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NEW EXACT TAYLOR'S EXPANSIONS WITHOUT THE REMAINDER: APPLICATION TO FINANCE

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ABSTRACT. We present new *exact* Taylor's expansions with fixed coefficients and without the remainder. We apply the method to the portfolio model.

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1. INTRODUCTION

In this paper, we provide an *exact* Taylor's expansion [5] with *constant* coefficients and without the remainder. In doing so, we provide an explicit (closed) form of the remainder. Needless to say, this pioneering contribution is extremely useful in many applications, such as the areas of optimization, integration, and partial differential equations PDEs, since it transforms any arbitrary function to an exact explicit quasi-linear form. Consequently, this method will simplify the solutions to cumbersome integrals, PDEs and optimization problems (see [1] for potential applications). Consequently, this contribution can be applied to many areas in operations research, such as optimization, stochastic models, and statistics.

We apply our approach to the heat equation and the reaction-convection-diffusion equation. In doing so, we use it to transform a (non-linear) PDE to a non-differential equation or an ordinary differential equation ODE. We also apply our method to the dynamic portfolio model in finance. In so doing, unlike previous literature, we derive an explicit solution to the investor's optimal portfolio when the utility function is unknown.

2. THE NEW EXPANSIONS

Theorem 2.1. *A sufficiently differentiable function with a compact support $f(.)$ is given by*

(i) $f(x) = a_1 + a_2 x + a_3 (x - c_2) \ln (x - c_2)$,

(i) $f(x,y) = a_4 + \mathring{c} + \bar{c}x + x\left\{\dot{a}_1 + \dot{a}_2 y + \dot{a}_3 (y - \dot{c}) \ln (y - \dot{c})\right\} + \ddot{a}_2 y + \ddot{a}_3 (y - \ddot{c}) \ln (y - \ddot{c}) + c_5 \left[(x - c_3) \ln (x - c_3) - x\right]$,

where c and a are constants.

Proof. (i) Consider these Taylor's expansions

$$f(x) = f(c_1) + f'(c_1)(x - c_1) + R_1(x), c_1 \neq 0, \tag{2.1}$$

$$f(x) = f(c_2) + R_2(x), \tag{2.2}$$

where R is the remainder and c is a constant. Taking the derivatives of the remainders w.r.t. x yields

$$R_1'(x) = f'(x) - f'(c_1),$$

$$R_2'(x) = f'(x).$$

Thus,

$$R_2'(x) - R_1'(x) = f'(c_1). \tag{2.3}$$

Dividing both sides of (2.3) by $x - c_2$ and using the mean value theorem, we obtain

$$\frac{R_2'(x) - R_1'(x)}{x - c_2} = \frac{f'(c_1)}{x - c_2}.$$

Using the mean value theorem and the fact that $R_2'(x)$ is a liner function of $R_1'(x)$, we obtain

$$R_2''(u) = \frac{R_2'(u) - R_2'(c_2)}{u - c_2} = \frac{f'(c_1)}{u - c_2}, c_2 \leq u \leq x, x - c_2 > 0.$$

Integrating yields

$$R(x) = \int_{c_2}^{x} \int_{c_2}^{x} \frac{f'(c_1)}{u - c_2} du du =$$

$$\hat{c} + \bar{c}x + f'(c_1)\left[(x - c_2) \ln (x - c_2) - x\right]. \tag{2.4}$$

Substituting (2.4) into (2.2) , we obtain

$$f(x) = f(c_2) + \hat{c} + \bar{c}x + f'(c_1)\left[(x - c_2)\ln(x - c_2) - x\right].$$

We can rewrite the above equation as

$$f(x) = a_1 + a_2 x + a_3 (x - c_2) \ln (x - c_2), \tag{2.5}$$

where a is a constant.

Multiple-variable functions:

As an example, we use a two-variable function, however the extension to a multiple-variable function is straightforward. As before we consider these Taylor's expansions

$$f(x,y) = f(c_1,c_2) + f_x(c_1,c_2)(x - c_1) + f_y(c_1,c_2)(y - c_2) + R_1(x,y), \tag{2.6}$$

$$f(x,y) = f(c_3,c_4) + R_2(x,y). \tag{2.7}$$

Taking the partial derivatives of the remainders w.r.t. x yields

$$R_{1x}(x,y) = f_x(x,y) - f_x(c_1,c_2),$$

$$R_{2x}(x,y) = f_x(x,y).$$

Therefore

$$R_{2x}(x,y) - R_{1x}(x,y) = f_x(c_1,c_2).$$

Using the procedure in the previous section, we obtain

$$R_{2xx}(.) = \frac{f_x(c_1,c_2)}{u - c_3}.$$

Integrating yields

$$R_2(.) = \int_{c_3}^{x}\int_{c_3}^{x} \frac{f_x(c_1,c_2)}{u - c_3} du du =$$

$$\hat{c} + \bar{c}x + g(y)x + h(y) + c_5\left[(x - c_3)\ln(x - c_3) - x\right], \tag{2.8}$$

where $c_5 \equiv f_x(c_1,c_2)$. Using (2.5) , we obtain

$$g(y) = \dot{a}_1 + \dot{a}_2 y + \dot{a}_3 (y - \dot{c})\ln(y - \dot{c}), \tag{2.9}$$

$$h(y) = \ddot{a}_1 + \ddot{a}_2 y + \ddot{a}_3 (y - \ddot{c})\ln(y - \ddot{c}). \tag{2.10}$$

Substituting $(2.9) - (2.10)$ into (2.8) , we obtain

$$\begin{aligned} R_2(x,y) &= \mathring{c} + \bar{c}x + x\left\{\dot{a}_1 + \dot{a}_2 y + \dot{a}_3 (y - \dot{c})\ln(y - \dot{c})\right\} + \\ &\quad \ddot{a}_2 y + \ddot{a}_3 (y - \ddot{c})\ln(y - \ddot{c}) + \\ &\quad c_5\left[(x - c_3)\ln(x - c_3) - x\right]. \end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.7) yields

$$\begin{aligned} f(x,y) &= f(c_3,c_4) + \mathring{c} + \bar{c}x + x\left\{\dot{a}_1 + \dot{a}_2 y + \dot{a}_3 (y - \dot{c})\ln(y - \dot{c})\right\} + \\ &\quad \ddot{a}_2 y + \ddot{a}_3 (y - \ddot{c})\ln(y - \ddot{c}) + c_5\left[(x - c_3)\ln(x - c_3) - x\right]. \end{aligned} \tag{2.12}$$

∎

3. THE PORTFOLIO MODEL AND THE HJB PDE

We first provide a brief description of the portfolio model (see [2, 3] and [4]), among others). The risk-free asset price process is given by $S_0 = e^{\int_t^T r_s ds}$, where $r_t \in C_b^2(R)$ is the risk-free rate of return. The dynamics of the risky asset price are given by

$$dS_s = S_s\left(\mu_s ds + \sigma_s dW_s\right), \tag{3.1}$$

where $\mu_s \in C_b^2(R)$ and $\sigma_s \in C_b^2(R)$ are the rate of return and the volatility, respectively; W_s is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t\leq s\leq T}$ is the augmentation of filtration.

The wealth process is given by

$$X_T^\pi = x + \int_t^T \left\{rX_s^\pi + (\mu_s - r_s)\pi_s\right\} ds + \int_t^T \pi_s\sigma_s dW_s, \tag{3.2}$$

where x is the initial wealth, $\{\pi_s, \mathcal{F}_s\}_{t\leq s\leq T}$ is the portfolio process, with $E\int_t^T \pi_s^2 ds < \infty$. The trading strategy $\pi_s \in \mathcal{A}(x, y)$ is admissible.

The investor's objective is to maximize the expected utility of the terminal wealth

$$V(t, x) = \underset{\pi}{Sup} E\left[U\left(X_T^\pi\right) \mid \mathcal{F}_t\right],$$

where $V(.)$ is the value function, $U(.)$ is a continuous, bounded and strictly concave utility function. Under well-known assumptions, the value function satisfies the Hamilton-Jacobi-Bellman PDE

$$V_t + rxV_x + \pi_t^*\left(\mu_t - r_t\right)V_x +$$

$$\frac{1}{2}\pi_t^{*2}\sigma_t^2 V_{xx} = 0; V(T, x) = U(x).$$

The optimal portfolio is given by

$$\pi_t^* = -\frac{(\mu_t - r_t)V_x}{\sigma_t^2 V_{xx}}. \tag{3.3}$$

Using (2.12) yields

$$\begin{aligned} V(x,t) &= \check{c}_1 + \check{c}_2 x + \check{c}_3 t + x\left\{\dot{a}_1 + \dot{a}_2 t + \dot{a}_3 (t - \dot{c})\ln(t - \dot{c})\right\} + \\ &\quad \ddot{a}_3(t - \ddot{c})\ln(t - \ddot{c}) + c_5\left[(x - c_6)\ln(x - c_6)\right]. \end{aligned} \tag{3.4}$$

Taking the partial derivatives of (3.4) yields

$$V_x(x,t) = \check{c}_2 + \left\{\dot{a}_1 + \dot{a}_2 t + \dot{a}_3(t - \dot{c})\ln(t - \dot{c})\right\} + c_5\left[\ln(x - c_6) + 1\right], \tag{3.5}$$

$$V_{xx}(x,t) = \frac{c_5}{x - c_6}. \tag{3.6}$$

Substituting this into (3.3) yields (and without loss of generality letting $t = 0$)

$$\pi_t^* = -\frac{(x - c_6)(\mu_t - r_t)\left(a_5 + c_5\left[\ln(x - c_6)\right]\right)}{\sigma_t^2}. \tag{3.7}$$

Another significant contribution of this paper is that the optimal portfolio explicitly depends on the wealth x, regardless of the functional form of the utility (even for an exponential utility). It is well known that a major criticism of the exponential utility is that the optimal portfolio does not depend on the wealth.

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