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BIRKHOFF-JAMES ORTHOGONALITY AND BEST APPROXIMANT IN $L^1(X)$

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ABSTRACT. Let X be a complex Banach space and let (X, ρ) be a positive measure space. The Birkhoff-James orthogonality is a generalization of Hilbert space orthogonality to Banach spaces. We use this notion of orthogonality to establish a new characterization of Birkhoff-James orthogonality of bounded linear operators in $L^1(X, \rho)$ also implies best approximation has been proved.

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1. INTRODUCTION

The problem of best approximation has a long history and gives rise to a lot of notions and techniques useful in functional analysis. The usual framework of this theory consists in Banach or Hilbert spaces because the geometry of these spaces, via Birkhoff-James orthogonality or orthogonality with respect to the inner product, yields the support for results of existence and uniqueness for elements of best approximation, see e.g. [15].

The notion of orthogonality is an arbitrary normed space, with the norm not necessarily coming from an inner product, may be introduced in various ways. One of the possibilities is the definition introduced by Birkhoff [3], in order to generalize the concept of orthogonality in inner product spaces. Over the years, Birkhoff-James orthogonality has been undoubtedly established as an important concept in the study of geometry of normed linear spaces by virtue of its rich connection with several geometric properties of the space, like strict convexity, uniform convexity, smoothness etc.

Let X be a complex Banach space and let (X, ρ) be a positive measure space. M denote a closed subspace of X. Let $f \in L^1(X)\diagdown\overline{M}$. Then there exists a unique best approximant g to f from M if and only if

$$\|f - g\| \leq \|f - h\| \text{ for all } h \in M.$$

We recall that f is said to be orthogonal to M, written $f \perp M$, if and only if

$$\forall\lambda \in \mathbb{C} : \|f\| \leq \|f + \lambda g\| \text{ for all } g \in M.$$

In this paper, our aim is to establish a new characterization of Birkhoff-James orthogonality of bounded linear operators in $L^1(X, \rho)$ also implies best approximation has been proved.

Definition 1.1. Let $L^p(X), 1 < p < \infty$, and $L^q(X)$ the Dual space. Let M be a closed subspace of $L^p(X)$, we recall that $f \in L^p(X)$ is orthogonal to M, written $f \perp M$, if and only if

$$\|f\|_p \leq \|f + g\|_p \quad \text{for all } g \in M.$$

Theorem 1.1. *Let M be a closed subspace of $L^p(\Omega), 1 < p < \infty$, $f \in L^p(X)$ is orthogonal to M if and only if*

$$\int_X g\,|f|^{p-1}\, sign(f) dx = 0 \text{ for all } g \in M.$$

Proof. See [6]. ∎

Definition 1.2. Let $(X, \|.\|)$ be an arbitrary Banach space. The φ-Gateaux derivative of the norm at f in the direction g is defined as

$$D_{\varphi,f}(g) = \lim_{t\to 0^+} \frac{\|f + te^{i\varphi}g\| - \|f\|}{t}.$$

2. MAIN RESULT

Proposition 2.1. *If the function $H_{f,g}(t) = \|f + te^{i\varphi}g\|$ is convex, then the following statements are hold:*

i) $D_{\varphi,f}(g)$ is subadditive and positively homogeneous functional on X.
ii) $D_{\varphi,f}(g) \leq \|g\|$.
iii) $D_{\varphi,f}(e^{i\theta}g) = D_{\varphi+\theta,f}(g)$.

Proof. i)We have

$$\left\|f+te^{i\varphi}(g+h)\right\| \leq \left\|\frac{f}{2}+te^{i\varphi}g\right\| + \left\|\frac{f}{2}+te^{i\varphi}h\right\|.$$

Taking the limit as $t \to 0^+$, we obtain

$$\begin{aligned} D_{\varphi,f}(g+h) &= \lim_{t\to 0^+}\frac{\|f+te^{i\varphi}(g+h)\|-\|f\|}{t} \leq \lim_{t\to 0^+}\frac{\|f+2te^{i\varphi}g\|+\|f+2te^{i\varphi}h\|-2\|f\|}{2t} \\ &= D_{\varphi,f}(g)+D_{\varphi,f}(h). \end{aligned}$$

Positive homogeneity is obvious.

ii) It is easy to see that

$$\left|\left\|f+te^{i\varphi}g\right\|-\|f\|\right| \leq \left\|f+te^{i\varphi}g-f\right\| = t\|g\|.$$

Taking the limit as $t \to 0^+$, we get

$$D_{\varphi,f}(g)=\lim_{t\to 0^+}\frac{\|f+te^{i\varphi}(g)\|-\|f\|}{t} \leq \|g\|.$$

iii) The proof is obvious. ∎

Theorem 2.2. *Let $(X,\|.\|)$ be an arbitrary Banach space. If the function $f \in X$ is orthogonal to $g \in X$, then*

$$\inf_{\varphi} D_{\varphi,f}(g) \geq 0.$$

Proof. Let f be orthogonal to g, i.e.

$$\forall\lambda \in \mathbb{C}\ \ \|f\| \leq \|f+\lambda g\| \quad \text{for all } g \in M.$$

Then

$$\frac{\|f+te^{i\varphi}g\|-\|f\|}{t} \geq 0 \qquad \text{for all } t>0,$$

and passing to the limit as $t \to 0^+$, we obtain

$$\inf_{\varphi} D_{\varphi,f}(g) \geq 0.$$

∎

Theorem 2.3. *Let M be linear subspace of $L^1(X)$ and $f \in L^1(X)\diagdown\overline{M}$, then g is a best $L^1(X)$ approximant to f from M if and only if*

$$\left|\int_{f(x)=g(x)} e^{-i\theta(x)}h(x)d\rho(x)\right| \leq \int_{f(x)\neq g(x)} |h(x)|\,d\rho(x) \text{ for all } h \in M,$$

where

$$(f-h)(x)=|f-h|\,e^{-i\theta(x)}.$$

Proof. See [12]. ∎

Definition 2.1. Let M be a linear closed subspace of $L^1(X)$ and let $S(X)=\{\varphi\in X\diagup\|\varphi\|\leq 1\}$. $f\in L^1(X)$ is orthogonal to M if and only if, there exists a function $\varphi \in S(X)$, such that

i) $\int\limits_X f\varphi dx = \int\limits_X |f|\,dx.$

ii) $\int\limits_X \varphi h dx = 0,$ for all $h \in M$.

Remark 2.1. In the particular case, for $x \in \mathcal{K}er f$ has measure zero. The function $f \in L^1(X)$ is orthogonal to M if and only if

$$\int_X (sign f)h(x)d\rho(x) = 0, \text{ for all } h \in M.$$

Theorem 2.4. *Let M be linear closed subspace of $L^1(X)$. The function $f \in L^1(X)$ is orthogonal to $g \in M$ if and only if*

$$\left| \int_{\{g\neq 0\}} e^{-i\theta(x)} f(x)d\rho(x) \right| \leq \int_{\{g=0\}} |f(x)|\, d\rho(x),$$

where

$$f(x) = |f(x)|\, e^{i\theta(x)}.$$

Proof. We have, in $L^1(X)$

$$D_{\varphi,g}(f) = \mathcal{R}e \left\{ \int_{\{g\neq 0\}} e^{i\varphi} e^{-i\theta(x)} f(x)d\rho(x) \right\} + \int_{\{g=0\}} |f(x)|\, d\rho(x).$$

Since

$$\lim_{\rho\to 0} \frac{|g(x) + \rho e^{i\varphi} f(x)| - |g(x)|}{\rho} = \begin{cases} \cos(\varphi - \theta(x)) + \psi(x)\, |f(x)| & g(x) \neq 0, \\ |f(x)|\,, & g(x) = 0, \end{cases}$$

and also

$$\frac{|g(x) + \rho e^{i\varphi} f(x)| - |g(x)|}{\rho} \leq |f(x)|\,.$$

Thus, we get $f \perp g$ if and only if

$$\inf_{\varphi} \mathcal{R}e \left\{ \int_{f\neq 0} e^{i\varphi} e^{-i\theta(x)} f(x)d\rho(x) \right\} + \int_{f=0} |f(x)|\, dx \geq 0.$$

However, the infimum will be attained for that φ, for which

$$e^{i\varphi} \int_{f\neq 0} e^{-i\theta(x)} f(x)d\rho(x) = - \left| \int_{f\neq 0} e^{-i\theta(x)} f(x)d\rho(x) \right|,$$

and the result follows. ∎

Corollary 2.5. *Let M be a linear closed subspace of $L^1(X)$, then the following assertions are equivalent*

i) The function $f \in L^1(X)$ is orthogonal to $g \in M$ if and only if

$$\left| \int_{\{g\neq 0\}} e^{-i\theta(x)} f(x)d\rho(x) \right| \leq \int_{\{g=0\}} |f(x)|\, d\rho(x).$$

ii) g is a best $L^1(X)$ approximant to f from M if and only if

$$\left| \int_{f(x)=g(x)} e^{-i\theta(x)} h(x) d\rho(x) \right| \leq \int_{f(x)\neq g(x)} |h(x)|\, d\rho(x) \; for\ all\ h \in M,$$

where

$$(f-h)(x) = |f-h|\, e^{-i\theta(x)}$$

iii) The function $f \in L^1(X)$ is orthogonal to $g \in M$ if and only if

$$e^{i\varphi} \int_{f\neq 0} e^{-i\theta(x)} f(x) d\rho(x) = - \left| \int_{f\neq 0} e^{-i\theta(x)} f(x) d\rho(x) \right|.$$

Proof. i)$\Longrightarrow$ ii)

Let $f \in L^1(X)$ is orthogonal to $g \in M$, then

$$\left| \int_{\{g\neq 0\}} e^{-i\theta(x)} f(x) d\rho(x) \right| \leq \int_{\{g=0\}} |f(x)|\, d\rho(x).$$

Taking

$$g(x) = f_1(x) - h_1(x).$$

Then f_1 is a best $L^1(X)$ approximant to h_1 from M.

iii) $\Longrightarrow$ i)

The function $f \in L^1(X)$ is orthogonal to $g \in M$ implies

$$e^{i\varphi} \int_{f\neq 0} e^{-i\theta(x)} f(x) d\rho(x) = - \left| \int_{f\neq 0} e^{-i\theta(x)} f(x) d\rho(x) \right|.$$

Taking $\varphi = 0$, then

$$\left| \int_{\{g\neq 0\}} e^{-i\theta(x)} f(x) d\rho(x) \right| \leq \int_{\{g=0\}} |f(x)|\, d\rho(x).$$

iii) $\Longrightarrow i \Rightarrow$ ii)

i) $\Rightarrow$ iii), using Theorem 2.4. ∎

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