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ON COMMUTATOR OF ALUTHGE TRANSFORMS AND FUGLEDE-PUTNAM PROPERTY

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ABSTRACT. We deal with the well-known Fuglede-Putnam theorem and related FP-property. We show that if (A, B) has the FP-property, then so has $(\tilde{A}_{(1,t_1)}, \tilde{B}_{(1,t_2)})$ where $0 \leq t_1, t_2 \leq 1$ are arbitrary. We first prove that $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$ if and only if $AX = XB$ for all X, whenever (A, B) has the FP-property. We prove some similar results for $(\tilde{A}^{(*)}_{(1,t_1)}, \tilde{B}^{(*)}_{(1,t_2)})$ instead of $(\tilde{A}_{(1,t_1)}, \tilde{B}_{(1,t_2)})$ as well. Also we introduce the sequence of generalized iterations of Aluthge transform of operators and express some results for this notion associated to the FP-property.

Key words and phrases: Aluthgeh transform, Polar decomposition, Self-commutator, Positive semidefinite, Fuglede–Putnam's theorem.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{B}(\mathcal{H},\mathcal{K})$ be the algebra of all bounded linear operators between complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and let $\mathbb{B}(\mathcal{H})$ denote $\mathbb{B}(\mathcal{H},\mathcal{H})$. A subspace $M\subseteq\mathcal{H}$ is said to be a reducing subspace of $A\in\mathbb{B}(\mathcal{H})$ if $AM\subseteq M$ and $A^*M\subseteq M$. Evidently In the case when $\mathcal{H}$ is finite dimensional we could identify $\mathbb{B}(\mathcal{H})$ and $M_n(\mathbb{C})$, the space of all $n\times n$ complex matrices. A matrix A is called positive semidefinite when $A\geq 0$ where $\geq$ is the well-known Heinz–Löwner order. We write $A>0$, to mean A is invertible and positive semidefinite. In this case we say that A is positive definite. For $A\in M_n(\mathbb{C})$ there is a unitary matrix U and a positive semidefinite matrix P such that $A=UP$. The right hand side of this equality is called polar decomposition of A. Note that the positive part P is uniquely determined by $P=|A|=(A^*A)^{\frac{1}{2}}$. When A is invertible, $U=AP^{-1}$ is also unique. Polar decomposition for operators similar to that of matrices exists. The difference is that the operator U in $A=U|A|$ is a partial isometry. For every $r\geq t\geq 0$, the generalized (r,t)-Aluthge transform $\tilde{A}_{(r,t)}$ of A is defined by

$$\tilde{A}_{(r,t)}=|A|^tU|A|^{r-t},$$

whenever $A=U|A|$ is a polar decomposition and the generalized $*$-Aluthge transform $\tilde{A}^{(*)}_{(r,t)}$ of A is defined by

$$\tilde{A}^{(*)}_{(r,t)}=|A^*|^tU|A^*|^{r-t}.$$

If $r=1$ and $t=\frac{1}{2}$, then $\tilde{A}_{(r,t)}$ is denoted by $\tilde{A}$ and is called the Aluthge transform of A.

We notify here that this notion was firstly introduced by Aluthge in [1] during the review of properties of p-hyponormal operators, and today it has been become a powerful tool in the operator theory. An operator $A\in\mathbb{B}(\mathcal{H})$ is normal if $A^*A=AA^*$ and is p-hyponormal, for some $0<p\leq 1$, if $|A|^{2p}\geq|A^*|^{2p}$.

Let $A\in\mathbb{B}(\mathcal{H})$ and $B\in\mathbb{B}(\mathcal{K})$. A pair (A,B) is said to have the Fuglede-Putnam property if $AX=XB$ implies $A^*X=XB^*$.

The famous Fuglede-Putnam theorem, see [2], asserts that if A and B are normal operators, then (A,B) has FP-property. There exist many generalization of this theorem which most of them go into relaxing the normality of A and B, see [7]. The next lemma is concerned with the Fuglede-Putnam theorem and we need it in the future.

Lemma 1.1. [7] *Let $A\in\mathbb{B}(\mathcal{H})$ and $B\in\mathbb{B}(\mathcal{K})$. Then the following assertions equivalent.*

(i) *The pair (A,B) has the Fuglede-Putnam property.*
(ii) *If $AX=XB$, then $\overline{R(X)}$ reduces A, $ker(X)^\perp$ reduces B and $A|_{\overline{R(X)}}$, $B|_{ker(X)^\perp}$ are unitarily equivalent normal operators.*

In this paper we look for the response of the following question; under what conditions on engaged operators, any one of $AX=XB$, $\tilde{A}^{(*)}_{(r,t_1)}X=X\tilde{B}^{(*)}_{(r,t_2)}$ and $\tilde{A}_{(r,t_1)}X=X\tilde{B}_{(r,t_2)}$ implies the other ones. Another problem, considered in this paper, relates to relationships between pairs (A,B), $(\tilde{A}^{(*)}_{(r,t_1)},\tilde{B}^{(*)}_{(r,t_2)})$ and $(\tilde{A}_{(r,t_1)},\tilde{B}_{(r,t_2)})$, through FP-property. In fact it is probed when does the FP-property for any one of which yield that for others. These problems were firstly raised in [5] for $\tilde{A}$ and $\tilde{B}$ instead of $\tilde{A}^{(*)}_{(r,t)}$, $\tilde{B}^{(*)}_{(r,t)}$, $\tilde{A}_{(r,t)}$ and $\tilde{B}_{(r,t)}$. In [6] the author continued this conclusion in that article. This article is in fact a continuation of [3, 5, 6]. We consider a vary general case in the sense that we have allowed indices in two sides to be freely chosen in interval $[0,1]$. We also introduce the sequence of generalized iterations of Aluthge transform of operators and express some result for that, associated to the problem stated above.

2. FUGLEDE-PUTNAM TYPE THEOREMS AND ALUTHGE TYPES TRANSFORMS

In this section we speak some on well-known Fuglede-Putnam theorem. We mainly interested in relationship between pair (A, B) of operators and pairs $(\tilde{A}^{(*)}_{(r,t_1)}, \tilde{B}^{(*)}_{(r,t_2)})$ and $(\tilde{A}_{(r,t_1)}, \tilde{B}_{(r,t_2)})$, for appropriate parameters r, t_1, t_2, of their associated Aluthge transforms through this theorem. We start with the following theorem in which we generalize some results of [5]. The proof is exactly the same. Its elementary proof is dropped.

Theorem 2.1. *Let A and B be two invertible operators with the polar decompositions $A = U|A|$ and $B = V|B|$ such that (A, B) has the FP-property and let r, t_1 and t_2 be some positive numbers with $1 \geq t_1 \geq 0$ and $1 \geq t_2 \geq 0$. The following assertions hold*

(i) *if X is an operator with $AX = XB$, then $|A|^r X = X|B|^r$, $|A^*|^r X = X|B^*|^r$, $UX = XV$, $U^*X = XV^*$ and;*
(ii) *if X is an operator such that $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$, then $\tilde{A}^*_{(1,1-t_1)}X = X\tilde{B}^*_{(1,1-t_2)}$.*
(iii) *if X is an operator such that $\tilde{A}^{(*)}_{(1,t_1)}X = X\tilde{B}^{(*)}_{(1,t_2)}$, then $(\tilde{A}^{(*)}_{(1,1-t_1)})^*X = X(\tilde{B}^{(*)}_{(1,1-t_2)})^*$.*

Remark 2.1. For invertible operators A and B if (A, B) has FP- property, then so are $(\tilde{A}, \tilde{B})$ and $(\tilde{A}^{(*)}, \tilde{B}^{(*)})$. Note that the method used in this theorem for concluding the FP-property of $(\tilde{A}, \tilde{B})$ from that of (A, B), is not effective for the pair $(\tilde{A}_{(r,t)}, \tilde{B}_{(r,t)})$ or $(\tilde{A}^{(*)}_{(r,t)}, \tilde{B}^{(*)}_{(r,t)})$. Fortunately this conclusion is valid and will be proved in the sequel , in another method even in more general case(Theorem 2.6 below).

Proposition 2.2. *Suppose A and B are invertible operators. The pair $(\tilde{A}^{(*)}, \tilde{B}^{(*)})$ has the FP-property if and only if so does $(\tilde{A}, \tilde{B})$.*

Proof. In theorem 2.4 of [5], it is shown that the FP-property for $(\tilde{A}, \tilde{B})$ is equivalent to the following requirement

$$U^2X = XV^2 \tag{2.1}$$

for all X with $AX = XB$. Thus we have to just show that the FP-property for $(\tilde{A}^{(*)}, \tilde{B}^{(*)})$ is equivalent to (2.1) which is done in the similar method as theorem 2.4 of [5] and stating of which is somehow redundant. ∎

Now, we present some results concerning a problem which is closely related to Fuglede-Putnam-Aluthge problem discussed in [4].

Theorem 2.3. *Let $r > t > 0$ and $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ and let the pair (A, B) has the FP property. Then $AX = XB$ implies*

(i) $\tilde{A}_{(r,t_1)}X = X\tilde{B}_{(r,t_2)}$;
(ii) $\tilde{A}^*_{(r,t_1)}X = X\tilde{B}^*_{(r,t_2)}$;
(iii) $\tilde{A}^{(*)}_{(r,t_1)}X = X\tilde{B}^{(*)}_{(r,t_2)}$;
(iv) $(\tilde{A}^{(*)}_{(r,t_1)})^*X = X(\tilde{B}^{(*)}_{(r,t_2)})^*$.

Proof. Firstly we note that $AX = XB$ and the fact that (A, B) has the FP-property ensure $A'X = XB'$ where $A' = U|A|^r$ and $B' = V|B|^r$ by using a routine application of functional calculus. Since (A, B) has FP-property and $AX = XB$ by Lemma 1.1 we have that $\overline{R(X)}$ reduces A, $ker(X)^\perp$ reduces B and $A|_{\overline{R(X)}}$, $B|_{ker(X)^\perp}$ are unitarily equivalent normal operators. Let

$$A = N \oplus T \text{ on } \mathcal{H} = \overline{R(X)} \oplus R(X)^\perp$$

and

$$B = M \oplus S \text{ on } \mathcal{K} = ker(X)^{\perp} \oplus ker(X),$$

where N and M are unitarily equivalent normal operators. Let $N = U_1|N|$ and $M = V_1|M|$ be the polar decomposition. Thus $N' = U_1|N|^r$ and $M' = V_1|M|^r$ are the polar decomposition and N' and M' are normal operators and

$$A' = N' \oplus T' \text{ on } \mathcal{H} = \overline{R(X)} \oplus R(X)^{\perp}$$

and

$$B' = M' \oplus S' \text{ on } \mathcal{K} = ker(X)^{\perp} \oplus ker(X),$$

Let

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to $\mathcal{H} = \overline{R(X)} \oplus R(X)^{\perp}$ and $\mathcal{K} = ker(X)^{\perp} \oplus ker(X)$. From $A'X = XB'$ we can conclude

$$N'X_1 = X_1M'.$$

On the other hand we have that

$$\tilde{A}_{(r,t_1)}X = \tilde{A'}_{(1,\frac{t_1}{r})}X = \begin{bmatrix} N'X_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$X\tilde{B}_{(r,t_2)} = X\tilde{B'}_{(1,\frac{t_2}{r})} = \begin{bmatrix} X_1M' & 0 \\ 0 & 0 \end{bmatrix}$$

which by (2.2) establish (i). The items (ii),(iii) and (iv) are accomplished similar to (i). ∎

In the following lemma we generalized theorem 4.2 of [5]. The proof used in [5] works good enough for that of ours and we state it for the sake of completeness.

Lemma 2.4. *Let $0 < t_1, t_2 \leq 1$ and let $A \in \mathbb{B}(\mathcal{H})$ be invertible and $B \in \mathbb{B}(\mathcal{K})$ be arbitrary operators and let the pair (A, B) has the Fuglede-Putnam property. If any one of*

(i) $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$ *and*

(ii) $\tilde{A}^{(*)}_{(1,t_1)}X = X\tilde{B}^{(*)}_{(1,t_2)}$

takes place, then $AX = XB$.

Proof. Assume (i) and let $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$. Let $A = U|A|$ and $B = V|B|$ be the polar decomposition of A and B, respectively. Let $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$ and let $W = |A|^{-t_1}X|B|^{t_2}$. Since $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$, we have

$$\begin{aligned} |A|^{-t_1}\tilde{A}_{(1,t_1)}X|B|^{t_2} &= |A|^{-t_1}X\tilde{B}_{(1,t_2)}|B|^{t_2} \\ |A|^{-t_1}|A|^{t_1}U|A|^{1-t_1}X|B|^{t_2} &= |A|^{-t_1}X|B|^{t_2}V|B|^{1-t_2}|B|^{t_2} \\ U|A|(|A|^{-t_1}X|B|^{t_2}) &= (|A|^{-t_1}X|B|^{t_2})V|B|, \end{aligned}$$

so

$$AW = WB.$$

Hence by hypothesis and Lemma 1.1, $\overline{R(W)}$ reduces A, $ker(W)^{\perp}$ reduces B and $A|_{\overline{R(W)}}$, $B|_{ker(W)^{\perp}}$ are normal operators. Therefore

$$A = N \oplus T \text{ on } H = \overline{R(W)} \oplus R(W)^{\perp}$$

and

$$B = M \oplus S \ on \ K = ker(W)^{\perp} \oplus ker(W),$$

where N and M are unitarily equivalent normal operators. Since A is invertible then so are N and T. Also, since N and M are unitarily equivalent, M is invertible. Let

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \text{ and } W = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to $H = \overline{R(W)} \oplus R(W)^{\perp}$ and $K = ker(W)^{\perp} \oplus ker(W)$. Clearly $|A|^{-1} = |N|^{-1} \oplus |T|^{-1}$. It follows from $W = |A|^{-t_1} X |B|^{t_2}$ that

$$\begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} |N|^{-t_1} X_{11} |M|^{t_2} & |N|^{-t_1} X_{12} |S|^{t_2} \\ |T|^{-t_1} X_{21} |M|^{t_2} & |T|^{-t_1} X_{22} |S|^{t_2} \end{bmatrix}.$$

Hence $X_{12}|S|^{t_2} = 0$, $X_{12} = 0$, $X_{22}|S|^{t_2} = 0$. So $X_{21}\tilde{S}_{(1,t_2)} = 0$ and $X_{22}\tilde{S}_{(1,t_2)} = 0$. Then $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$ implies that

$$\begin{bmatrix} NX_{11} & NX_{12} \\ 0 & \tilde{T}_{(1,t_1)}X_{22} \end{bmatrix} = \begin{bmatrix} X_{11}M & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $X_{12} = 0$ and $X_{22} = 0$. Since $\tilde{A}_{(1,t_1)} = N \oplus \tilde{T}_{(1,t_1)}$ and $\tilde{B}_{(1,t_2)} = M \oplus \tilde{S}_{(1,t_2)}$ and $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$ and $X = X_{11} \oplus 0$, we have $NX_{11} = X_{11}M$ and this, in turn, implies that $AX = XB$.
In other case the proof is similar and omitted. ∎

The following theorem is concluded easily from the Lemma 2.4 and Theorem 2.3

Theorem 2.5. *Let* $0 \leq t_1, t_2 \leq 1$. *Let* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$ *be invertible operators and let the pair* (A, B) *have the Fuglede-Putnam property. Then the following assertions are equivalent*

(i) $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$;
(ii) $\tilde{A}^{(*)}_{(1,t_1)}X = X\tilde{B}^{(*)}_{(1,t_2)}$;
(iii) $AX = XB$.

Now we show that the FP-property is spread from the tuple (A, B) to tuples $(\tilde{A}_{(1,t_1)}, \tilde{B}_{(1,t_2)})$ and $(\tilde{A}^{(*)}_{(1,t_1)}, \tilde{B}^{(*)}_{(1,t_2)})$. Note that the indices have been chosen so much freely.

Theorem 2.6. *Let A and B be two invertible operators. If (A, B) has the FP- property, then so have $(\tilde{A}_{(1,t_1)}, \tilde{B}_{(1,t_2)})$ and $(\tilde{A}^{(*)}_{(1,t_1)}, \tilde{B}^{(*)}_{(1,t_2)})$ for all $0 < t_1, t_2 \leq 1$.*

Proof. Let $\tilde{A}_{(1,t_1)}X = X\tilde{B}_{(1,t_2)}$ for some X. Then $AX = XB$ by the previous theorem. So $\tilde{A}_{(1,t_1)}W = W\tilde{B}_{(1,t_2)}$ where $W = |A|^{t_1}X|B|^{-t_2}$. Again the previous theorem implies that $AW = WB$ therefore $A^*W = WB^*$ by FP-property of (A, B). Hence

$$|A|U^*|A|^{t_1}X|B|^{-t_2} = |A|^{t_1}X|B|^{-t_2}|B|V^*$$

which is $\tilde{A}^*_{(1,t_1)}X = X\tilde{B}^*_{(1,t_2)}$.

The FP- property for the pair $(\tilde{A}^{(*)}_{(1,t_1)}, \tilde{B}^{(*)}_{(1,t_2)})$ could be proved in the similar way. ∎

Remark 2.2. Here we draw the reader's attention to the indices of generalized Aluthge transforms in tow sides of all equalities in Theorems 2.1 and 2.4 and Corollary 2.6 which are different.

Problem 1. When does $(\tilde{A}_{(r,t_1)}, \tilde{B}_{(r,t_2)})$ have the FP-property, where $0 \leq t_1, t_2 \leq r$

Definition 2.1. Let $\hat{\alpha} = \{\alpha_n\}$ be a sequence in interval $[0,1]$. The sequence of generalized iterations of Aluthge transform of operator A is defined inductively by

$$\Delta_{\widehat{\alpha_1}}(A) = \Delta_{\alpha_1}(A) = \tilde{A}_{(1,\alpha_1)}$$

and

$$\Delta_{\widehat{\alpha_i}}(A) = \Delta_{\alpha_i}(\Delta_{\widehat{\alpha_{i-1}}}(A))$$

for $i \geq 2$.

The sequence of generalized iterations of $*$-Aluthge transform of operator A, denoted by $\Delta^{(*)}_{\widehat{\alpha_i}}(A)$, is defined similarly.

Corollary 2.7. *Let A and B be two invertible operators. If (A,B) has the FP- property, then so have $(\Delta_{\widehat{\alpha_i}}(A), \Delta_{\widehat{\beta_i}}(B))$ and $(\Delta^{(*)}_{\widehat{\alpha_i}}(A), \Delta^{(*)}_{\widehat{\beta_i}}(B))$ for all sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$.*

Corollary 2.8. *Let A and B be two invertible operators and X be an arbitrary operator and let sequences $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$. If (A,B) has the FP- property and either $\Delta_{\widehat{\alpha_i}}(A)X = X\Delta_{\widehat{\beta_i}}(B)$ or $\Delta^{(*)}_{\widehat{\alpha_i}}(A)X = X\Delta^{(*)}_{\widehat{\beta_i}}(B)$, then $AX = XB$.*

Proof. We prove that $\Delta_{\widehat{\alpha_i}}(A)X = X\Delta_{\widehat{\beta_i}}(B)$ implies $AX = XB$. The other is done similarly. By Corollary 2.7 we have that $(\Delta_{\widehat{\alpha_{i-1}}}(A), \Delta_{\widehat{\beta_{i-1}}}(B))$ has the FP-property which by $\Delta_{\widehat{\alpha_i}}(A)X = X\Delta_{\widehat{\beta_i}}(B)$ and using Lemma 2.4 we reach to $\Delta_{\widehat{\alpha_{i-1}}}(A)X = X\Delta_{\widehat{\beta_{i-1}}}(B)$. Going on in this process we could prove $AX = XB$. ∎

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