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# SOME PROPERTIES OF QUASINORMAL, PARANORMAL AND $2-k^*$ PARANORMAL OPERATORS

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ABSTRACT. In the beginning of this paper some conditions under which an operator is partial isometry are given. Further, the class of $2-k^*$ paranormal operators is defined and some properties of this class in Hilbert space are shown. It has been proved that an unitarily operator equivalent with an operator of a $2-k^*$ paranormal operator is a $2-k^*$ paranormal operator, and if is a $2-k^*$ paranormal operator, that commutes with an isometric operator, then their product also is a $2-k^*$ paranormal operator.

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## 1. INTRODUCTION

We will denote with $H$ the Hilbert space and with L(H) the space of all bounded linear operators defined in Hilbert space $H$. The operator $T \in L(H)$ is said to be paranormal, or an operator from (N) class, if $\| Tx \|^2 \leq \| T^2x \|, (x \in H, \| x \|= 1)$, and $^*$ paranormal if, $\| T^*x \|^2 \leq \| T^2x \|, (x \in H, \| x \|= 1)$. The operator $T \in L(H)$ is called $k-$paranormal or from $(N; k)$ class, if $\| Tx \|^k \leq \| T^kx \|, (x \in H, \| x \|= 1)$ and $T \in L(H)$ is called $k^*-$paranormal or from $(N; k^*)$ class, if $\| T^*x \|^k \leq \| T^kx \|, (x \in H, \| x \|= 1)$. It has been shown than every paranormal operator is a $k-$paranormal operator [5]. An operator $T \in L(H)$ is said to be a $2 - k^*$ paranormal operator if $\| T^{*2}x \|^k \leq \| T^kx \|^2, (x \in H, \| x \|= 1)$ and an operator $T \in L(H)$ is said to be a $2 - k$ paranormal operator if $\| T^2x \|^k \leq \| T^kx \|^2, (x \in H, \| x \|= 1)$. Operator $T$ belongs to the $(M, k)$ class, if and only if $T^{*k}T^k \geq (T^*T)^k (k \geq 2)$ [2].

**Theorem 1.1.** *( Hölder-MrCarthy inequality)* [4] *Let $A$ be a positive linear operator on a Hilbert space $H$. Then the following properties (i), (ii) and (iii) hold.*

*(i)* $(A^\lambda x, x) \geq (Ax, x)^\lambda (\lambda > 1, x \in H, \| x \|= 1)$.

*(ii)* $(A^\lambda x, x) \leq (Ax, x)^\lambda (\lambda \in [0, 1], x \in H, \| x \|= 1)$.

*(iii) If $A$ is invertible, then*

$$(A^\lambda x, x) \geq (Ax, x)^\lambda (\lambda < 0, x \in H, \| x \|= 1).$$

Moreover (i), (ii) and (iii) are equivalent to the following (i)', (ii)' and (iii)', respectively.

(i)' $(A^\lambda x, x) \geq (Ax, x)^\lambda \| x \|^{2(1-\lambda)} (\lambda > 1, x \in H)$.

(ii)' $(A^\lambda x, x) \leq (Ax, x)^\lambda \| x \|^{2(1-\lambda)} (\lambda \in [0, 1], x \in H)$.

(iii)' If $A$ is invertible, then

$$(A^\lambda x, x) \geq (Ax, x)^\lambda \| x \|^{2(1-\lambda)} (\lambda < 0, x \in H).$$

An operator $U \in L(H)$ is said to be a partial isometry operator if there exists a closed subspace $M$, such that $\| U(x) \|=\| x \|$ for any $x \in M$, and $Ux = 0$ for any $x \in M^\perp$ . $M$ is said to be the initial space of $U$ and $N = R(U)$ is said to be the final space of $U$. Operator $U \in L(H)$ is isometry if and only if $U$ is partial isometry and $M = H$, and $U$ is unitary if and only if $U$ is partial isometry and $M = N = H$ . Operator $T \in L(H)$ is said to be a subnormal operator if $T$ has a normal extension $N$, that is, there exists a normal operator $N$ on a larger Hilbert space $K \supset H$ such that $Nx = Tx$ for all $x \in H$.

**Theorem 1.2.** [4] *Let $U$ be a partial isometry operator on a Hilbert space $H$ with the initial space $M$ and the final space $N$ . Then the following (i), (ii) and (iii) hold;*

*(i) $UP_M = U$ and $U^*U = P_M$.*

*(ii) $N$ is a closed subspace of $H$ .*

*(iii) $U^*$ is a partial isometry with the initial space $N$ and the final space $M$ , that is, $U^*P_N = U^*$ and $UU^* = P_N$ .*

**Theorem 1.3.** [4] *If $T$ is an idempotent and contraction operator $\| T \| \leq 1$, then $T$ is a projection.*

**Corollary 1.4.** [4] *(i) If $T$ is an idempotent normaloid operator, then $T$ is a projection.*

*(ii) If $T$ is an idempotent paranormal operator, then $T$ is a projection.*

**Theorem 1.5.** [4] *If $T$ is a contraction operator, and satisfies*

$$T^k = T$$

*for some positive integer $T \geq 2$, then $T^{k-1}$ is a projection.*

## 2. SOME CONDITIONS UNDER WHICH AN OPERATOR IS PARTIAL ISOMETRY

**Theorem 2.1.** *Let $T \in L(H)$ such that*

$$(T^*T)^k = T^*T (k \geq 2).$$

*Then $T$ is a partial isometry.*

*Proof.* Since $(T^*T)^k = T^*T$, we have

$$\begin{aligned}
&\|T^*T\|^k = \|T^*T\| \\
\Rightarrow &\|(\sqrt{T^*T})^2\|^k = \|T\|^2 \\
\Rightarrow &\|\sqrt{T^*T}\|^{2k} = \|T\|^2 \\
\Rightarrow &\|\sqrt{T^*T}\|^k = \|T\| \\
\Rightarrow &\|U\sqrt{T^*T}\|^k = \|T\| \\
\Rightarrow &\|T\|^k = \|T\| \\
\Rightarrow &\|T\| = 1,
\end{aligned}$$

where $U$ is taken from the polar form of the operator $T$. It means that $T$ is contraction. By Theorem 1.2, we conclude that $(T^*T)^{k-1}$ is projection. Further on,

$$\sigma((T^*T)^{k-1}) = \sigma(P) = \{0, 1\}.$$

Since $T^*T$ is positive we will have

$$\sigma(T^*T) = \{0, 1\}.$$

We denote with $Q = (T^*T)^2 - T^*T$. Then the spectrum of the operator $Q$ is

$$\sigma(Q) = \sigma((T^*T)^2 - T^*T) = \{\lambda^2 - \lambda : \lambda = 0, \lambda = 1\} = \{0\},$$

which means that $Q$ is a quasinilpotent hermitian operator. Therefore $Q = 0$ and $(T^*T)^2 = T^*T$. Since $T^*T$ is idempotent and contraction by Theorem 1.3 $T^*T = P_1$, where $P_1$ is a projection, it means that $T$ is a partial isometry. ∎

**Theorem 2.2.** *Let $T \in L(H)$ be a quasinormal operator, such that*

$$T^k = T (k \geq 2).$$

*Then $T$ is partial isometry.*

*Proof.* From $T^k = T$, we have $T^{*k} = T^*$. Multiplying these equations we have

$$T^{*k}T^k = T^*T.$$

Since $T$ is a quasinormal we obtain

$$(T^*T)^k = T^*T.$$

By Theorem 2.1 $T^*T$ is a projection, it means that $T$ is a partial isometry. ∎

## 3. $2-k^*$ PARANORMAL OPERATORS

Let $T$ and $S$ be operators on Hilbert spaces $H_1$ and $H_2$ respectively. $T$ is said to be unitarily equivalent to $S$ if there exists unitary operator $U$ from $H_1$ to $H_2$ such that $S = UTU^*$. When an operator $T$ commutes with $S$ and $S^*$, we say that $T$ doubly commutes with $S$. In [5] authors Y. Park and Ch. Ryoo have proved the following theorem regarding the $k^*-$ paranormal operators.

**Theorem 3.1.** *A unitarily equivalent operator to the $k^*-$ paranormal operator is a $k^*-$ paranormal operator.*

The theorem is true for the $2-k^*$ paranormal operator, also.

**Theorem 3.2.** *If $T \in L(H)$ is a $2-k^*$ paranormal operator, then the unitarily equivalent operator of the operator $T$, is also $2-k^*$ paranormal operator.*

*Proof.* Let's assume that $S = U^*TU$, where $T$ is a $2-k^*$ paranormal operator and $U$ is an unitary operator. For every $x \in H$, we have

$$\|S^{*2}x\|^k = \|(U^*TU)^{*2}x\|^k = \|(U^*T^*U)^2x\|^k = \|U^*T^{*2}Ux\|^k$$

$$= \|T^{*2}Ux\|^k \le \|T^kUx\|^2 = \|U^*T^kUx\|^2 = \|(U^*TU)^kx\|^2 = \|S^kx\|^2.$$

From which we have that $S$ is a $2-k^*$ paranormal operator. ∎

**Lemma 3.3.** *If $T \in L(H)$ is a $2-k$ paranormal operator, then the unitarily equivalent operator of the operator $T$, is also a $2-k$ paranormal operator.*

*Proof.* The proof is similar to the Theorem 3.2 ∎

**Theorem 3.4.** *If $T$ is an isometry and $T^*$ is a $2-k^*$ paranormal operator, then $T$ is unitary.*

*Proof.* Since $T$ is isometry and $T^*$ is $2-k^*$ paranormal, then $T$ is $2-k^*$ paranormal. On the other hand every isometry is hyponormal operator therefore for every $x \in H$, we have

$$\begin{aligned}\|x\|^k = \|T^*Tx\|^k \le \|T^2x\|^k &\le \|T^{*k}x\|^2\|x\|^{k-2} \\ &\le \|TT^{*k-1}x\|^2\|x\|^{k-2} = \|T^{*k-2}x\|^2\|x\|^{k-2} \\ &\le \|T^*x\|^2\|x\|^{k-2} \le \|Tx\|^2\|x\|^{k-2} = \|x\|^k.\end{aligned}$$

From this we have that $\|Tx\| = \|x\|$ and $\|x\| = \|T^*x\| (x \in H)$ or $T^*T = I$ and $TT^* = I$, respectively. Consequently, $T$ is unitary. ∎

**Theorem 3.5.** *Let $T \in L(H)$ be a $2-k^*$ paranormal operator, which commutes with on isometry $S$. Then $TS$ is a $2-k^*$ paranormal operator.*

*Proof.* Let $x \in H, \|x\| = 1$. Then,

$$\begin{aligned}\|(TS)^{*2}x\|^k &= \|S^{*2}T^{*2}x\|^k \le \|SS^*T^{*2}x\|^k \\ &= \|S^*T^{*2}x\|^k \le \|T^{*2}x\|^k \le \|T^kx\|^2 \\ &= \|ST^kx\|^2 = \|S^kT^kx\|^2 = \|(TS)^kx\|^2.\end{aligned}$$

Therefore, $TS$ is a $2-k^*$ paranormal operator. ∎

**Lemma 3.6.** *Let $T \in L(H)$ be a $2-k$ paranormal operator, which commutes with an isometry $S$. Then $TS$ is a $2-k$ paranormal operator.*

*Proof.* The proof is similar to the Theorem 3.5. ∎

**Theorem 3.7.** *Let $T$ and $S$ are $2-k^*$ paranormal and double-commuting operators. If*

$$\|T^{*2}x\|\|S^kx\| \leq \|T^{*2}S^kx\|(x \in H, k \geq 2)$$

*then, $TS$ is a $2-k^*$ paranormal operator.*

*Proof.* Assume that

$$\|T^{*2}x\|\|S^kx\| \leq \|T^{*2}S^kx\|(x \in H, k \geq 2).$$

Since $T$ and $S$ are double-commuting and $2-k^*$ paranormal operators, then for every $x \in H$ and $k \geq 2$, we have

$$\begin{aligned}
&\|(TS)^{*2}x\|^k\|T^{*2}x\|^{k-2}\|S^kx\|^{k-2}\\
=&\|(S)^{*2}(T)^{*2}x\|^k\|T^{*2}x\|^{k-2}\|S^kx\|^{k-2}\\
=&\|S^{*2}T^{*2}x\|^k\|T^{*2}S^kx\|^{k-2}\\
\leq&\|S^kT^{*2}x\|^2\|T^{*2}x\|^{k-2}\|T^{*2}S^kx\|^{k-2}\\
=&\|T^{*2}S^kx\|^k\|T^{*2}\|^{k-2}\\
\leq&\|T^kS^kx\|^2\|T^{*2}x\|^{k-2}\|S^kx\|^{k-2}\\
=&\|(TS)^kx\|^2\|T^{*2}x\|^{k-2}\|S^kx\|^{k-2}.
\end{aligned}$$

Hence

$$\|(TS)^{*2}x\|^k \leq \|(TS)^kx\|^2(x \in H, k \geq 2).$$

This means that $TS$ is a $2-k^*$ paranormal operator. ∎

**Lemma 3.8.** *Let $T$ and $S$ are $2-k$ paranormal and double-commuting operators. If*

$$\|T^2x\|\|S^kx\| \leq \|T^2S^kx\|(x \in H, k \geq 2)$$

*then, $TS$ is a $2-k$ paranormal operator.*

**Theorem 3.9.** *If $T$ is $2-k^*$ paranormal operator, then $r(T) \leq \|T^k\|^{\frac{1}{k}}$.*

*Proof.* Let $T$ be a $2-k^*$ paranormal operator. Then

$$\|T^{*2}x\|^k \leq \|T^kx\|^2(x \in H, \|x\| = 1).$$

From the last inequality, we have

$$\|T^2\|^k = \|T^{*2}\|^k \leq \|T^k\|^2.$$

Hence

$$\|T^{2n}\|^k \leq \|\underbrace{T^2T^2\cdots T^2}_{n-times}\|^k \leq \|T^2\|^{nk} \leq \|T^k\|^{2n},$$

respectively

$$\|T^{2n}\|^{\frac{1}{2n}} \leq \|T^k\|^{\frac{1}{k}}.$$

Acting with limit on both sides when $n \to \infty$ we have $r(T) \leq \|T^k\|^{\frac{1}{k}}$. ∎

**Theorem 3.10.** *If $T$ is a normal operator, then for $k \geq 2, T$ is a $2-k^*$ paranormal operator.*

*Proof.* The fact that the operator $T$ is normal and according to Theorem 1.1, for any $x \in H$ have

$$\begin{aligned}\|T^k x\|^2 &= (T^k x|T^k x) = (T^{*k}T^k x|x) = ((T^*T)^k x|x)\\ &\geq ((T^*T)^2 x|x)^{\frac{k}{2}} = (T^2T^{*2}x|x)^{\frac{k}{2}} = \|T^{*2}\|^k (k \geq 2).\end{aligned}$$

∎

From above inequality it follows that the class of normal operators is contained in the class of $2-k^*$ paranormal operators $(k \geq 2)$.

**Lemma 3.11.** *If $T$ is a normal operator, then for $k \geq 2, T$ is a $2-k$ paranormal operator.*

By Lemma 3.11 follows that the class of normal operators is contained in the class of $2-k$ paranormal operators $(k \geq 2)$.

**Lemma 3.12.** *If $T$ is a $2-k^*$ paranormal operator, then*

$$T^* \in cllas(M,2) \Rightarrow T \in cllas(M,k)^* (k \leq 2).$$

*holds true.*

*Proof.* Since $T^* \in cllas(M,2)$, then

$$T^2T^{*2} \geq (TT^*)^2 \tag{3.1}$$

and since $T$ is a $2-k^*$ paranormal operator, we have

$$\|T^k x\|^2 \geq \|T^{*2}x\|^k (x \in H, \|x\| = 1)$$

or

$$\|T^k x\|^4 \geq \|T^{*2}x\|^{2k} (x \in H, \|x\| = 1). \tag{3.2}$$

From (3.1)(3.2) and by Theorem 1.1 for $k \leq 2$, we have

$$(T^{*k}T^k x|x)^2 \geq (T^2T^{*2}x|x)^k \geq \quad (((TT^*)^2 x|x)^{\frac{k}{2}})^2 \geq ((TT^*)^k x|x)^2 (x \in H, \|x\| = 1)$$

hence

$$(TT^*)^k \leq T^{*k}T^k (k \leq 2)$$

and finally $T \in cllas(M,k)^* (k \leq 2)$. ∎

**Lemma 3.13.** *If $T$ is a $2-k$ paranormal and hyponormal operator, then*

$$T \in cllas(M,k)^* (k \leq 2).$$

*Proof.* Since $T$ is hyponormal operator, we have

$$\|T^* x\| \leq \|Tx\| (x \in H),$$

or

$$\|TT^* x\| \leq \|T^2 x\| (x \in H).$$

Since $T$ is a $2-k$ paranormal operator, we have

$$\|TT^*x\|^{2k} \leq \|T^2 x\|^{2k} \leq \|T^k x\|^4 (x \in H, \|x\| = 1).$$

consequently,

$$(((T^*T)^2 x|x)^{\frac{k}{2}})^2 \leq (T^{*k}T^k x|x)^2 (x \in H, \|x\| = 1). \tag{3.3}$$

By Theorem 1.1 for $k \leq 2$, we have

$$((TT^*)^k x|x)^2 \leq (((T^*T)^2 x|x)^{\frac{k}{2}})^2 (x \in H, \|x\| = 1). \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$(TT^*)^k \leq T^{*k}T^k.$$

This means that $T \in cllas(M,k)^*(k \leq 2)$. ∎

**Lemma 3.14.** *If $T$ is a $2-k^*$ paranormal operator and $T^*$ is a paranormal operator, then $T$ is a hyponormal operator.*

*Proof.* Since $T$ is $2-k^*$ paranormal operator, then

$$\|T^{*2}x\|^{2k} \leq \|T^k x\|^4 (x \in H, \|x\| = 1)$$

consequently,

$$(T^2T^{*2}x|x)^k \leq (T^{*k}T^k x|x)^2 (x \in H, \|x\| = 1). \tag{3.5}$$

On the other hand, since $T^*$ is a paranormal operator, we have

$$\|T^{*2}x\| \geq \|T^*x\|^2 (x \in H, \|x\| = 1)$$

or

$$(T^2T^{*2}x|x)^k \geq (TT^*x|x)^{2k} (x \in H, \|x\| = 1) \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$(TT^*x|x)^{2k} \leq (T^2T^{*2}x|x)^k \leq (T^{*k}T^k x|x)^2 (x \in H, \|x\| = 1) \tag{3.7}$$

By Theorem 1.1 we have

$$((T^*T)^k x|x)^2 \leq (TT^*x|x)^{2k} (k \leq 1) \tag{3.8}$$

Finally, from (3.7) and (3.8) it follows that

$$(TT^*)^k \leq T^{*k}T^k (k \leq 1).$$

For $k = 1$ we have

$$TT^* \leq T^*T$$

Therefore $T$ is a hyponormal operator. ∎

**Lemma 3.15.** *An operator $T$ is $2-k^*$ paranormal, if and only if,*

$$z^2 + 2z(T^2T^{*2}x|x)^{\frac{k}{4}} + (T^{*k}T^k x|x) \geq 0 (z \in \mathbb{R}, x \in H, \|x\| = 1).$$

*Proof.* Using the definition of $2-k^*$ paranormal operator, we have

$$\|T^{*2}x\|^k \leq \|T^k x\|^2 (x \in H, \|x\| = 1),$$

or

$$4\|T^{*2}x\|^k - 4\|T^k x\|^2 \leq 0 (x \in H, \|x\| = 1).$$

By the above relation we obtain

$$z^2 + 2z\|T^{*2}x\|^{\frac{k}{2}} + \|T^k x\|^2 \geq 0 (z \in \mathbb{R}, x \in H, \|x\| = 1)$$

Expressing the norm through the inner product we obtain the required inequality

$$z^2 + 2z(T^2T^{*2}x|x)^{\frac{k}{4}} + (T^{*k}T^k x|x) \geq 0 (z \in \mathbb{R}, x \in H, \|x\| = 1). \quad ∎$$

**Theorem 3.16.** *If $T$ is a $2-k^*$ paranormal operator and if $Tx=\alpha x$, then $T^{*2}=\overline{\alpha}^2 x(k\neq 2, x\in H, \|x\|=1, \alpha\in\mathbb{C})$.*

*Proof.* If operator $T$ is $2-k^*$ paranormal, then by Lemma 3.15 we have

$$z^2+2z(T^2T^{*2}x|x)^{\frac{k}{4}}+(T^{*k}T^kx|x)\geq 0(z\in\mathbb{R}, x\in H, \|x\|=1).$$

For $k=4n\neq 2$, we have

$$z^2+2z(T^2T^{*2}x|x)^n+(T^{*4n}T^{4n}x|x)\geq 0(z\in\mathbb{R}, x\in H, \|x\|=1),$$

respectively

$$z^2+2z(T^2T^{*2}x|x)^n+(T^{4n}x|T^{4n}x)\geq 0(z\in\mathbb{R}, x\in H, \|x\|=1).$$

From $Tx=\alpha x(\alpha\in\mathbb{C})$, we have

$$z^2+2z(T^2T^{*2}x|x)^n+(\alpha^{4n}x|\alpha^{4n}x)\geq 0(z\in\mathbb{R}, x\in H, \|x\|=1)$$

$$\begin{aligned}&\Rightarrow (T^2T^{*2}x|x)^{2n}\leq|\alpha^{4n}|^2(x\in H, \|x\|=1)\\&\Rightarrow (T^{*2}x|T^{*2}x)^{2n}\leq|\alpha|^{8n}(x\in H, \|x\|=1)\\&\Rightarrow \|T^{*2}x\|^{4n}\leq|\alpha|^{8n}(x\in H, \|x\|=1)\\&\Rightarrow \|T^{*2}x\|^{2}\leq|\alpha|^{4}(x\in H, \|x\|=1)\end{aligned}$$

From the last inequality we have

$$\|T^{*2}x-\overline{\alpha}^2x\|^2=(T^{*2}x-\overline{\alpha}^2x|T^{*2}x-\overline{\alpha}^2x)=\|T^{*2}x\|^2-|\alpha|^4\leq 0(x\in H, \|x\|=1)$$

This means that

$$\|T^{*2}x-\overline{\alpha}^2x\|\leq 0(x\in H, \|x\|=1).$$

Hence

$$T^{*2}x-\overline{\alpha}^2x=0(x\in H, \|x\|=1)$$

or

$$T^{*2}x=\overline{\alpha}^2x(x\in H, \|x\|=1, \alpha\in\mathbb{C}).$$

By which we have proved the Theorem. ∎

**Lemma 3.17.** *Let $T$ be a two sides weighted shift operator with weighted sequence $(\alpha_n)$. Then the operator $T$ is $2-k^*$ paranormal, if and only if*

$$|\alpha_{n-1}|^k|\alpha_{n-2}|^k\leq|\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+k-1}|^2.$$

**Exercise 1.** *Let $T$ be a two sides weighted shift operator with weights defined as follows:*

$$\alpha_n=\begin{cases}1/3, & n\leq -1\\ 1, & n=0\\ 1/3, & n=1\\ 3, & n=2\\ 1/9, & n=3\\ 9, & n\geq 4.\end{cases}$$

After some computations, we conclude that this operator is $2-k^*$ paranormal for $k\geq 2$ but not for $k=3$. This means that this operator is not $2-3^*$ paranormal. The above exercise shows that there exists $2-k^*$ paranormal operator that is not $2-(k+1)^*$ paranormal.

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