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POLYANALYTIC FUNCTIONS ON SUBSETS OF $\mathbb{Z}[i]$

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ABSTRACT. For positive integers q we consider the kernel of the powers L^q where L is one of three kinds of discrete analogues of the Cauchy-Riemann operator. The first two kinds are well-studied, but the third kind less so. We give motivations for further study of the third kind especially since its symmetry makes it more appealing for the cases $q \geq 2$. From an algebraic perspective it makes sense that the chosen multiplication on the kernels is compatible with the choice of pseudo-powers. We propose such multiplications together with associated pseudo-powers. We develop a proof-tool in terms of certain sets of uniqueness.

Key words and phrases: Monodiffric functions; Pseudo-powers; q-analytic functions; q-polyanalytic functions.

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CONTENTS

1. INTRODUCTION

The purpose of this paper is to describe the kernel of the powers three kinds of discrete analogues of the Cauchy-Riemann operator on subsets of $\mathbb{Z}[i]$. Members of the kernels are called q-polyanalytic functions (or polyanalytic functions of order q). We also propose, based upon previous work choices of multiplication together with associated pseudo-powers and we develop a proof-tool and use it to e.g. investigate a question posed by Kiselman [12].

Some of the pioneers of the investigation of analogues of complex analytic functions on $\mathbb{Z}[i]$ were Isaacs [8], [9], Ferrand [7] and Duffin [4],[5].

In Isaacs [8] the *monodiffric functions of the first kind* on a the discrete complex plane, where defined square-wise as those that where annihilated by a certain first order linear difference operator, in particular a complex-valued function f on $\mathbb{Z}[i]$, is monodiffric of the first kind on a square with vertices $\{z, z+1, z+i, z+i+1\}$, whose lower left point is $z \in \mathbb{Z}[i]$, if and only if f satisfies,

$$f(z+1) - f(z) = \frac{f(z+i) - f(z)}{i} \tag{1.1}$$

We shall say that f is monodiffric of the first kind *at* z if and only if f satisfies equation 1.1. In this paper we shall say that a function f in the discrete complex plane is *monodiffric functions of the second kind* at $z \in \mathbb{Z}[i]$, if and only if

$$\frac{f(z+1+i) - f(z)}{i+1} = \frac{f(z+i) - f(z+1)}{i-1} \tag{1.2}$$

Ferrand [7] (who uses a discrete version of Moreras theorem) used the term *preholomorphic* for the monodiffric functions of the second kind. In this paper we shall say that a function f in the discrete complex plane is *monodiffric functions of the third kind* at $z \in \mathbb{Z}[i]$,

$$f(z+1) - f(z-1) = \frac{f(z+i) - f(z-i)}{i} \tag{1.3}$$

The monodiffric functions of the third kind (these where also introduced by Isaacs [8], p.179) appear less frequently in the literature, and then they are not referred to as monodiffric functions of the third kind.

In this paper we shall be interested in powers of the operators in equation 1.1, 1.2 and 1.3 respectively. To avoid confusion we point out that in Kurowski [14], the functions that we here call monodiffric of the third kind, are called monodiffric of the second kind. We have chosen not to adapt that terminology and instead use the terminology used by e.g. Kiselman [11], [12], regarding monodiffric functions of the first and second kind. We shall in Section 6 motivate why monodiffric functions of the third kind deserve attention of their own.

Definition 1.1 (q-polyanalytic functions (polyanalytic functions of order q))**.** Define for complex-valued functions f on $\mathbb{Z}[i]$,

$$L_1 f(z) := f(z+1) - f(z) + i(f(z+i) - f(z)) \tag{1.4}$$

$$L_2 f(z) := f(z) + if(z+1) - f(z+1+i) - if(z+i) \tag{1.5}$$

$$L_3 f(z) := f(z+1) - f(z-1) + i(f(z+i) - f(z-i)) \tag{1.6}$$

We define, for a given positive integer q, and a fixed $j \in \{1,2,3\}$, a complex-valued function $f \colon \mathbb{Z}[i] \to \mathbb{C}$ to be:
q-polyanalytic (or *polyanalytic of order q*) *of the first kind* at $z \in \mathbb{Z}[i]$ if

$$L_1^q f(z) = 0 \tag{1.7}$$

q-polyanalytic (or *polyanalytic of order q*) *of the second kind* at $z \in \mathbb{Z}[i]$ if

$$L_2^q f(z) = 0 \tag{1.8}$$

and *q-polyanalytic* (or *polyanalytic of order q*) *of the third kind* at $z \in \mathbb{Z}[i]$ if

$$L_3^q f(z) = 0 \tag{1.9}$$

If the condition holds true at each point of a subset $S \subseteq \mathbb{Z}[i]$ where the defining operator is defined, then we say that f is q-polyanalytic (or polyanalytic of order q) of the first, second or third kind, respectively on S and when it is clear from the context what S is we simply say that f is q-polyanalytic (or polyanalytic of order q) of the first, second or third kind respectively.

Remark 1.1. Obviously a complex-valued function is monodiffric of the j:th kind at $z \in \mathbb{Z}[i]$, if and only it is 1-polyanalytic (or polyanalytic of order 1) of the j:th kind at $z \in \mathbb{Z}[i]$, where $j \in \{1,2,3\}$. The use of q-monodiffric is already taken in the literature, see Tu [18], p.237, where, for a given $p \in (0,1]$, a complex-valued function f on $D := \{z \in \mathbb{C} \colon (\operatorname{Re} z \in j\mathbb{N}) \wedge (\operatorname{Im} z \in j\mathbb{N})\}$, is defined to be p-monodiffric at $w \in D$ if $f(w) = (i-1)f(w) + f(w+ip) - if(w+p)$. The term n-analytic (as a variant to polyanalytic of order n) has been present in classical complex analysis for the members of the kernel of $\overline{\partial}^n$ at least since 1970, see e.g. Bosch & Kraijkiewvicz [2] and more recent work by e.g. Ramazanov [16], [17], Cuckovic & Le [3] and Fedorovskiy [6]. The term polyanalytic seems more useful only to describe the general case when the order is not of particular interest. The term q-analytic would be preferred by the author also in the context of this paper, but we have been informed that it is, in discrete analysis, used with other meaning. The term polyanalytic of order q, under the given circumstances of this article, has two direct draw-backs. The first being that when specifying the kind, possible phrasings like polyanalytic of order q of the j:th kind, are longer (than q-polyanalytic of the j:th kind) and can be ambiguous. Furthermore, as in the case in the analogous theory in classical complex analysis, authors like Avanissian & Traoré [1] worked with generalizations to higher complex dimension (in the sense that the order q must be replaced by a multi-index α) it is most likely that generalizations will be studied on powers of $\mathbb{Z}[i]$ in the discrete theory, in which case one must replace the order q, by a multi-index, and a short and distinct terminology would then be α-analytic or α-polyanalytic.

For a formalistic motivation of the monodiffric functions of the third kind, we shall in Section 6 start from more basic structures and construct the archetype for the operator and in doing so we believe adjacency is useful in motivating the defined operators and furthermore some level of structure such as multiplication is a priori not required. We shall in Section 6 motivate the first order linear operator given by equation 1.3 by analogy to the differential geometric definition of holomorphic functions and we believe that its powers will be natural analogues of $\overline{\partial}^q$. In Section 3 we shall solve for the q-polyanalytic functions (or polyanalytic functions of order q) of the three kinds respectively. In Section 4 we propose, based upon previous work, some natural multiplicative structures for the function spaces over $\mathbb{Z}[i]$, for each of the three kinds of analytic functions. In relation to pseudo-powers associated to such multiplicative structures, we investigate a question posed by Kiselman [12].

2. INITIAL COMPARISON BETWEEN THE SECOND AND THIRD KIND

Kiselman [11], Sec 3, has astutely pointed out that at the level of ideas the operators defined by equation 1.2 and 1.3 are quite similar. Kurowski [13], p.1, makes a remark that we interpret to be of similar sort to Kiselman's point. It is clear however that the solution spaces defined by the operators L_2, L_3 are not, in any formal rigorous way, equivalent. This can easily be displayed by example. See Proposition 5.1 for a more comprehensive result regarding any pair among L_1, L_2, L_3.

Example 2.1. *Let $z \in \mathbb{Z}[i]$, and let $V := \{z, z+1, z+2, z+2+i, z+2+2i, z+1+2i, z+2i, z+1, z+1+i\}$. Now relative to V, the points $z, z+1, z+1+i, z+i$, are the only points where L_2 is defined whereas $z+1+i$ is the only point where L_3 is defined. To this end, set $f(z+2+i) = 1, f(z+i) = f(z+1+2i) = f(z+1) = 0$ (making sure that $L_3 f(z+1+i) \neq 0$) and set $f(z+1+i) = 0$. Then there are four undefined values $f(z), f(z+2), f(z+2+2i), f(z+2i)$, and we invoke four conditions $L_2 f(z) = L_2 f(z+1) = L_2 f(z+1+i) = L_2 f(z+i) = 0$, giving,*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \begin{bmatrix} f(z) \\ f(z+2) \\ f(z+2+2i) \\ f(z+2i) \end{bmatrix} =$$

$$\begin{bmatrix} -if(z+1) + f(z+1+i) + if(z+i) \\ -f(z+1) - if(z+2+i) + if(z+1+i) \\ -f(z+1+i) - if(z+2+i) + if(z+1+2i) \\ -f(z+i) - if(z+1+i) + f(z+1+2i) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Which gives,

$$\begin{bmatrix} f(z) \\ f(z+2) \\ f(z+2+2i) \\ f(z+2i) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \\ -1 \\ 0 \end{bmatrix}$$

Thus, we have a complex-valued function, f, on V such that $L_2 f(z) = L_2 f(z+1) = L_2 f(z+1+i) = L_2 f(z+i) = 0$ but $L_3 f(z+1+i) \neq 0$.

We choose $z = 0$. Set $f(w) = 0$, for all $w \in V'$ where $V' = \{z = x+iy \in \mathbb{Z}[i] \setminus V : x \cdot y = 0\}$. This uniquely defines an extension of f that satisfies $L_2 f = 0$ at all points of $\mathbb{Z}[i]$. Indeed consider the point $(2+k)+i$ where $k = 1$. The value $f(2+k+i)$ is uniquely determined by the value at $2+k, 2+k-1, (2+k-1)+i$ which are all known for $k = 1$. Once the value at $f(2+k+i)$ is determined the process can be repeated by replacing k with $k+1$, thus uniquely determining f on the set $V \cup \{y = 1, x \geq 3\}$. Obviously we can then iteratively repeat the process row-wise until we have f uniquely determined on the upper right quadrant. Analogously, we have the unique extension of f that satisfies $L_2 f = 0$, to each the three remaining quadrants by the same iterative process (See Section 3 for a formalization of this process). This yields an extension of f that is an entire 1-polyanalytic of the second kind, and clearly no extension of f can be 1-polyanalytic of the third kind at $z+1+i$. Proposition 5.1 below gives a more comprehensive result on these matters.

2.1. **A note on integrals.** A *polygon*, Γ, in the complex plane $\mathbb{C}$ consists of a set of N edges $[a_0, a1], [a_1, a_2], \ldots, [a_{N-1}, a_0]$, where $a_j, j = 0, \ldots, N$ are given points in $\mathbb{C}$. In particular it is determined by the ordered set of vertices $(a_0, \ldots, a_N) \in \mathbb{C}^N$. A function f defined on Γ is called *piecewise affine* if f is affine on each segment $[a_j, a_{j+1}]$ with the possible exception of the points that belong to two or more segments. Γ is called *closed* if $a_0 = a_N$. Let f be piecewise affine on a polygon Γ determined by $(a_0, \ldots, a_N) \in \mathbb{C}^N$. Then (see e.g. Kiselman [12], p.2)

$$\int_\Gamma f(z)dz = \frac{1}{2}\sum_{j=1}^{N}(f(a_j) + f(a_{j-1}))(a_j - a_{j-1}) \tag{2.1}$$

Definition 2.1 (Integral of complex functions along polygons)**.** Let f be a complex-valued function on a polygon Γ determined by $(a_0, \ldots, a_N) \in \mathbb{C}^N$. We define the integral of f along Γ as

$$\int_\Gamma f(z)dz = \frac{1}{2}\sum_{j=1}^{N}(f(a_j) + f(a_{j-1}))(a_j - a_{j-1}) \tag{2.2}$$

Let f be a complex-valued function on $\mathbb{Z}[i]$. Let $p_0 \in \mathbb{Z}[i]$, and denote by Γ_{p_0} the closed polygon defined by the ordered set of vertices $(a_1, a_2, a_3, a_4) := (p_0, p_0+1, p_0+1+i, p_0+i)$, i.e. moving counter-clockwise.

It is easy to verify that f is 1-polyanalytic of the second kind at p_0 iff

$$\int_{\Gamma_{p_0}} \tilde{f}(z)dz = 0 \tag{2.3}$$

where $\tilde{f}$ is the unique piecewise affine function on the closed polygon Γ_{p_0} such that $\tilde{f}(z) = f(z)$ for $z \in \{p_0, p_0+1, p_0+i, p_0+(1+i)\}$. Indeed $\int_{\Gamma_{p_0}} f = (f(p_0+1+i)+f(p_0+1))i + (f(p_0+i) + f(p_0+1+i))(-1) + (f(p_0)+f(p_0+i))(-i) + (f(p_0+1)+f(p_0)) = (1-i)f(p_0) + (1+i)f(p_0+1) + (-1-i)f(p_0+i) + (i-1)f(p_0+1+i) = (1-i)(f(p_0)+if(p_0+1) - f(p_0+1+i) - if(p_0+i)) = (1-i)L_2 f(p_0) = 0$. (In fact a stronger result holds true, see Remark 2.1).
As we have pointed out the condition of equation 1.2 at p_0 does not involve all four adjacent points to p_0 but it actually involves the point p_0+i+1 which is not adjacent to p_0 in the usual sense. Considering the operator L_3 which at z involves precisely the four points $z \pm i, z \pm 1$, we have the following.

Observation 2.1. Let f be a complex-valued function on $\mathbb{Z}[i]$, let $p_0 \in \mathbb{Z}[i]$, and denote by Γ_{p_0} the closed polygon defined by the ordered set of vertices $(p_0+1, p_0+i, p_0-1, p_0-i)$. Then,

$$L_3 f(p_0) = 0 \Leftrightarrow \int_{\Gamma_{p_0}} \tilde{f}(z)dz = 0 \tag{2.4}$$

where $\tilde{f}$ is the unique piecewise affine function on the closed polygon Γ_{p_0} such that $\tilde{f}(z) = f(z)$ for $z \in \{p_0+1, p_0+i, p_0-1, p_0-i\}$.

Proof. Using the notation from equation 2.3 together with equation 2.1, we have

$$
\begin{aligned}
(2.5)\quad 2\int_{\Gamma_{p_0}} f(z)dz = 0 \Leftrightarrow (f(p_0+i)+f(p_0+1))(i-1)+ \\
(f(p_0-1)+f(p_0+i))(-1-i)+(f(p_0-i)+f(p_0-1))(1-i)+ \\
(f(p_0+1)+f(p_0-i))(1+i) = \\
-2f(p_0+i)+2if(p_0+1)-2if(p_0-1)2f(p_0-i) = \\
-\frac{2}{i}(i(f(p_0+i)-f(p_0-i))+f(p_0+1)-f(p_0-1)) = \\
-\frac{2}{i}L_3 f(p_0) = 0 \Leftrightarrow L_3 f(p_0) = 0
\end{aligned}
$$

∎

Definition 2.2 (Zig-zag polygons)**.** A polygon determined by the ordered set of (possibly infinite) points $(a_0,\ldots,a_N)$,(or $(a_0,\ldots)$) $a_j \in \mathbb{Z}[i]$, $j = 0,\ldots,N$ (or $j = 0,1,\ldots$) is called a *zig-zag polygon* if $a_j - a_{j-1} \in \{1 \pm i, -1 \pm i\}$, $j = 1,\ldots,N$ (or $j = 0,1,\ldots$) some positive integer N. It is *non-selfintersecting* if $a_k \neq a_l$ for $k \neq l$ except possibly for $(k,l) \in \{(0,N),(N,0)\}$. It is further called *closed* if it has $N-1$ points where $a_0 = a_N$. A point of any subset $\omega \subset \mathbb{Z}[i]$ is called an *interior point* if and only if all four of its adjacent points of first order belong to ω. The set of interior points is denoted $\mathring{\omega}$. A *subset with zig-zag boundary*, $\omega \subset \mathbb{Z}[i]$ is the union of a possibly infinite set of points together with their sets of adjacent points of first order, such that the set of non-interior points can be ordered to yield a non-selfintersecting zig-zag polygon. Such ω is called a *domain with zig-zag boundary* if each pair of interior points can be connected by a zig-zag polygon contained in $\mathring{\omega}$ and it is called a *simple domain with zig-zag boundary* if each pair of non-interior points can be connected by a zig-zag polygon in the set of non-interior points. A point is called *zig-zag even* (*odd*) if it can be connected by a zig-zag polygon to 0 (1). We also call 0 zig-zag even and we call 1 zig-zag odd.

Obviously no zig-zag even point can be connected to a zig-zag odd point by a zig-zag polygon, in particular a zig-zag polygon consists either entirely of zig-zag even points or entirely of zig-zag odd points.

Proposition 2.2. *Let f be a complex-valued function on $\mathbb{Z}[i]$. Let $\Omega \subset \mathbb{Z}[i]$ be a simple domain with zig-zag boundary. The function f satisfies $L_3 f(z) = 0$ on $\mathring{\Omega}$ if and only if for any closed non self-intersecting zig-zag polygon $\gamma \subset \Omega$ defined by an ordered set of vertices*

$$\int_\gamma f(z)dz = 0 \tag{2.6}$$

Proof. First of all, a function on a domain with zig-zag boundary satisfies $L_3 f(z) = 0$ on $\mathring{\Omega}$ if and only if $L_3 f(z) = 0$ on $\mathring{\omega}$ for any domain with zig-zag boundary $\omega \subseteq \Omega$, with finitely many elements. Hence we only need to prove the statement for the case of finitely many interior points. We use induction in the number, n, of interior points (see Definition 2.2) of the discrete domain, Ω, with Γ as its zig-zag boundary, where Γ is defined by the ordered set of points $(a_0,\ldots,a_{N-1})$. The case $n = 1$ is precisely equation 2.4. Assume $n > 1$ and that the result holds true for the case of $n-1$ interior points. Since n is finite we can find an interior point, $z_0+i = x_0+iy_0$, such that y_0 is minimal and finite. In particular the points z_0-1+i, z_0, z_0+1+i belong to Ω but are not interior points and we can assume $a_{n_0} = z_0$. We can assume that the boundary is traversed counter-clockwise as vanishing of the integral will be independent with reversed direction. First consider when also the point $z_0 + 2i$ is not an interior point of Ω, i.e. Γ contains the ordered subsequence $z_0+2i, z_0-1+i, z_0, z_0+1+i$. The set $\Omega' := \Omega \setminus \{z_0, z_0+i, z_0-1+i\}$, is again a domain with zig-zag boundary, Γ', defined by $(a_0, \ldots, a_{n_0-3}, z_0+2i, z_0+1+i, a_{n_0+2}, \ldots, a_{N-1})$. The only domains in Ω with boundary γ that is a closed non self-intersecting zig-zag polygon, such that γ does not also lie in Ω', is either simply the polygon, $\hat{\gamma}_{z_0}$, defined by $(z_0+2i, z_0-1+i, z_0, z_0+1+i)$ or γ is defined by $(b_0, \ldots, z+2i, z_0-1+i, z_0, z_0+1+i, \ldots, b_{M-1})$ for some positive integer M and points b_j in Ω', in particular does not contain z_0+i. (Note that the case where γ contains z_0+i cannot occur because the point z_0+2i is assumed not to be an interior point). The first case is handled by a translated version of equation 2.4. So assume the second case. Then letting γ' be the closed non self-intersecting zig-zag polygon $\gamma' \subseteq \Omega'$ defined by $(b_0, \ldots, z_0+2i, z_0+1+i, \ldots, b_{M-1})$ (i.e. the two points z_0-1+i, z_0 are removed) we have

$$
\begin{aligned}
(2.7)\quad 2\left(\int_{\gamma'} - \int_{\gamma}\right) f(z)dz &= (f(z_0+2i) + f(z_0+1+i))(i-1)- \\
&[(f(z_0+2i) + f(z_0-1+i))(i+1) + (f(z_0-1+i) + f(z_0))(-1+i)+ \\
&(f(z_0) + f(z_0+1+i))(-1-i)] = \\
&-2f(z_0+2i) + 2f(z_0) + 2if(z_0+1+i) - 2if(z_0-1+i) = \\
&-\frac{2}{i}(if(z_0+2i)) - if(z_0) + f(z_0+1+i) - f(z_0-1+i)) \\
&= -\frac{2}{i}L_3 f(z_0+i) = 0
\end{aligned}
$$

But Ω' has $n-1$ interior points ($z+i$ is not part of the set) thus by the induction hypothesis $\int_{\gamma'} f(z)dz = 0$ for any closed non self-intersecting zig-zag polygon $\gamma' \subseteq \Omega'$. This takes care of the case when z_0+2i is not an interior point.

Now assume that z_0+2i is an interior point of Ω (in particular this implies that $z+3i, z+2i\pm 1$ belong to Ω).

Then the set $\Omega' := \Omega \setminus \{z_0, z_0+i\}$ is again a domain with zig-zag boundary, Γ', defined by $(a_0, \ldots, z_0-1+i, z_0+2i, z_0+1+i, \ldots, a_{N-1})$, but Ω' has $n-1$ interior points thus by the induction hypothesis $\int_{\gamma'} f(z)dz = 0$ for any closed non self-intersecting zig-zag polygon $\gamma' \subseteq \Omega'$.

It is easy to see that the only closed non self-intersecting zig-zag polygon $\gamma \subseteq \Omega$ which does not also lie in Ω', is one defined by either $(b_0, \ldots, z_0-1+i, z_0, z_0+1+i, \ldots, b_{M-1})$ or $(b_0, \ldots, z_0-1+2i, z_0+i, z_0+1+2i, \ldots, b_{M-1})$ for some positive integer M and points b_j in Ω'.

For each such γ define the associated closed non self-intersecting zig-zag polygon $\gamma' \subseteq \Omega'$ by $(b_0, \ldots, z_0-1+i, z_0+2i, z_0+1+i, \ldots, b_{M-1})$ in the first case and by $(b_0, \ldots, z_0-1+$

$2i, z_0+3i, z_0+1+2i, \ldots, b_{M-1})$ in the second case. In the first case we have

$$(2.8)\quad 2\left(\int_{\gamma'} - \int_{\gamma}\right) f(z)dz = (f(z_0+2i)+f(z_0-1+i))(1+i)+$$
$$(f(z_0+1+i)+f(z_0+2i))(1-i)-[(f(z_0)+f(z_0-1+i))(1-i)+$$
$$(f(z_0+1+i)+f(z_0))(1+i)] = (f(z_0+2i)+f(z_0-1+i))(1+i)+$$
$$(f(z_0+1+i)+f(z_0+2i))(1-i)+(f(z_0)+f(z_0-1+i))(i-1)+$$
$$(f(z_0+1+i)+f(z_0))(-1-i) = -2f(z_0)+2if(z_0-1+i)-2if(z_0+1+i)+$$
$$2f(z_0+2i) = \frac{2}{i}(i(f(z_0+2i)-f(z_0))+f(z_0+1+i)-f(z_0-1+i)) =$$
$$\frac{2}{i}L_3 f(z_0+i) = 0$$

In the second case, if z_0+3i is not an interior point then we can repeat the procedure applied to the case when z_0+2i was not an interior point. If z_0+3i is an interior point the same calculations as in equation 2.8, but translated one step in the $\operatorname{Im} z$-directions yields

$$(2.9)\qquad 2\left(\int_{\gamma'} - \int_{\gamma}\right) f(z)dz = \frac{2}{i}L_3 f(z_0+2i) = 0$$

This proves the induction step. This completes the proof. ∎

Remark 2.1. Kiselman [12] defines a polygon determined by the ordered set $(a_0, \ldots, a_N)$, $a_j \in \mathbb{Z}[i]$, $j = 0, \ldots, N$ to be a *4-curve* if $a_j - a_{j-1} \in \{\pm 1, \pm i\}$, $j = 1, \ldots, N$ and it is a well-known result see e.g. Isaacs [8], p.183, that if f is a monodiffric function of the first kind then

$$(2.10)\qquad \int_{\gamma} f(z)dz = 0$$

for each closed (non-selfintersecting) 4-curve γ. The corresponding result for monodiffric functions of the second kind also holds true (see e.g. Duffin [5], Corollary 2.1.1).

3. CHARACTERIZING THE SET OF q-POLYANALYTIC FUNCTIONS

In this section we shall obtain the kernels of the powers of the operators L_1, L_2, L_3 which in turn give the defining difference equations for q-polyanalytic functions on $\mathbb{Z}[i]$.

For background on solving finite difference equations see e.g. Mickens [15] and Jordan [10]. For this particular section we shall in the interest of conformity with previous literature use some special notations.

3.1. **First kind, $q = 1$.** For a complex-valued function f on $\mathbb{Z}\times\mathbb{Z}$ (or $\mathbb{Z}[i]$ in which case we shall still write $f(k,l)$ instead of $f(k+il)$ where $(k,l)\in\mathbb{Z}^2$), we use the notation

$$(3.1)\qquad E_1 f(k,l) := f(k+1,l) - f(k,l), \quad E_2 f(k,l) := f(k,l+1) - f(k,l)$$

This implies that a function is monodiffric of the first kind on $\mathbb{Z}[i]$ if at each $k+il\in\mathbb{Z}$, we have

$$(3.2)\qquad E_1 f(k,l) = (iE_2 - I + i) f(k,l)$$

where I denotes the identity operator. For $k \geq 0$, this can be is solved by the symbolic method of Boole (see e.g. Jordan p.616) to yield,

$$(3.3)\qquad f(k,l) = (iE_2 - I + i)^k \phi(l) = \sum_{j=0}^{k} i^j \binom{k}{j} E_2^j \phi(l)$$

where ϕ is an arbitrary function. Hence $f(k,l)$ is monodiffric on $\mathbb{Z}[i]$ if and only if at each point $k+il \in \mathbb{Z}[i], k \geq 0$, we have,

$$f(k,l) = \sum_{j=0}^{l} i^j \binom{k}{j} \phi(l+j) \tag{3.4}$$

for an arbitrary ϕ.

Obviously, given a function f defined for $\{z \in \mathbb{Z}[i] \colon \operatorname{Re} z \geq 0\}$, and 1-polyanalytic of the first kind, to obtain an extension to $\mathbb{Z}[i]$ that is 1-polyanalytic of the first kind it is sufficient to know the values of f on $\{z \in \mathbb{Z}[i] \colon \operatorname{Re} z < 0\}$.

Observation 3.1. Let f be 1-polyanalytic of the first kind on $\mathbb{Z}[i]$, let $z = x+iy$ denote the standard coordinate in $\mathbb{Z}[i]$, and let $p_0 \in \mathbb{Z}[i]$. Then f is uniquely determined by its values on $D_0 := \{z \in \mathbb{Z}[i] \colon \operatorname{Re} z = \operatorname{Re} p_0\} \cup \{z \in \mathbb{Z}[i] \colon (\operatorname{Re} z < \operatorname{Re} p_0) \wedge (\operatorname{Im} z = \operatorname{Im} p_0)\}$ Also any proper subset of $D' \subset D$ is not a set of uniqueness (in the sense that there are two different 1-polyanalytic functions on $\mathbb{Z}[i]$ that agree on D').

Proof. We can use the same procedure as that in Example 2.1 but applied to L_1 instead of L_2 namely start from the set, D_0. Define iteratively the sets $D_j, j \in \mathbb{N}$, as follows: Let D_{j+1} be the set of all points $s \in \mathbb{Z}[i]$ satisfying $s \in \{w, w+1, w+i\}$ for some $w \in \mathbb{Z}[i]$ such that precisely one point of the set $\{w, w+1, w+i\}$ does not belong to D_j. Then $\bigcup_j D_j = \mathbb{Z}[i]$. f can be iteratively extended to each D_j by assigning the value of f at $s \in D_j \setminus D_{j-1}$ to be determined by the equation $f(w+1) + if(w+i) - (1+i)f(w) = 0$. Obviously replacing the value of f at a point of D_0 yields a different extension to $\mathbb{Z}[i]$. This completes the proof. ∎

Remark 3.1 (Uniqueness of extension)**.** In the proof of Observation 3.1, 3.2,3.4 respectively, the procedure for obtaining the extension of f from its values on the set D_0 can only be done in one way. In particular, the only extension of a function that vanishes on D_0 is the function that vanishes identically on $\mathbb{Z}[i]$. In other words, if two functions F_1, F_2 on $\mathbb{Z}[i]$, satisfy for some $j = 1,2,3$, $L_j F_1 \equiv 0$, and $L_j F_2 \equiv 0$ then

$$F_1|_{D_0} = F_2|_{D_0} \Leftrightarrow F_1 \equiv F_2 \tag{3.5}$$

Definition 3.1. For a fixed $j \in \{1,2,3\}$, we say that a set $D \subset \mathbb{Z}[i]$ is a *set of uniqueness* (with respect to L_j) if any function f satisfying $L_j f \equiv 0$ on $\mathbb{Z}[i]$ and $f|_D \equiv 0$ must vanish identically. It is a *minimal* set of uniqueness if it does not properly contain any other set of uniqueness.

3.2. **Second kind,** $q = 1$**.** From a purely theoretical perspective we can find two parametrized independent solutions in the kernel of L_2 as follows. We use Lagrange's method (see e.g. Mickens Section 5.3, p.186) in order to find a particular solution that depends on a parameter. Set $\tilde{f}(k,l) := f(k-1, l-1)$, $(k,l) \in \mathbb{Z}^2$. Obviously, $\tilde{f}$ is 1-polyanalytic if and only if f is 1-polyanalytic. we can write the defining equations of f being 1-polyanalytic of the second kind according to,

$$(E_1^2 E_2 - I + iE_2^2 E_1 - iI)\tilde{f}(k,l) = 0, \quad \forall (k,l) \in \mathbb{Z}^2 \tag{3.6}$$

Set $\phi(E_1, E_2) := E_1^2 E_2 - I + iE_2^2 E_1 - iI$. To find a particular solution we look for those of the form $\lambda^k \mu^l$, and we consider the equation, $\phi(\lambda, \mu) = 0$, i.e.,

$$\lambda^2 \mu + i\mu^2 \lambda - 1 - i = 0 \tag{3.7}$$

Denote by $\lambda_j(\mu)$ the two roots of this equation, which yields the two particular solutions $(\lambda_j(\mu))^k \mu^l, j = 1,2$. By linearity the sum of all such expressions for all possible values of

μ will also be solutions. Let $D_j(\mu)$ be arbitrary functions of μ, $j = 1, 2$. This yields two independent solutions (1-polyanalytic functions of the second kind)

$$\tilde{f}_j(k,l) := \int_{(\operatorname{Re}\mu, \operatorname{Im}\mu)\in\mathbb{R}^2} D_j(\mu)(\lambda_j(\mu))^k \mu^l d\operatorname{Re}\mu d\operatorname{Im}\mu, \quad j = 1, 2 \tag{3.8}$$

However from a practical point of view, equation 3.8 are rather intractable and intangible and more work is required to determine whether every 1-polyanalytic function of the second kind corresponds to such a sum. We can instead, as in the case of 1-polyanalytic functions of the first kind, consider minimal determining sets (minimal sets of uniqueness) in order to describe the kernel of L_2.

Observation 3.2. Let f be 1-polyanalytic of the second kind on $\mathbb{Z}[i]$, let $z = x + iy$ denote the standard coordinate in $\mathbb{Z}[i]$, and let $p_0 \in \mathbb{Z}[i]$. Then f is uniquely determined by its values on $D_0 := \{z \in \mathbb{Z}[i] \colon (\operatorname{Re} z - \operatorname{Re} p_0)(\operatorname{Im} z - \operatorname{Im} p_0) = 0\}$ Also any proper subset of $D' \subset D$ is not a set of uniqueness (in the sense that there are two different 1-polyanalytic functions on $\mathbb{Z}[i]$ that agree on D').

Proof. We can easily use the procedure in Example 2.1 as follows. Start from the set, D_0. Define iteratively the sets $D_j, j \in \mathbb{N}$, by letting D_{j+1} be the set of all points $s \in \mathbb{Z}[i]$ satisfying $s \in \{w+1, w-1, w+i, w-i\}$ for some $w \in \mathbb{Z}[i]$ such that precisely 1 point of the set $\{w, w+1, w+1+i, w+i\}$ does not belong to D_j. Then $\bigcup_j D_j = \mathbb{Z}[i]$. f can be iteratively extended to each D_j by assigning the value of f at $s \in D_j \setminus D_{j-1}$ to be determined by the equation $f(w+1) + if(w+1) - f(w+1+i) - if(w+i) = 0$. Obviously replacing the value of f at a point of D_0 yields a different extension to $\mathbb{Z}[i]$. This completes the proof. ∎

3.3. **Third kind,** $q = 1$. Let $f(k,l) = u(k,l) + iv(k,l)$ be a 1-polyanalytic function of the third kind (where u and v are the real and imaginary parts of f respectively). We will start by explaining from a theoretical point of view how one could go about solving for the 1-polyanalytic function of the third kind, however as will be clear this approach can be non-tractable in practice which is why we then give a description of the kernel of L_3 also in terms of minimal sets of uniqueness.
For the theoretical perspective we start by noting that $L_3 f(k,l) = 0$ is equivalent to the pair of equations

$$u(k+1,l) - u(k-1,l) = v(k,l+1) - v(k,l-1) \tag{3.9}$$

$$u(k,l+1) - u(k,l-1) = v(k-1,l) - v(k+1,l) \tag{3.10}$$

Now equation 3.10 yields

$$v(k+1,l) = v(k-1,l) - u(k,l+1) + u(k,l-1) \tag{3.11}$$

If we replace (k,l) by $(k-1,l+1)$ and $(k-1,l-1)$ respectively in equation 3.11 we get the pair of equations

$$v(k,l+1) = v(k-2,l+1) - u(k-1,l+2) + u(k-1,l) \tag{3.12}$$

$$v(k,l-1) = v(k-2,l-1) - u(k-1,l) + u(k-1,l-2) \tag{3.13}$$

Now equations 3.12 and 3.13 combined with equation 3.9 yield

$$\begin{aligned} u(k+1,l) - u(k-1,l) = {} & v(k-2,l+1) - v(k-2,l-1) + \\ & u(k-1,l-2) - u(k-1,l+2) + 2u(k-1,l) \end{aligned} \tag{3.14}$$

i.e.

$$u(k+1,l)-u(k-1,l)=u(k-3,l)-u(k-1,l)+ \\ u(k-1,l-2)-u(k-1,l+2)+2u(k-1,l) \tag{3.15}$$

Hence a partial difference equation for a real-valued function in two variables. To solve it we shall use Laplace's method of generating functions (see Jordan [10], p.607). First let

$$\tilde{u}(k,l):=u(k-3,l-2) \tag{3.16}$$

so that equation 3.15 becomes

$$\tilde{u}(k+4,l+2)-\tilde{u}(k+2,l+2)-\tilde{u}(k,l+2)+\tilde{u}(k+2,l+2)- \\ \tilde{u}(k+2,l)+\tilde{u}(k+2,l+4)-2\tilde{u}(k+2,l+2)=0 \tag{3.17}$$

Let $a_{r,s}$ denote the coefficient of $\tilde{u}(k+r,l+s)$ in equation 3.17, e.g. $a_{4,2}=1, a_{2,2}=-1$, etc. Denote the generating function of $\tilde{u}$ by,

$$U(t,t_1):=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\tilde{u}(k,l)t^k t_1^l \tag{3.18}$$

Then we can deduced (see Jordan [10], p.608) that, using the notation

$$w(k,t_1):=\sum_{l=0}^{\infty}\tilde{u}(k,l)t_1^l \tag{3.19}$$

we have

$$\sum_{s=0}^{4}\sum_{r=0}^{4}a_{r,s}t_1^{4-s}\left(w(k+r,t_1)-t_1^0\tilde{u}(k+r,0)-t_1^1\tilde{u}(k+r,1)-\cdots- \\ t_1^{s-1}\tilde{u}(k+r,s-1)\right)=0 \tag{3.20}$$

Now equation 3.20 is a linear difference equation in k (for the function $w(k,t_1)$ with t_1 fixed) with constant coefficients (in the sense that they are independent of the variable k) of order 4 and it contains already 4 arbitrary functions of k

$$\phi_j(k):=\tilde{u}(k,j-1), \quad j=1,2,3,4 \tag{3.21}$$

To be clear set

$$K(t_1,k):=\sum_{s=0}^{4}\sum_{r=0}^{4}a_{r,s}t_1^{4-s}\left(t_1^0\tilde{u}(k+r,0)+t_1^1\tilde{u}(k+r,1)+\cdots+ \\ t_1^{s-1}\tilde{u}(k+r,s-1)\right) \tag{3.22}$$

$$A(t_1):=\sum_{s=0}^{4}a_{4,s}t_1^{4-s}, \quad B(t_1,r):=-\sum_{s=0}^{4}a_{r,s}t_1^{4-s} \tag{3.23}$$

and write equation 3.20 as

$$A(t_1)w(k+4,t_1)+\sum_{r=0}^{3}B(t_1,r)w(k+r,t_1)=K(t_1,k) \tag{3.24}$$

The general solution to such an equation is given by the sum $S_{h,t_1} + S_{p,t_1}$ where S_{h,t_1} is the general solution to the homogeneous problem (K replaced by 0) and S_{p,t_1} is a particular solution. Since the coefficients of the homogeneous equation are independent of k, the S_{h,t_1} can be obtained via the roots of the characteristic equation

$$\theta(\lambda) := A(t_1)\lambda^4 + \sum_{r=0}^{3} B(t_1, r)\lambda^r = 0 \tag{3.25}$$

If there are κ different roots, say $\lambda_1, \dots, \lambda_\kappa$ of multiplicities $\tau_1, \dots, \tau_\kappa$ then

$$S_{h,t_1}(k) = \sum_{\sigma=1}^{\kappa} \left(H_{\sigma,1}(t_1) + H_{\sigma,2}(t_1)k + \cdots + H_{\sigma,\tau_\kappa}(t_1)k^{\tau_\kappa - 1} \right) \tag{3.26}$$

where the $H_{\sigma,\tau}(t_1)$ are arbitrary functions of t_1. A particular solution is usually found by an Ansatz e.g. using the method of Section 3.2. The expansion of $w(k, t_1)$ into a power-series in t_1 will yield $\tilde{u}(k, l)$, the arbitrary functions of t_1 will after expansion yield 4 arbitrary functions of l. In this way one could determine all possible solutions, $\tilde{u}(k, l)$, to equation 3.17. This in turn yields the starting real part u via equation 3.16. Finally, given u we have the following.

Observation 3.3. Let f be 1-polyanalytic of the third kind, $u := \operatorname{Re} f, v := \operatorname{Im} f$. Then f is uniquely determined by the values of u on $\mathbb{Z}[i]$ together with the values of v on a set of the form $\{p_0, p_0 + 1, p_0 + i, p_0 + 1 + i\}$, for some point $p_0 \in \mathbb{Z}[i]$.

Proof. Let z denote the standard coordinate in $\mathbb{Z}[i]$. By equation 3.9 and 3.10 we have,

$$v(q_0 + i) = -v(q_0 - i) - (u(q_0 + 1) - u(q_0 - 1)) \tag{3.27}$$

$$v(q_0 + 1) = v(q_0 - 1) - (u(q_0 + 1) - u(q_0 - 1)) \tag{3.28}$$

Hence having the two values of v at $p_0 = q_0 - 1$ and $p_0 + 1 = q_0$, we can obtain v on the set $S_1 := \mathbb{Z}[i] \cap \{\operatorname{Im} z = \operatorname{Im} p_0\}$, via equation 3.28. Similarly we obtain v on the set $S_1 := \mathbb{Z}[i] \cap \{\operatorname{Im} z = \operatorname{Im} p_0 + 1\}$, via the values of v at $p_0 + i$ and $p_0 + 1 + i$. Analogously, given v at the two adjacent points $p_0 + i$ and p_0, equation 3.27 yields v on the set $S_3 := \mathbb{Z}[i] \cap \{\operatorname{Re} z = \operatorname{Re} p_0\}$, whereas the two adjacent points $p_0 + 1 + i$ and $p_0 + 1$, yields v on the set $S_4 := \mathbb{Z}[i] \cap \{\operatorname{Re} z = \operatorname{Re} p_0 + 1\}$. This process can now be iterated for each subset of $\bigcup_{j=1}^4 S_j$, of the form $\{w_0, w0 + 1, w_0 + i, w_0 + 1 + i\}$, for some point $w_0 \in \mathbb{Z}[i]$. This completes the proof. ∎

Now, for the sake of practicality, we can also for the 1-polyanalytic functions of the third kind, consider sets of uniqueness in order to describe the kernel of L_3.

Observation 3.4. Let f be 1-polyanalytic of the third kind on $\mathbb{Z}[i]$, let $z = x + iy$ denote the standard coordinate in $\mathbb{Z}[i]$, and let $p_0, q_0 \in \mathbb{Z}[i]$ such that p_0 (q_0) is zig-zag even (odd). Then f is uniquely determined by its values on $D_0 := D_0^{\text{even}} \cup D_0^{\text{odd}}$, where D_0^{even} can be either $\{p_0\} \cup \{p_0 + \sum_{j=0}^k ((-1)^j + i), k \in \mathbb{Z}\}$ or $\{p_0\} \cup \{p_0 + \sum_{j=0}^k (1 + (-1)^j i), k \in \mathbb{Z}\}$ and D_0^{odd} can be either $\{q_0\} \cup \{q_0 + \sum_{j=0}^k ((-1)^j + i), k \in \mathbb{Z}\}$ or $\{q_0\} \cup \{q_0 + \sum_{j=0}^k (1 + (-1)^j i), k \in \mathbb{Z}\}$

Proof. Start from the set, D_0. Define iteratively the sets $D_j, j \in \mathbb{N}$, by letting D_{j+1} be the set of all points $s \in \mathbb{Z}[i]$ satisfying $s \in \{w + 1, w - 1, w + i, w - i\}$ for some $w \in \mathbb{Z}[i]$ such that precisely 3 points of the set $\{w + 1, w - 1, w + i, w - i\}$ belongs to D_j. $\bigcup_j D_j = \mathbb{Z}[i]$ and f can be iteratively extended to each D_j by assigning the value of f at $s \in D_j \setminus D_{j-1}$ to be determined by the equation $f(w + 1) - f(w - 1) + if(w + i) - if(w - i) = 0$. Obviously replacing the value of f at a point of D_0 yields a different extension to $\mathbb{Z}[i]$. This completes the proof. ∎

3.4. **The set of q-polyanalytic functions when $q>1$.** Now that we have seen for each of the kernels of L_1, L_2, L_3 how to obtain any member by choosing appropriate values on a minimal set of uniqueness we can use these members in order to solve for the members of the kernels of L_1^q, L_2^q, L_3^q for $q>1$. Define the convolution of two complex-valued functions f, g on $\mathbb{Z}[i]$ by

$$(f*g)(z) := \sum_{w\in\mathbb{Z}[i]} f(w)g(z-w), z\in Z[i], \tag{3.29}$$

Observation 3.5. Assume that we have an operator $\mathcal{E}_j$ acting on complex-valued functions on $\mathbb{Z}[i]$ and satisfying $L_j(\mathcal{E}_j g)(z)=g(z)$. If f is a q-polyanalytic function of the j:th kind on $\mathbb{Z}[i]$, we have $L_j^q f=0 \Rightarrow L_j(L_j^{q-1}f)=0$ so $g_1 := L_j^{q-1}f$, is a 1-polyanalytic function of the j:th kind. Since $L_j(\mathcal{E}_j g_1)=g_1$ this means that the function $G_2 := \mathcal{E}_j g_1 + g_2$ is 2-polyanalytic of the j:th kind satisfying $L_j^2 G_2 = g_1$, for any function g_2 that is 1-polyanalytic of the j:th kind. Continuing in this fashion we obtain a function $G_q = \mathcal{E}_j^{q-1}g_1 + \mathcal{E}_j^{q-2}g_2 + \cdots + \mathcal{E}_j g_{q-1} + g_q$, where the g_k are arbitrary 1-polyanalytic functions of the j:th kind, such that $L_j^q G_q = 0$. Hence we can obtain any q-polyanalytic function of the j:th kind starting from 1-polyanalytic functions of the j:th kind and we have in the previous section described how all 1-polyanalytic functions of the j:th kind can be determined.

Hence in order to determine the q-polyanalytic functions of the j:th kind for $q>1$ we only need to find the appropriate operators $\mathcal{E}_j$ associated to the operators L_j, $j=1,2,3$.

We start with $\mathcal{E}_1$. It was resolved by Isaacs [8], p.194. Define $B_+ := \{z\in\mathbb{Z}[i]\colon 1-\operatorname{Re} z \le \operatorname{Im} z \le 0\}$, $B_- := \{z\in\mathbb{Z}[i]\colon 1-\operatorname{Im} z \le \operatorname{Re} z \le 0\}$. Define the operator Q_1 as follows: $Q_+(z)=0$ when $\operatorname{Re} z\le 0$, $Q_+(1)=1$, $Q_+(1+iy)=0$ for $y\neq 0$ and then define recursively $Q_+(z)$ for $\operatorname{Re} z = p+1, p>1$, by $Q_+(p+1+iy) = (1+i)Q_+(p+iy) - iQ_+(p+i(y+1))$. Analogously define Q_- such that $Q_-(z)=0$ for $\operatorname{Im} z\le 0$, $Q_-(i)=1$, $Q_-(x+i)=0$ for $x\neq 0$ and then recursively $Q_-(z)$ for $\operatorname{Im} z = p+1, p>1$, by $Q_-(x+i(p+1)) = (1-i)Q_-(x+ip) + iQ_-(x+1+ip)$. Kiselman [11], showed in the proof of Theorem 4.2, that the following operator satisfies the wanted conditions in Observation 3.5,

$$\mathcal{E}_1\colon f\mapsto Q_+ + *(\chi f) + Q_- * (1-\chi)f \tag{3.30}$$

In the case of the second kind Let χ denotes the characteristic function of the set $A_+ := \{\operatorname{Re} z + \operatorname{Im} z \ge 0\}$, and set

$$S_+(x+iy) = i^{y-x}d(x,y), \quad x+iy\in Z[i] \tag{3.31}$$

where $d(x,y), (x,y)\in\mathbb{Z}^2$, is defined as 0 when $x\le -1$ or when $y\le -1$, as 1 when $(x,y)=(0,0)$, and for $(x,y)\in\mathbb{N}^2\setminus\{(0;0)\}$ by the recursion formula $d(x,y)=d(x-1,y)+d(x-1,y-1)+d(x,y-1)$. Kiselman [12], showed that S_+ is a fundamental solution supported in $A_+ := \{(\operatorname{Re} z\ge 1)\wedge(\operatorname{Im} z\ge 1)\}$, and pointed out that there is a natural analogue of S_+ but whose support is $A_- := \{(\operatorname{Re} z\le 0)\wedge(\operatorname{Im} z\le 0)\}$ (instead of A_+) and the existence of which is proved similarly. Kiselman [12], showed in the proof of Theorem 4.2, that the following operator satisfies the wanted conditions in Observation 3.5,

$$\mathcal{E}_2\colon f\mapsto S_+ * (\chi f) + S_- * (1-\chi)f \tag{3.32}$$

Finally in the case of the third kind, define for a complex-valued f on $\mathbb{Z}[i]$,

$$L_3' f(z) := f(z+1) - f(z-1) - if(z+i) + if(z-i) \tag{3.33}$$

and denote by $\Delta_{(2)}$ the operator which acts according to

$$\Delta_{(2)}f(z) := (L_3 \circ L_3')f(z) = f(z+2) + f(z-2) + f(z+2i) + f(z-2i) - 4f(z) \tag{3.34}$$

We call this operator the *two-step* discrete Laplacian.

Define

$$\mathcal{E}_3 \colon f \mapsto f * (L_3'\mathscr{G}) \tag{3.35}$$

where we choose $\mathscr{G}$ to be the following

$$\mathscr{G}(m,n) := \frac{1}{2\pi}\int_0^{2\pi} d\psi\,(1 - \exp(i(\pm m\phi \pm n\psi)))\,(4 - 2\cos\phi - 2\cos\psi)^{-1} \tag{3.36}$$

satisfying (see van der Pol [20])

$$\Delta_{(2)}\mathscr{G}(m+in) = \delta_{(m,n)} \tag{3.37}$$

where $\delta_{(m,n)} = 1$ if $m = n = 0$ and $\delta_{(m,n)} = 0$, otherwise. Then in light of $L_3(L_3'\mathscr{G}) = \Delta_{(2)}\mathscr{G}$, we have $L_3(\mathcal{E}_3 f)(z) = L_3(f*(L_3'\mathscr{G}))(z) = \sum_{w\in\mathbb{Z}[i]} L_3 f(w)(L_3'\mathscr{G})(z-w) = \sum_{w\in\mathbb{Z}[i]} f(w)(L_3(L_3'\mathscr{G})(z-w))$, $z \in Z[i]$.

Now given a fixed $w \in \mathbb{Z}[i]$, we have $(L_3(L_3'\mathscr{G})(z-w)) = \Delta_{(2)}\mathscr{G}(z-w)$ and the two-step Laplacian commutes with translation $\tau_w \colon z \mapsto z - w$, i.e. $\Delta_{(2)}\tau_w \circ \mathscr{G}(z) = \tau_w \circ (\Delta_{(2)}\mathscr{G})(z)$. Thus $\Delta_{(2)}\mathscr{G}(z-w)$ equals 1 if $z = w$ and 0 otherwise. Hence $\mathcal{E}_3$ satisfies the wanted conditions in Observation 3.5.

Remark 3.2. In this paper we are considering homogeneous equations of the form $L_j^q f = 0$, where $j \in \{1,2,3\}$ and q a fixed positive integer. We mention, as a remark, that Tu [19], p.46, presented the following statement: Let n be a positive integer, let $c_0, \dots, c_{n-1}$ be arbitrary constants and let $a_1, \cdots, a_n$ be distinct roots of $a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = 0$. Then the general solution to $\sum_{j=1}^n c_{j-1}\frac{1}{2^j}L_1^j F = 0$ is $F(z) = \sum_{i=1}^n B_i e^{a_i, z}$, $B_i, i = 1, \cdots, n$, are arbitrary constants and $e^{a,x} = (1+a)^x(1+ia)^y$, for $z = x+iy$, and $a \in \mathbb{C}$.
Clearly any equation of the form $\frac{1}{2^n}L_1^n F = 0$ would have associated to it, $c_{n-1} = 1, c_j = 0, j < n-1$, i.e. the equation $a^n + 1\cdot a^{n-1} = a^{n-1}(a+1) = 0$, which has n distinct roots only in the case of $n = 1$ or $n = 2$.

If $n = 1, c_0 = 0$ then $a_1 = 0$ is the only root of $a + c_0 = 0$, and we are considering (up to multiplication by the constant $1/2$) precisely the equation for monodiffric functions of the first kind, $L_1 f = 0$. However the function $\sum_{i=1}^1 B_1 e^{0,z} = B_1 1^{x+y} = B_1$.

If $n = 2$ then the roots are $a_1 = 0$ and $a_2 = -1$, thus giving the function $F(z) = B_1 e^{0,z} + B_2 e^{-1,z}$ which, for $x \neq 0$, can be evaluated as $B_1 1^{x+y} + B_2 \cdot 0 \cdot (1-i)^y = B_1$. Obviously a complex constant is not the general solution to $L_1 f = 0$ or $L_1^2 f = 0$. We conclude that the statement in Tu [19] is not meant to apply to equations of the form $L_1^q f = 0$ for any positive integer q.

4. SOME NATURAL MULTIPLICATIONS AND PSEUDO-POLYNOMIALS

We believe that from an algebraic perspective it makes sense that some notion of multiplication is used such that the corresponding notion of (pseudo-)polynomial will be 1-polyanalytic, and that the multiplication would need to be given by a binary relation, distributive over addition. Obviously being both left and right distributive as well as associative and abelian would be satisfying properties as well, however such requirements are overly restrictive given the circumstances. Multiplicative structures that are non-associative do occur in modern research but non-distributive ones seem to be rare which is why we have required, at minimum, distributivity. Natural analogues of polynomials in the function spaces that satisfy our above requirements, more or less exist in the literature, for the case of monodiffric functions of the first and second kind already. We shall similarly introduce such analogues for 1-polyanalytic functions of the third kind.

4.1. **First kind.** The analogues of polynomials that we endorse, when it comes to multiplication in the function spaces, are finite linear combinations of what is called pseudo-powers, $z^{(j)}$, $j \in \mathbb{N}$. In the case of 1-polyanalytic functions of the first kind, these where introduced by Isaacs [9] using a multiplication, between the coordinate function $x+iy \mapsto x+iy$, and another complex function $f\colon \mathbb{Z}[i] \to \mathbb{C}$, according to,

$$(x+iy) \odot_1 f(x+iy) := xf((x-1)+iy) + iyf(x+i(y-1)) \tag{4.1}$$

and $c \odot_1 f := cf$ for constants $c \in \mathbb{C}$.

For two complex-valued functions f, g on $\mathbb{Z}[i]$ we define

$$(g \odot_1 f)(x+iy) := \operatorname{Re} g(x+iy) f((x-1)+iy) + i \operatorname{Im} g(x+iy) f(x+i(y-1)) \tag{4.2}$$

and $c \odot_1 f := cf$ for constants $c \in \mathbb{C}$.

Example 4.1. *We have $z^2 = (x^2-y^2)+2ixy$, so that $z \odot z^2 = x((x-1)^3-(x-1)y^2+2iy(x-1)) + iy(x^2-(y-1)^2) + 2ix(y-1))$ whereas $z^2 \odot_1 z = (x^2-y^2)(x-1+iy) + i2xy(x+i(y-1))$ which implies $z \odot_1 z^2(2) = 2-4i \neq 4 = z^2 \odot_1 z$. Hence the multiplication $\odot_1$ is not commutative. Furthermore note that $z \odot_1 z = x(x-1+iy)+iy(x+i(y-1)) = (x^2-x+y-y^2)+i2xy$, thus $(z \odot_1 z) \odot_1 z = (x^2-x+y-y^2)(x-1+iy)+i2xy(x+i(y-1))$ whereas $z \odot_1 (z \odot_1 z) = x((x-1)^2-x+1+y-y^2)+i2(x-1)y)+iy((x^2-x+y-1-(y-1)^2)+i2x(y-1))$, giving $(z \odot_1 z) \odot_1 z(-1) = -4 \neq -1(4+1+1)+i2 = z \odot_1 (z \odot_1 z)(-1)$. Hence $\odot_1$ is non-associative.*

The multiplication $\odot_1$ is obviously distributive over addition but as we have seen not abelian, and therefore yields two different kinds of pseudo-powers which we shall call *left* and *right* polynomials respectively. The left pseudo-monomials are defined recursively according to

$$z^{0,\odot_1,l} := 1, \quad z^{j+1,\odot_1,l} = z \odot_1 z^{j,\odot_1,l}, \quad j \in \mathbb{Z}_+ \tag{4.3}$$

whereas the right pseudo-powers are defined according to

$$z^{0,\odot_1,r} := 1, \quad z^{j+1,\odot_1,r} = z^{j,\odot_1,r} \odot_1 z, \quad j \in \mathbb{Z}_+ \tag{4.4}$$

By distributivity we obtain natural extension to left (right) *pseudo-polynomials* $P_{1,\text{left}}$ ($P_{1,\text{right}}$) of degree N, according to

$$P_{1,\text{left}}(z) = \sum_{j=0}^{N} c_j z^{j,\odot_1,l}, \quad \left(P_{1,\text{right}}(z) = \sum_{j=0}^{N} c_j z^{j,\odot_1,r}\right) \tag{4.5}$$

where the c_j are complex constants.

We call complex multiples of pseudo-powers, *pseudo-monomials*.

Proposition 4.1. *Let f be a 1-polyanalytic function of the first kind on $\mathbb{Z}[i]$. Then $z \odot_1 f(z)$ is a 1-polyanalytic function of the first kind on $\mathbb{Z}[i]$.*

Proof. Denote $x = \operatorname{Re} z, y = \operatorname{Im} z$. We have $z \odot_1 f(z) = xf(x-1+iy) + iyf(x+i(y-1))$. Thus

$$
\begin{aligned}
(4.6) \quad L(z \odot_1 f(z)) &= (x+1)f(z) - f(x-1+iy) + iyf(x+1+i(y-1)) - \\
& iyf(x+i(y-1)) + i[xf(x-1+i(y+1) - xf(x-1+iy) + i(y+1)f(z) - \\
& iyf(x+i(y-1)] = xf(z) + f(z) - xf(z-1) + iyf(f+1-i) - \\
& iyf(z-i) + ixf(z-1+i) - ixf(z-1) - yf(z) - f(z) + \\
& yf(z-i) = y(-f(z) + f(z-i) + i(f(z+1-i) - f(z-i)) + \\
& x(f(z) - f(z-1) + i(f(z-1+i) - f(z-1)) = \\
& -\frac{y}{i} L_1 f(z-i) + xL_1 f(z-1) = 0
\end{aligned}
$$

This completes the proof. ∎

Since the pseudo-powers $z^{j,\odot_1,l}$ are defined iteratively and obviously $f(z) = z$ is 1-polyanalytic of the first kind, we immediately have the following.

Corollary 4.2. *The pseudo-power $z^{j,\odot_1,l}$ is, for each $j \in \mathbb{N}$, a 1-polyanalytic function of the first kind.*

4.2. **Second kind.** Also in the case of 1-polyanalytic functions of the second kind, there exists in previous literature a natural multiplication (see e.g. Duffin & Petersson [5], p.626) that we endorse. We define the following binary relation, $\odot_2$, on the space of complex valued functions on $\mathbb{Z}[i]$,

$$
(4.7) \qquad (g \odot_2 f)(z) := \int_{\gamma_{g(z)}} f(w)dw
$$

where $\gamma_{g(z)}$ is a 4-curve with initial point $a_0 = 0$ and end point $a_N = g(z)$ for some positive integer N. As a direct consequence of equation 2.10 (path-independence), the number $(g \odot_2 f)(z)$ is independent of the choice of different, non-self-intersecting 4-curves, γ, that share initial and end point. Clearly, $\odot_2$ is distributive over addition. It is however not abelian or associative.

Example 4.2. *That $\odot_2$ is not abelian take $f(z) = z^2, g(z) = z$ and calculate $2(g \odot_2 f)(-1) = (1+0)(-1-0) = -1$, whereas $2(f \odot_2 g)(-1) = (1-0)(1-0) = 1$. It is not associative which can be seen by setting $A(z) := 2(g \odot_2 g)(z)$, and noting that $A(-1) = (-1+0)(-1-0) = 1$, $A(0) = 0$, and $((g \odot_2 g) \odot_2 g)(-1) = (1-0)(1-0) = 1$, whereas $(g \odot_2 (g \odot_2 g))(-1) = (A(-1) + A(0))(-1-0) = -1$.*

The multiplication $\odot_2$ therefore yields two different kinds of candidates for so called *pseudo-monomials*, namely we introduce the left pseudo-powers

$$
(4.8) \qquad z^{(0),l} := 1, \quad z^{(j+1),l} = z \odot_3 z^{(j),l}, j \in \mathbb{Z}_+
$$

whereas the right pseudo-powers are defined according to

$$
(4.9) \qquad z^{(0),r} := 1, \quad z^{(j+1),r} = z^{(j),r} \odot_3 z, j \in \mathbb{Z}_+
$$

Up to multiplication by j, the left pseudo-monomials can be found in e.g. Duffin & Petersson [5], p.626, there denoted $z^{(j)}$ (which means that $jz^{(j+1),l} =: z^{(j)}$, for $j > 0$) and they are known to be 1-polyanalytic of the second kind, see e.g. Theorem 2.6, Duffin & Petersson [5] (note that the the reason the multiple j does not appear in our definition is that our definition arises as a consequence of a more general multiplication whereas the $z^{(j)}$ are stand-alone definitions). It is precisely the *left* pseudomonimials that we endorse (as natural analogues of powers of z in the case of holomorphic functions) in the case of 1-polyanalytic functions of the second kind.

4.3. **Third kind.** Let v_1 be a zig-zag even point and let v_1 (v_2) be a zig-zag even (odd) point in $\mathbb{Z}[i]$. Let $\Gamma_{0,v_1}(\Gamma_{1,v_2})$ be a zig-zag polygon staring at $0(1)$ with end point $v_1(v_2)$. Obviously for any zig-zag polygon $\Gamma^+(\Gamma^-)$ from a point a(b) to a point b(a) we have for any complex-valued function f on $\mathbb{Z}[i]$, $\int_{\Gamma^-} f(z)dz = -\int_{\Gamma^+} f(z)dz$. As a consequence of Proposition 2.2 (zig-zag path-independence) the numbers

$$\text{(even)} \int_0^{v_1} f(w)dw := \int_{\Gamma_{0,v_1}} f(w)dw, \ \text{(odd)} \int_1^{v_2} f(w)dw := \int_{\Gamma_{1,v_2}} f(w)dw \tag{4.10}$$

are independent of the choice of zig-zag polygon Γ_{0,v_1} (Γ_{1,v_2}) as long as they are zig-zag polygons starting at $0(1)$ with end point $v_1(v_2)$. Define, for each function $f\colon \mathbb{Z}[i] \to \mathbb{C}$, the number

$$\int_{\text{zig-zag},z} f(w)dw := \begin{cases} \text{(even)} \int_0^z f(w)dw & , \text{ if } z \text{ is zig-zag even} \\ \text{(odd)} \int_1^z f(w)dw & , \text{ if } z \text{ is zig-zag odd} \end{cases} \tag{4.11}$$

Obviously, by zig-zag path-independence we can now define integration along an arbitrary zig-zag polygon Γ, with starting point a_0 and end point a_N (for a positive integer N) namely $\int_\Gamma f(w)dw = \int_{\text{zig-zag},a_N} f(w)dw - \int_{\text{zig-zag},a_0} f(w)dw$.

Proposition 4.3. *The function $F(z) := \int_{\textit{zig-zag},z} f(w)dw$ is 1-polyanalytic of the third kind whenever f is.*

Proof. Without loss of generality, assume z is zig-zag even. we chose paths from $z-1$ to $z+1$ namely $(z-1, z-i, z+1)$, and from $z-i$ to $z+i$ $(z-i, z-1, z+i)$, and as a consequence of Proposition 2.2 (zig-zag path-independence) we can write

$$\begin{aligned}
2\int_{\text{zig-zag},z} f(w)dw &= 2\left(\int_{\text{zig-zag},z+1} f(w)dw - \int_{\text{zig-zag},z-1} f(w)dw\right) + \\
&\quad i2\left(\int_{\text{zig-zag},z+i} f(w)dw - \int_{\text{zig-zag},z-i} f(w)dw\right) \\
&= 2\left(\int_{\text{zig-zag},z+1} f(w)dw - \int_{\text{zig-zag},z-1} f(w)dw\right) + \\
&\quad i2\left(\int_{\text{zig-zag},z+i} f(w)dw - \int_{\text{zig-zag},z-i} f(w)dw\right) = \\
&(f(z+1)+f(z-i))((z+1)-(z-i)) + (f(z-i)+f(z-1))((z-i)-(z-1))+ \\
&i(f(z+i)+f(z-1))((z+i)-(z-1)) + i(f(z-1)+f(z-i))((z-1)-(z-i)) = \\
&(f(z+1)+f(z-i))(1+i) + (f(z-i)+f(z-1))(1-i)+ \\
&i(f(z+i)+f(z-1))(1+i) + +i(f(z-1)+f(z-i))(-1+i) = \\
&(1+i)f(z+1) + (-i-1)f(z-1) + (i-1)f(z+i) + (1-i)f(z-i)) = \\
&(1+i)(f(z+1)-f(z-1)) + i(1+i)f(z+i) - i(1+i)f(z-i) = \\
&(1+i)\cdot L_3(f(z)) = 0
\end{aligned} \tag{4.12}$$

This completes the proof. ∎

We define the following binary relation, $\odot_3$, on the space of complex valued functions on $\mathbb{Z}[i]$,

$$(g \odot_3 f)(z) := \int_{\text{zig-zag},g(z)} f(w)dw \tag{4.13}$$

Again, $\odot_3$ is distributive over addition but not abelian and not associative (see example 4.3, and therefore yields two different (see Example 4.3) kinds of monomials,

$$z^{[0],l} := 1, \quad z^{[j+1],l} = z \odot_3 z^{[j],l}, j \in \mathbb{Z}_+ \tag{4.14}$$

whereas the right monomials are defined according to

$$z^{[0],r} := 1, \quad z^{[j+1],r} = z^{[j],r,1} \odot_3 z, j \in \mathbb{Z}_+ \tag{4.15}$$

By Proposition 4.3 the pseudo-polynomials obtained via the left-monomials are all q-polyanalytic of the third kind.

Example 4.3. *To see that $\odot_3$ is non-abelian take $g(z) \odot_3 f(z)$ at the point $z = 1+i$, where $f(z) = z^2, g(z) = z$. We have $g(z) \odot_3 f(z) = \int_{\text{zig-zag},z} w^2 dw$, whereas $f(z) \odot_3 g(z) = \int_{\text{zig-zag},z^2} wdw$. Hence at $z = 1+i$ we have $z^2 = 2i$ so that $f(z) \odot_3 g(z) = ((1+i)^2+0)(1+i-0) = (1+i)^2 = 2i$, and $g(z) \odot_3 f(z) = \int_{\text{zig-zag},2i} wdw = (2i+(1+i))(2i-(1+i)) + ((1+i)+0)((1+i)-0) = (2i+1)(i-1)+(1+i)^2 = -3+i \neq f(z) \odot_3 g(z)$.*

To see that $\odot_3$ is non-associative, set $A(z) := g(z) \odot_3 g(z) = \int_{\text{zig-zag},z} wdw$. Then $(g(z) \odot_3 g(z)) \odot_3 g(z) = \int_{\text{zig-zag},A(z)} wdw = (A(a_N) + A(a_{N-1}))(a_N - a_{N-1}) + \cdots + (A(a_1) + A(a_0))(a_1 - a_0)$, for a zig-zag polygon with ordered set or vertices $(a_N, \ldots, a_0)$, where $a_{j+1} \in \{a_j \pm (1+i), a_j \pm (1-i)\}$. Choose $z = 1+i$. Then we have a path of integration with only two vertices namely $a_1 = 1+i, a_0 = 0$. Now $g(z) \odot_3 (g(z) \odot_3 g(z)) = \int_{\text{zig-zag},z} A(w)dw$. Let $\Gamma_w := (b_{M,w}, \ldots, b_{0,w})$, be a zig-zag polygon of minimal length such that $b_{M,w} = w$. Then $A(w) = (A(b_{M,w}) + A(b_{M-1,w}))(b_{M,w} - b_{M-1,w}) + \cdots + (A(b_{1,w}) + A(b_{0,w}))(b_{1,w} - b_{0,w})$, Now we have $A(0) = 0, A(1+i) = \int_{\text{zig-zag},1+i} wdw = (1+i)^2 = 2i$. Hence $(g \odot_3 (g \odot_3 g))(1+i) = \int_{\text{zig-zag},1+i} A(w)dw = (A(1+i) + A(0))(1+i-0) = 2i(1+i) = 2i-2$, whereas $((g \odot_3 g) \odot_3 g)(1+i) = \int_{\text{zig-zag},A(1+i)} wdw = \int_{\text{zig-zag},2i} wdw = ((1+i)+0)((1+i)-0) + (2i+(1+i))(2i-(1+i)) = (1+i)^2 + (3i+1)(i-1) = -4 \neq (g \odot_3 (g \odot_3 g))(1+i)$.

Remark 4.1. Note the slight difference in notation for the pseudo-monomials $z^{(j),l}$ and $z^{[j],l}$ used to separate between the case of the second and third kind respectively.

5. THE SETS OF SECTION 3 USED AS A PROOF-TOOL

5.1. **Pairwise inequivalence of the three kinds.** Here is a proposition that illustrates one way that the minimal sets of uniqueness from Section 3 can be useful.

Proposition 5.1. *Denote for $j = 1, 2, 3$, by Ker(L_j) the set of complex-valued functions on $\mathbb{Z}[i]$ that are annihilated by L_j. Then Ker$(L_k) \backslash$Ker$L_l \neq \emptyset$ for $k \neq l$.*

Proof. By Observation 3.1 the set $D = \{z \colon 0 \leq \operatorname{Re} z \leq 1\}$ is a minimal determining set for functions annihilated by L_3. Define f on D according to $f(0) = 1, f(1) = -i, f(i+1) = f(i) = 0$, and $f(z) = 0$ otherwise. Then we know that f has a unique extension (see Remark 3.1) to $\mathbb{Z}[i]$ which is 1-polyanalytic of the third kind. However $L_2 f(0) = 2 \neq 0$ and $L_1 f(0) = -2i-1 \neq 0$. This takes care of the cases Ker$(L_3) \backslash$Ker(L_1) and Ker$(L_3) \backslash$Ker(L_2). By Observation 3.2 we know that $D = \{z \in \mathbb{Z}[i] \colon \operatorname{Re} z \operatorname{Im} z = 0\}$ is a minimal set of uniqueness for any function f satisfying $L_2 f \equiv 0$. On the other hand, by Observation 3.1, we know that

$D' := D \setminus \{z \in \mathbb{Z}[i] \colon (\operatorname{Re} z = 0) \wedge (\operatorname{Im} z > 0)\}$ is a minimal determining set for any function f satisfying $L_1 f \equiv 0$. Defining a function g on D according to $g(z) = f(z)$ for $z \in D'$ and $g(z) = f(z) + 1$ for $z \in D \setminus D'$, we know that g determines uniquely an extension to $\mathbb{Z}[i]$ that is annihilated by L_2. On the other hand g cannot be annihilated by L_1 because it does not coincide with the unique extension from the set D' for functions that are annihilated by L_1. Furthermore define a function h on D' according to $h(-i) = 1$ and $h(z) = 0$ otherwise. Then by Observation 3.1 h has unique (see Remark 3.1) extension to $\mathbb{Z}[i]$ that is annihilated by L_1. However $L_3 h(0) = -i \neq 0$. Furthermore we know that the extension of h satisfies $g(-i+1) = 1 + i$, hence $L_2 h(-i) = 1 + i(1+i) = i \neq 0$. This takes care of the cases Ker(L_2) $\neq$Ker(L_1), Ker(L_1) $\neq$Ker(L_3) and Ker(L_1) $\neq$Ker(L_2) respectively. Finally define the function $G(z)$ on D according to $G(1) = 1, G(-1) = -1, G(i) = 0, G(-i) = 0$ and $G(z) = 0$ otherwise. By Observation 3.2, G determines uniquely an extension to $\mathbb{Z}[i]$ that is annihilated by L_2 but by construction $L_3 G(0) = 2 \neq 0$. This takes care of the case Ker(L_2) $\neq$Ker(L_3), This completes the proof. ∎

5.2. **A question posed by Kiselman.** A complex function on $\mathbb{Z}[i]$, is said to have a representation in terms of a *pseudo-power (Maclaurin) series* if there is a series $\sum_{j=0}^{\infty} c_j z^{(j),l}$, where the c_j are complex constants, which pointwise coincides with the given complex function. Let $\mathcal{P}$ denote the set of complex-valued functions on $\mathbb{Z}[i]$ which can be expressed in terms of a Maclaurin series. Kiselman [12], Sec 3, p.5, posed the question whether or not each function that is annihilated by L_2 on $\mathbb{Z}[i]$ has a pseudo-power (Maclaurin) series expansion in terms of pseudo-powers associated to the functions annihilated by L_2 on $\mathbb{Z}[i]$ (these pseudo-powers are, up to multiplication by their integer powers, given in Section 4). Given our previous work, we may make a small statement regarding Kiselman's question.

Proposition 5.2. *(i)⇒(ii) where:*

(i) There exists no Maclaurin series of the form $P(z) = \sum_{j=0}^{\infty} c_j z^{(4j),l}$, where the c_j are complex constants, such that $c_0 \neq 0$ but P vanishes on the coordinate axes except at 0.
(ii) Ker$(L_2) \neq \mathcal{P}$

Proof. Since we already know from Section 4 that $\mathcal{P}$ $\subseteq$Ker(L_2), it is sufficient for obtaining validity of (ii), to prove that Ker(L_2) has a minimal set of uniqueness which properly contains a set of uniqueness with respect to $\mathcal{P}$.

Let us first look at some conditions we know each member of $\mathcal{P}$ must satisfy. Recall that by Observation 3.2, a complex-valued function P on $\mathbb{Z}[i]$ is uniquely determined (see Remark 3.1) by its values on the set

$$D := \{z \in \mathbb{Z}[i] \colon \operatorname{Re} z \operatorname{Im} z = 0\} \tag{5.1}$$

i.e. any arbitrary choice of complex values on D will render an extension function that is annihilated by L_2 on $\mathbb{Z}[i]$.

Let $P(z) = \sum_{j=0}^{\infty} d_j z^{(j),l}$, where the d_j are complex constants We split the Maclaurin series P into 4 sums

$$P = P_1 + P_2 + P_3 + P_4 \tag{5.2}$$

where P_1 and P_2 both consist only of even pseudo-powers according to

$$P_1(z) := \sum_{j=0}^{\infty} d_{4j} z^{(4j),l}, \quad P_2(z) := \sum_{j=0}^{\infty} d_{4j+2} z^{(4j+2),l} \tag{5.3}$$

and P_3 and P_4 both consist only of odd pseudo-powers according to

$$P_3(z) := \sum_{j=0}^{\infty} d_{4j+1} z^{(4j+1),l}, \quad P_4(z) := \sum_{j=0}^{\infty} d_{4j+3} z^{(4j+3),l} \tag{5.4}$$

Introduce the following notation for the restrictions to D

$$p_k = P_k|_D, \quad k = 1, 2, 3, 4 \tag{5.5}$$

For $x := \operatorname{Re} z, y := \operatorname{Im} z$, we will show that

$$z^{(j),l}|_{y=0}(-x) = (-1)^j z^{(j),l}|_{y=0}(x) \tag{5.6}$$

This can be done by induction as follows. For $j = 1$ this is trivial. Assume equation 5.6 holds true for $j = n - 1$. Let γ_t denote the 4-curve on the x-axis, starting at 0 and with end point $t + i0$. Recall that if $G(x) := G(x + 0i)$ is a complex function on the x-axis then

$$\begin{aligned} 2\int_{\gamma_x} G(t) &= (G(0) + G(1))(1 - 0) + \cdots + \\ &(G(x-1) + G(x))(x - (x-1)) = G(0) + G(x) + 2\sum_{t=1}^{x-1} G(t) \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} 2\int_{\gamma_{-x}} G(t) &= (G(0) + G(-1))(-1 - 0) + \cdots + \\ &(G(-(x-1)) + G(-x))(-x - (-x+1)) = -G(0) - G(-x) - 2\sum_{t=-1}^{-(x-1)} G(-t) \end{aligned} \tag{5.8}$$

Hence

$$G(t) = \pm G(-t), 0 \le t \le x \Longrightarrow \int_{\gamma_{-x}} G(t) = \pm \int_{\gamma_x} G(t) \tag{5.9}$$

Now writing the left hand side of equation 5.6 for $j = n$ as

$$z^{(n),l}|_{y=0}(-x) = \int_{\gamma_{-x}} z^{(n-1),l}|_{y=0}(t) dt \tag{5.10}$$

the induction hypothesis (provided by equation 5.6 for $j = n - 1$) implies that the integrand in the right hand side of equation 5.10 satisfies one of the options (depending upon j) of the left hand side of equation 5.9, which implies

$$\int_{\gamma_{-x}} z^{(n-1),l}|_{y=0}(t) dt = \int_{\gamma_{-x}} (-1)^{n-1} z^{(n-1),l}|_{y=0}(-t) dt \tag{5.11}$$

Finally note that

$$2\int_{\gamma_{-x}} G(-t) = (G(0)+G(1))(-1-0)+\cdots+ (G(x-1)+G(x))(-x-(-x+1)) = -G(0)-G(x)-2\sum_{t=1}^{x-1} G(t) \tag{5.12}$$

which applied to equation 5.11 gives

$$\int_{\gamma_{-x}} (-1)^{n-1} z^{(n-1),l}|_{y=0}(-t)dt = (-1)^{n-1}(-1)\int_{\gamma_x} z^{(n),l}|_{y=0}(t)dt \tag{5.13}$$

This completes the induction step and thus proves equation 5.6.

By equation 5.6

$$p_1(-x) = p_1(x), \quad p_2(-x) = p_2(x), \quad x \in \mathbb{Z} \tag{5.14}$$

$$p_3(-x) = -p_3(x), \quad p_4(-x) = -p_4(x), \quad x \in \mathbb{Z} \tag{5.15}$$

Also the counterpart to equation 5.6 for the y-axis instead of the x-axis can be verified analogously

$$z^{(j),l}|_{x=0}(-iy) = (-1)^j z^{(j),l}|_{x=0}(iy) \tag{5.16}$$

which yields

$$p_1(-iy) = p_1(iy), \quad p_2(-iy) = p_2(iy), \quad y \in \mathbb{Z} \tag{5.17}$$

$$p_3(-iy) = -p_3(iy), \quad p_4(-iy) = -p_4(iy), \quad y \in \mathbb{Z} \tag{5.18}$$

Next we note that if κ_{it} denotes the 4-curve on the y-axis, starting at 0 and with end point $0+it$, then

$$2\int_{\kappa_{it}} z = (0+i\cdot 1)(i\cdot 1-0)+\cdots+ (i(t-1))+it)(it-i(t-1)) = i2\int_{\gamma_t} z \tag{5.19}$$

(where γ_t, as before, denotes the 4-curve on the x-axis, starting at 0 and with end point $t+i0$) which after repeated application implies

$$z^{(j),l}|_{y=0}(it_0) = i^j z^{(j),l}|_{x=0}(t_0), \quad t_0 \in \mathbb{Z} \tag{5.20}$$

By equation 5.20

$$p_1(it) = p_1(t), \quad p_2(it) = -p_2(t), \quad t \in \mathbb{Z} \tag{5.21}$$

$$p_3(it) = ip_3(t), \quad p_4(it) = -ip_4(t), \quad t \in \mathbb{Z} \tag{5.22}$$

Now consider the case when we additionally require

$$P|_{D\setminus\{0\}} \equiv 0, \quad P(0) \neq 0 \tag{5.23}$$

which implies

$$(p_1+p_2+p_3+p_4)(\pm x) = 0, \quad x \in \mathbb{Z}_+ \tag{5.24}$$

$$(p_1 + p_2 + p_3 + p_4)(\pm iy) = 0, \quad y \in \mathbb{Z}_+ \tag{5.25}$$

and since $P(0) = p_1(0)$,

$$(p_2 + p_3 + p_4)(0) = 0 \tag{5.26}$$

For a fixed $t \in \mathbb{Z}_+$, equations 5.14,5.15,5.17,5.18,5.21,5.22,5.24,5.25 render a homogeneous system of equations were the left hand side takes the matrix form

$$\begin{bmatrix}
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_1(-t) \\ p_2(-t) \\ p_3(-t) \\ p_4(-t) \\ p_1(it) \\ p_2(it) \\ p_3(it) \\ p_4(it) \\ p_1(-it) \\ p_2(-it) \\ p_3(-it) \\ p_4(-it)
\end{bmatrix} \tag{5.27}$$

The matrix of the homogeneous system of equations in 5.27 has rank 16 thus the only solution is the zero vector. Since $t \in \mathbb{Z}_+$ was arbitrary, this implies that $p_k \equiv 0$ on $D \setminus \{0\}$ for $k = 1, 2, 3, 4,$ and because each corresponding P_k is annihilated by L_2 together with the fact that D is a minimal determining set for functions annihilated by L_2 and equation 5.26, implies that

$$P_k \equiv 0, \quad k = 2, 3, 4 \tag{5.28}$$

For the Maclaurin series $P_1(z) = \sum_{j=0}^{\infty} d_{4j} z^{(4j),l}$, we have,

$$P_1|_{D \setminus \{0\}} \equiv 0 \tag{5.29}$$

If there does not exist such a P_1 satisfying $P_1(0) \neq 0$ then P_1 vanishes on D and therefore (as a consequence of $\mathcal{P} \subseteq$Ker(L_2) and D being a set of uniqueness) vanishes on $\mathbb{Z}[i]$. But that in turn implies that $P = P_1 + P_2 + P_3 + P_4$ vanishes on $\mathbb{Z}[i]$. Since P was an arbitrary member of $\mathcal{P}$ that vanished on $D \setminus \{0\}$, we conclude that the latter set is a set of uniqueness with respect to $\mathcal{P}$ and is also properly contained in D which is a minimal set of uniqueness with respect to Ker(L_2).

This completes the proof. ∎

Note that the proof involves examining whether or not a minimal determining set for 1-polyanalytic functions of the second kind turns out to contain, as a *proper* subset, a determining set with respect to the space of Maclaurin series. We know in turn, that for certain subset of the set of pseudo-power (Maclaurin) series, much smaller determining sets are known as the following Theorem shows.

Theorem 5.3 (Duffin & Petersson [5], p.637)**.** *Any member of the set of pseudo-power (Maclaurin) series* $\sum_{j=0}^{\infty} \frac{a_j}{j!} j z^{(j),l}$, *where the* a_j *are complex constants, such that* $\limsup |ja_j|^{1/j} < 2$, *is uniquely determined by its values on the non-negative real axis.*

We point out that a problem that is associated to determining sets is that of determining sets *relative* to a given subset of $\mathbb{Z}[i]$, by which we mean the following. Let $\Omega \subset \mathbb{Z}[i]$ be a finite discrete domain whose boundary, γ, is a non-selfintersecting closed polygon. Let $j \in \{1, 2, 3\}$. A natural question is: What are the minimal subset $D \subset \Omega$ such that $f|_D = 0 \Rightarrow f = 0$, for all 1-polyanalytic functions f of the j:th kind, on Ω. Having seen the proofs of Observation 3.1, 3.2, 3.4, the following is almost self-evident:
If f is 1-polyanalytic of the 1:th kind or of the second kind on $\mathring{\Omega}$, then $f|_\gamma = 0 \Rightarrow f \equiv 0$. This is an appealing feature that is lacking for the 1-polyanalytic functions of the third kind. If f is 1-polyanalytic of the third kind on $\mathring{\Omega}$ and γ is a closed non-selfintersecting zig-zag polygon that contains a zig-zag even (odd) point, then $f|_\gamma = 0 \Rightarrow f(z) = 0$ for all zig-zag even (odd) points of Ω. Weaker yet is the case when f is 1-polyanalytic of the third kind on $\mathring{\Omega}$, and γ is a general 4-curve.

Example 5.1. *Let* $\Omega := ((\sqrt{2}K + 1 - i) \cup (e^{i\pi/4}\sqrt{2}K)) \cap \mathbb{Z}[i]$, *where* $K = \{z \in \mathbb{C} \colon\ |\mathrm{Re}\, z| \leq 1, |\mathrm{Im}\, z| \leq 1\}$. *Then the boundary of* Ω *is the closed zig-zag polygon,* γ *determined by* $(-2, -1-i, -2i, -2i+1, -2i+2, -i+2, 2, 1+i, 2i, -1+i, -2)$. *Prescribing a complex-valued* f *on* Ω *to be* 0 *on* γ, *the condition that* $L_3 f = 0$ *only induces the value of* f *at* 0, *which means that we can for instance choose* $f(1)$ *arbitrarily and still be able to extend* f *to a solution for* $L_3 f = 0$ *on* Ω.

6. A MOTIVATION FOR THE DEFINITION OF q-POLYANALYTIC FUNCTIONS OF THE THIRD KIND

We believe that adjacency is useful in motivating the defined operators and furthermore some structure such as multiplication is a priori not required. For this reason we formalize our work using so called Gaussian structures. Recall that the notation $\mathbb{Z}[i]$ usually implies the *ring* of Gaussian integers (in particular with a priori given multiplication) and with no graph structure (i.e. no adjacency).

Definition 6.1 (Gaussian structure)**.** Let G be an additive abelian group. Also equip $G \times G$ with the additive group structure

$$(p_1, p_2) + (q_1, q_2) := (p_1 + q_1, p_2 + q_2) \tag{6.1}$$

$(p_1, p_2), (q_1, q_2) \in G \times G$. Define for each $(v_1, v_2) \in G \times G$,

$$\mathcal{J} := (v_1, v_2) \mapsto (-v_2, v_1). \tag{6.2}$$

Let G also be a directed graph with adjacency relation $\sim_G$. Define an extension, $\sim$, of the adjacency relation $\sim_G$, by defining for any pair of points $p, q \in G \times G$ such that, $p = (p_1, p_2)$, $p \neq q$: $q \sim p \iff p = q + \mathcal{J}^j((s_1 - p_1, 0))$ for some $j \in \mathbb{Z}_{\geq 0}$, and some $s_1 \sim_G p_1$. The structure, $\mathcal{G}$, so obtained is called the *Gaussian structure induced by* G. When $G = \mathbb{Z}$, we shall denote the Gaussian structure by $\mathcal{G}_{\mathbb{Z}}$.

It is clear that letting $G = \mathbb{Z}$ with adjacency being determined by ($q \sim p$, $q \neq p$) $\iff$ ($p \in \{q \pm 1\}$), and $\mathbb{Z}^2$ assumed to have the natural addition induced by $\mathbb{Z}$, we obtain a Gaussian structure which aside from its graph properties, can, when equipped with the usual multiplication, be identified with $\mathbb{Z}[i]$. Indeed, we have $G \times G = \mathbb{Z}^2$, and the map $\mathcal{J}$ can be identified with 90 degree clockwise rotation in the plane. However we are introducing graph properties (which are not a priori part of the definition of the Gaussian structure induced by $\mathbb{Z}$) which in the particular example of $\mathbb{Z}[i]$, implies $z \sim w$, and $z \neq w$, then $z_1 = w_1 \pm i$ or $z_2 = w_2 \pm 1$ and each point has precisely four adjacent points except itself. We may obviously introduce multiplication $(z_1, z_2) \cdot (w_1, w_2) := (z_1 w_1 - z_2 w_2, z_1 w_2 + z_2 w_1)$, and thus be able to identify the Gaussian structure, $\mathcal{G}_{\mathbb{Z}}$, induced by $G = \mathbb{Z}$ with $\mathbb{Z}[i]$ but with additional graph structure as above. Note however that we have not introduced a multiplication in our definitions.

Definition 6.2 (q-polyanalytic functions of the third kind on Gaussian structures)**.** Let $q \in \mathbb{Z}_+$ and let $\mathcal{G}$ be a Gaussian structure induced by a group G (in particular we have an adjacency relation $\sim$ on $G \times G$). Since G is directed we can assign to each ordered pair of adjacent points, s, t, $\lambda_{s,t} = 1$ ($\lambda_{s,t} = -1$) if the ordered pair is of positive (negative) direction. We define a complex-valued function $f \colon \mathcal{G} \to \mathbb{C}$ to be *q-polyanalytic of the third kind* at $p \in \mathcal{G}$ if and only if, $L_3 f(p) = 0$, where $L_3 f(p) = i \sum_{q \sim p, q_2 \neq p_2} f(q) \cdot \lambda_{p_2,q_2} + \sum_{q \sim p, q_1 \neq p_1} f(q) \cdot \lambda_{p_1,q_1}$.

In practice, we shall be working in the case where the inducing group G is $\mathbb{Z}$ and in such cases the other two kinds of 1-polyanalytic functions have equally natural formulations.

Definition 6.3. Let $q \in \mathbb{Z}_+$ and let $\mathcal{G}_{\mathbb{Z}}$ be the Gaussian structure induced by $\mathbb{Z}$. We define a complex-valued function $f \colon \mathcal{G} \to \mathbb{C}$ to be q-polyanalytic of the j:th kind at $z \in \mathcal{G}$ if and only if, $L_j^q f(z) = 0$, $j = 1, 2, 3$, where $L_1 f(z) := f(z+1) - f(z) + i(f(z+i) - f(z))$, $L_2 f(z) := f(z+1) - f(z-1) + i(f(z+i) - f(z-i))$, $L_3 f(z) := f(z+1+i) - f(z) + if(z+i) - if(z+1)$. If the condition holds true at each point of a subset $S \subseteq \mathcal{G}_{\mathbb{Z}}$ where the defining operator is defined, then we say that f is q-polyanalytic of the j:th kind on S and when it is clear from the context what S is we simply say that f is q-polyanalytic of the j:th kind.

Definition 6.4 (Order of adjacency)**.** Let G be a graph, with the adjacency relation $\sim$. Let $p \in G$. Denote $\text{adj}(p, 0) := \{p\}$, and define $\text{adj}(p, 1)$ as the set of points $\{r \in G \colon r \sim p\} \setminus \{p\}$. Iteratively define for each $k \in \mathbb{Z}_+$, $\text{adj}(p, k+1) = \{z \in G \colon z \sim q \text{ for some } q \in \text{adj}(p, k)\} \setminus \bigcup_{j=1}^{k-1} \text{adj}(p, j)$. The set $\text{adj}(p, k)$ will be called the set of points that are *adjacent of order k to p.*

From the perspective of graph theory, it may be notable that when applied to Gaussian structures, the defining operator for 1-polyanalytic functions of the second kind invokes second order adjacency when defining a discrete analogue of a first order operator and we note that the definition of 1-polyanalytic functions of the first kind does not does not use all first order adjacent point. In both cases, we find ourselves with rather skewed powers of the given operators in the sense that the q:th power of the operator at a point z, will involve points which lie unsymmetrically about z. This is not the case for the operator appearing in equation 1.3. We shall now give yet another motivation for 1-polyanalytic functions of the third kind, from the perspective of differential geometry.

Let M be a complex one-dimensional manifold, and let $f \colon M \to \mathbb{C}$ be a differentiable function. It is well-known that f is holomorphic on M if and only if df is $\mathbb{C}$-linear. Let $z = x + iy$, denote the standard complex coordinate for $\mathbb{C}$, and let $p \in M$. If J is the complex structure map on M then a basis for T_pM is given by $v = \frac{\partial}{\partial x}$, $Jv = \frac{\partial}{\partial y}$. Obviously, if df is $\mathbb{C}$-linear then $d_pf(Jv) = id_pf(v)$, for $v = \frac{\partial}{\partial x}$. Conversely, if $d_pf(iv) = id_pf(v)$ then $d_pf(\lambda v) = \lambda d_pf(v)$, and identifying the complex structure map J with multiplication by i, we see that for all $w \in T_pM$, $d_pf(\lambda w) = \lambda d_pf(w)$, i.e. d_pf is $\mathbb{C}$-linear.

Obviously, the real-linearity of d_pf together with the above implies that f satisfies the Cauchy-Riemann equations at p if and only if

$$d_pf(v) + id_pf(iv) = 0, \quad \forall v \in T_pM \tag{6.3}$$

and by definition d_pf is $\mathbb{R}$-linear so that,

$$2d_pf(v) = d_pf(v) - d_pf(-v), \quad \forall v \in T_pM \tag{6.4}$$

Hence

(6.5) $d_p f$ is $\mathbb{C}$-linear $\Leftrightarrow$

$$2d_p f(v) + i2d_p f(iv) = 0 \Leftrightarrow (d_p f(v) - d_p f(-v)) + i(d_p f(iv) - d_p f(-iv)) = 0$$

It is an analogue of these equations that we shall use to define a symmetric discrete operator whose q:th powers will be analogous to the powers $\overline{\partial}^q$.

Recall that if M is an n-dimensional smooth real manifold and $p \in M$, then we can define the set of tangent vectors at p (or *tangent space at* p) as the set of vectors v such that there exists a differentiable curve $\gamma\colon (-\epsilon, \epsilon) \to M$, some $\epsilon > 0$, $\gamma(0) = p$, such that $v = \frac{\partial \gamma}{\partial t}(0)$, and acts on the set of differentiable functions, defined on a neighborhood of p, according to $v(g) := \frac{\partial (f\circ\gamma)}{\partial t}(0)$, for differentiable $f\colon U \to \mathbb{C}$, $p \in U$, U an open neighborhood of p in M. The tangent space at p is denoted T_pM. Also for differentiable $f\colon M \to \mathbb{C}$, we define the differential map $d_pf\colon T_pM \to \mathbb{C}$, as $d_{\gamma(0)} f(\frac{\partial \gamma}{\partial t}(0)) = \frac{\partial (f\circ\gamma)}{\partial t}(0)$.

Definition 6.5. Let $\mathcal{G}$ be a graph and let $p \in \mathcal{G}$. A *path* Γ *through* p *in* G is an ordered set of points $\Gamma(j) = z_j \in G$, $j = -m_1, \dots, m_2$, for nonnegative integers m_1, m_2, such that $z_j \sim z_{j+1}$, $j = -m_1, \dots m_2 - 1$, and $p \in \{\Gamma(j), j = -m_1 + 1, \dots, m_2 - 1\}$. When the base point is not essential to the argument being made we shall simply use the term *path in* $\mathcal{G}$.

For each $p \in \mathcal{G}$, denote $T_p\mathcal{G} = \{v \in G\colon v = q - p,\ q \sim p\}$. This is the set of *tangents.* Obviously, the cardinality of $T_p\mathcal{G}$ may vary dependent upon the base point p.

Let f be a map $\mathcal{G} \to \mathcal{D}$, for an additive abelian group $\mathcal{D}$.

For each $p \in \mathcal{G}$, we have a map $d_pf\colon T_pG \to \mathcal{D}$, according to $v = (q - p) \mapsto f(q) - f(p)$. So there exists a path Γ containing p and q such that $d_pf(v) = f(\Gamma(j_0 + 1)) - f(p)$ where $\Gamma(j_0) = 0$.

Definition 6.6 (1-polyanalytic functions of the third on Gaussian structures)**.** Let $\mathcal{G}$ be the Gaussian structure induced by G, where G is an additive group.

Let R be an additive abelian group and let f be a function $\mathcal{G} \to R^2$, where R^2 is equipped with the component wise addition.

f is called a 1*-polyanalytic function of the third kind (with respect to the Gaussian structure* $\mathcal{G}$*, at* p*),* if (using the notation of Definition 6.5) we have

(6.6) $$d_pf(v) - d_pf(-v) + \mathcal{J}'(d_pf(\mathcal{J}v) - d_pf(-\mathcal{J}v)) = 0, v \in T_p\mathcal{G}$$

Where $\mathcal{J}'$, is defined by $\mathcal{J}'(A, B) = (-B, A)$, and $\mathcal{J}(v_1, v_2) = (-v_2, v_1)$.

From the definitions it is clear that this coincides with the case of 1-polyanalytic functions of the third kind from Definition 6.2, when e.g. $R = \mathbb{R}$, $G = \mathbb{Z}$.

Remark 6.1. Note that in defining our natural discrete analogue (L_3) of the Cauchy-Riemann operator, we have not needed to introduce a multiplicative structure on the domain space (the Gaussian structure), it has been sufficient with a group structure where we on the other hand have required that there exist adjacency (i.e. an additional graph structure).

REFERENCES

[1] V. AVANISSIAN and A. TRAORÉ, Sur les fonctions polyanalytiques de plusiers variables, *C. R. Acad. Sci. Paris,* Sér. A-B, **286**, no. 17 (1978), pp. 743-746.

[2] W. BOSCH, P. KRAIJKIEWICZ, The big Picard theorem for polyanalytic functions, *Proc. Amer. Math. Soc.*, **26** (1970), pp. 145-150.

[3] Z. CUCKOVIC, T. Le, Toeplitz operators on Bergman spaces of polyanalytic functions, *Bull. Lond. Math. Soc.*, **44** (2012), no. 5, pp. 961-973.

[4] R. J. DUFFIN, Basic properties of discrete analytic functions, *Duke Math. J.*, **23** (1956), pp. 335-364.

[5] R. J. DUFFIN, E. PETERSON, The discrete analogue of a class of entire functions, *J. Math. Anal. Appl.*, **21** (1968), pp. 619-642.

[6] K. YU, FEDORVSKIY, C^m approximation by polyanalytic polynomials on Compact Subsets of the Complex Plane, *Complex Anal. Oper. Theory*, **5**, (2011), pp. 671-681.

[7] J. FERRAND, Functions préharmoniques et fonctions préholomorphes, *Bull. Sci. Math.*, **68** (1944), second series, pp. 152-180.

[8] R. P. ISAACS, A finite difference function theory, *Univ. Nac. Tucuman Rev.*, 2 (1941), pp. 177-201.

[9] R. P. ISAACS, Monodiffric functions, *Nat. Bur. Standards Appl. Math.*, Ser. 18 (1952), pp. 257-266.

[10] C. JORDAN, *Calculus of Finite Differences*, 2nd ed., Chelsea, New York, 1950

[11] C. O. KISELMAN, Functions on discrete sets holomorphic in the sense of Isaacs, or monodiffric functions of the first kind, *Science in China, Series A, Mathematics*, **48** Supplement (2005), pp. 86-96.

[12] C. O. KISELMAN, Functions on discrete sets holomorphic in the sense of Ferrand, or monodiffric functions of the second kind, *Science in China, Series A, Mathematics*, **51**, No. 4 (2008), pp. 604-619.

[13] G. J. KUROWSKI, Further results in the theory of monodiffric functions, *Pacific J. Math.*, **18**, no. 1 (1966), pp. 139-147.

[14] G. J. KUROWSKI, Semi-discrete analytic functions, *Trans. Amer. Math. Soc.*, **106** (1963), pp. 1-18.

[15] R. MICKENS, *Difference Equations: Theory and Applications*, 2nd ed., CRC Press, Boca Raton, 1991

[16] A. K. RAMAZANOV, Representation of the space of polyanalytic functions as the direct sum of orthogonal subspaces, Application to rational approximations, *Mat. Zametki* **66** (1999), no. 5, pp. 741-759; translation in *Math. Notes*, **66** (1999), no. 5-6, (2000), pp. 613-627.

[17] A. K. RAMAZANOV, On the Structure of Spaces of Polyanalytic Functions, *Mat. Zametki*, **72** (2002), no. 5, pp. 750-764; translation in *Math. Notes*, **72**, no. 5-6 (2002), pp. 692-704.

[18] S. T. TU, A generalization of monodiffric function, *Hokkaido Math. J.* **12**, no. 2 (1983), pp. 237-243.

[19] S. T. TU, Higher order monodiffric difference equation, *Hokkaido Math. J.* **7**, no. 2 (1978), pp. 43-48.

[20] B. VAN DER POL, The finite-difference analogy of the periodic wave equation and the potential equation. In: Marc Kac, *Probability and Related Topics in Physical Sciences*, pp. 237-257, New York: Interscience Publishers.