''}(\mathbf{y}) - F_{3t}(\mathbf{y})}{\xi_t F_3(\mathbf{y})} = \frac{F_{1y''}(\mathbf{y}) - F_{3y}(\mathbf{y})}{\xi_y F_3(\mathbf{y})} = \frac{F_{2y''}(\mathbf{y}) - F_{3y'}(\mathbf{y})}{\xi_{y'} F_3(\mathbf{y})} = \frac{F_{2t}(\mathbf{y}) - F_{0y'}(\mathbf{y})}{\xi_{y'} F_0(\mathbf{y}) - \xi_t F_2(\mathbf{y})} =$$
$$\frac{F_{1t}(\mathbf{y}) - F_{0y}(\mathbf{y})}{\xi_y F_0(\mathbf{y}) - \xi_t F_1(\mathbf{y})} = \frac{F_{1y'}(\mathbf{y}) - F_{2y}(\mathbf{y})}{\xi_y F_2(\mathbf{y}) - \xi_{y'} F_1(\mathbf{y})}$$

and they are functions in $\xi := \alpha(t)\beta(y)\gamma(y')$. Moreover, the integrating factor is given by the formula

$$\mu(\xi) = \exp\left\{\int \frac{F_{0y''}(\mathbf{y}) - F_{3t}(\mathbf{y})}{\xi_t F_3(\mathbf{y})} d\xi\right\}.$$

Corollary 5.2. *Let $\mathbf{y} = (t, y, y', y'')$, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor $\mu(\xi) = \mu(\alpha(t)\beta(y)\delta(y''))$ where $\alpha(t)$, $\beta(y)$, and $\delta(y'')$ are differentiable functions; if*

$$\frac{F_{0y'}(\mathbf{y}) - F_{2t}(\mathbf{y})}{\xi_t F_2(\mathbf{y})} = \frac{F_{3y'}(\mathbf{y}) - F_{2y''}(\mathbf{y})}{\xi_{y''} F_2(\mathbf{y})} = \frac{F_{1y'}(\mathbf{y}) - F_{3y}(\mathbf{y})}{\xi_y F_3(\mathbf{y})} = \frac{F_{1y''}(\mathbf{y}) - F_{3y}(\mathbf{y})}{\xi_y F_3(\mathbf{y}) - \xi_{y''} F_1(\mathbf{y})} =$$
$$\frac{F_{3t}(\mathbf{y}) - F_{0y''}(\mathbf{y})}{\xi_{y''} F_0(\mathbf{y}) - \xi_t F_3(\mathbf{y})} = \frac{F_{1t}(\mathbf{y}) - F_{0y}(\mathbf{y})}{\xi_y F_0(\mathbf{y}) - \xi_t F_1(\mathbf{y})}$$

and they are functions in $\xi = \alpha(t)\beta(y)\delta(y'')$. Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{\int \frac{F_{0y'}(\mathbf{y}) - F_{2t}(\mathbf{y})}{\xi_t F_2(\mathbf{y})} d\xi\right\}.$$

Corollary 5.3. *Let $\mathbf{y} = (t, y, y', y'')$, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor $\mu(\xi) = \mu(\alpha(t)\gamma(y')\delta(y''))$ where $\alpha(t)$, $\gamma(y')$, and $\delta(y'')$ are differentiable functions; if*

$$\frac{F_{0y}(\mathbf{y}) - F_{1t}(\mathbf{y})}{\xi_t F_1(\mathbf{y})} = \frac{F_{2y}(\mathbf{y}) - F_{1y'}(\mathbf{y})}{\xi_{y'} F_1(\mathbf{y})} = \frac{F_{3y}(\mathbf{y}) - F_{1y''}(\mathbf{y})}{\xi_{y''} F_1(\mathbf{y})} \frac{F_{3t}(\mathbf{y}) - F_{0y''}(\mathbf{y})}{\xi_{y''} F_0(\mathbf{y}) - \xi_t F_3(\mathbf{y})} =$$
$$\frac{F_{2t}(\mathbf{y}) - F_{0y'}(\mathbf{y})}{\xi_{y'} F_0(\mathbf{y}) - \xi_t F_2(\mathbf{y})} = \frac{F_{2y''}(\mathbf{y}) - F_{3y'}(\mathbf{y})}{\xi_{y'} F_3(\mathbf{y}) - \xi_{y''} F_2(\mathbf{y})}$$

and they are functions in $\xi = \alpha(t)\gamma(y')\delta(y'')$. Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{\int \frac{F_{0y}(\mathbf{y}) - F_{1t}(\mathbf{y})}{\xi_t F_1(\mathbf{y})} d\xi\right\}.$$

Corollary 5.4. *Let $\mathbf{y} = (t, y, y', y'')$, and assume that Equation* (1.1) *is none exact differential equation. Then it admits an integrating factor $\mu(\xi) = \mu(\beta(y)\gamma(y')\delta(y''))$, where $\beta(y)$, $\gamma(y')$ and $\delta(y'')$ are differentiable functions; if*

$$\frac{F_{3t}(\mathbf{y})-F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})} = \frac{F_{2t}(\mathbf{y})-F_{0y'}(\mathbf{y})}{\xi_{y'}F_0(\mathbf{y})} = \frac{F_{1t}(\mathbf{y})-F_{0y}(\mathbf{y})}{\xi_{y}F_0(\mathbf{y})} = \frac{F_{1y'}(\mathbf{y})-F_{2y}(\mathbf{y})}{\xi_{y}F_2(\mathbf{y})-\xi_{y'}F_1(\mathbf{y})} =$$
$$\frac{F_{1y''}(\mathbf{y})-F_{3y}(\mathbf{y})}{\xi_{y}F_3(\mathbf{y})-\xi_{y''}F_1(\mathbf{y})} = \frac{F_{2y''}(\mathbf{y})-F_{3y'}(\mathbf{y})}{\xi_{y'}F_3(\mathbf{y})-\xi_{y''}F_2(\mathbf{y})}$$

and they are functions in $\xi := \beta(y)\gamma(y')\delta(y'')$. Moreover, the integrating factor is given by

$$\mu(\xi) = \exp\left\{\int \frac{F_{3t}(\mathbf{y})-F_{0y''}(\mathbf{y})}{\xi_{y''}F_0(\mathbf{y})}d\xi\right\}.$$

6. Concluding Remarks

In this paper, we investigated the existence of integrating factors of the following class of third order differential equations:

$$F_3(t,y,y',y'')y''' + F_2(t,y,y',y'')y'' + F_1(t,y,y',y'')y' + F_0(t,y,y',y'') = 0. \tag{6.1}$$

Particularly, we proved some results related to the existence of integrating factors of (6.1). We also presented some illustrative examples. We remark that these results not only useful for finding the integrating factors of (6.1) analytically but also computationally. In fact, we can check the validity of the conditions in these results by using the symbolic toolboxes in different mathematical softwares, e.g., MAPLE and MATLAB softwares. Also, by using these symbolic toolboxes, we can find an integrating factor of (6.1) using the explicit forms of the integrating factor given in our results. Moreover, by using the same argument in the proof of Theorem 2.1, we can derive an integrating factor of (6.1) in terms of $\xi = \alpha(t)+\beta(y)+\gamma(y')+\delta(y'')$, where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are differentiable functions.

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