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A NOTE ON CALDERÓN OPERATOR

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ABSTRACT. We have shown that the Calderón operator is bounded on Morrey Spaces on R^+. Also under certain conditions on the weight, the Hardy operator, the adjoint Hardy operator, and therefore the Calderón operator are bounded on the weighted Morrey spaces.

Key words and phrases: Adjoint Hardy operator, Calderón Operator, Hardy operator, Morrey space, Weighted Morrey space.

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1. INTRODUCTION AND PRELIMINARIES

If f is a measurable function defined on R^+, the Calderón operator S is defined by

$$\begin{aligned} S(f)(x) &= \int_0^\infty \min\left\{\frac{1}{x}, \frac{1}{t}\right\} f(t)dt \\ &= \frac{1}{x}\int_0^x f(t)dt + \int_x^\infty \frac{f(t)}{t}dt \\ &= H(f)(x) + H^*(f)(x). \end{aligned}$$

Here H is the classical Hardy operator and H^* is its adjoint operator. In [9], We have proved that the Hardy operator H is bounded on Morrey spaces, $L^{p,\lambda}(R^+)$. In this paper, by simple calculations, we have shown that H^* is also bounded on Morrey spaces, $L^{p,\lambda}(R^+)$. So the Calderón operator, S, as the sum of H and H^*, is bounded on Morrey spaces, $L^{p,\lambda}(R^+)$ as well.

In addition, we have considered the Hardy operator on weighted Morrey spaces and we have obtained that if the weight function w is nondecreasing, then with the condition M_p $(p > 1)$, the operator H is bounded on $L^{p,\lambda}(w)$ and the operator H^* is bounded on $L^{p,\lambda}(w)$, and hence the Calderón Operator S is bounded on $L^{p,\lambda}(w)$.

Let R^+ denote the set of all positive real numbers. For $p \in (0, \infty)$ and $0 < \lambda < 1$, Morrey space on R^+, $L^{p,\lambda}(R^+)$ consists of all measurable functions $f \in L^p_{loc}(R^+)$ with

$$\|f\|_{L^{p,\lambda}(R^+)} = \left(\sup_{I\subset R^+} \frac{1}{|I|^\lambda}\int_I |f(x)|^p dx\right)^{1/p} < \infty$$

where $I = (a, b] \subset R^+,\ \ 0 < a < b < +\infty$, is a bounded interval on R^+ and $|I|$ denotes the length of I.

Morrey space can be a part of a family that include L^p, BMO (the space of Bounded Mean Oscillation), and Hölder function spaces. It is well known now that if $1 \le p < \infty$, then $L^{p,0} = L^p$ and $L^{p,1} = L^\infty$. If $\lambda < 0$, $L^{p,\lambda} = \{0\}$ and if $\lambda > 1$, $L^{p,\lambda}$ is the space of $\frac{(\lambda-1)}{p}$-Hölder continuous functions. Therefore here is this paper, the Morrey space is defined to be $L^{p,\lambda}$ with $0 < \lambda < 1$.

Let w be a weight on $(0,\ \infty)$, i.e. w is a measurable function, $w > 0$ a.e. with respect to the Lebesgue measure. Then for $0 < \lambda < 1$ and $0 < p < \infty$, weighted Morrey space, $L^{p,\lambda}(w)$, contains all functions $f \in L^p_{loc}(w)$ such that

$$\|f\|_{L^{p,\lambda}(w)} = \left(\sup_{I\subset R^+} \frac{1}{|I|^\lambda}\int_I |f(x)|^p w(x)dx\right)^{1/p} < \infty$$

where $I = (a, b] \subset R^+,\ \ 0 < a < b < +\infty$, is a bounded interval on R^+ and $|I|$ denotes the length of I.

On R^+, the Muckenhoupt's condition M_p $(p > 1)$ is as follows.
There exists $C > 0$ such that for a.e. $x > 0$,

$$\left(\int_x^\infty \frac{w(t)}{t^p}dt\right)^{1/p} \left(\int_0^x w(t)^{-p'/p}dt\right)^{1/p'} \le C$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

From Theorem 2.2 in [9] we have the following which is used in the proof of Theorem 2.2 in this paper.

Lemma 1.1. *The Hardy operator H is bounded on $L^{p,\lambda}(R^+)$, that is,*

$$\|H(f)\|_{L^{p,\lambda}(R^+)} \le C_{p,\lambda}\|f\|_{L^{p,\lambda}(R^+)}$$

where $C_{p,\lambda} = \frac{p}{p+\lambda-1}$.

Throughout the whole note, C denotes a positive constant depending on p and λ only and C might be different at each occurance.

2. MAIN RESULTS

With the introduction and preliminaries, in this section we present all main results and their proofs and also give some useful remarks.

Theorem 2.1. *For* $1 \le p < \infty$, *the adjoint Hardy operator* H^* *is bounded on* $L^{p,\lambda}(R^+)$, *that is,*

$$\|H^*(f)\|_{L^{p,\lambda}(R^+)} \le C_{p,\lambda}\|f\|_{L^{p,\lambda}(R^+)}$$

where $C_{p,\lambda} = \frac{p}{1-\lambda}$.

Proof. For $x \in R^+$, let $t = \frac{x}{s}$, we rewrite the Hardy adjoint operator

$$H^*(f)(x) = \int_x^\infty \frac{f(t)}{t}dt = \int_0^1 f\left(\frac{x}{s}\right)\frac{ds}{s}.$$

Here $\int_0^1$ can be understood as $\lim_{\delta\to 0}\int_\delta^1$.

For any $0 < \lambda < 1$, $I = (a,b] \subset R^+$ $(0 < a < b < \infty)$, and $f \in L^{p,\lambda}(R^+)$, by Minkowski's inequality for integral and changing of variables, we have

$$\begin{aligned}
\left(\frac{1}{|I|^\lambda}\int_I |H^*(f)(x)|^p dx\right)^{1/p} &= \left(\frac{1}{(b-a)^\lambda}\int_a^b \left|\int_0^1 f\left(\frac{x}{s}\right)\frac{ds}{s}\right|^p dx\right)^{1/p} \\
&\le \left(\frac{1}{(b-a)^\lambda}\right)^{1/p}\left(\int_a^b \left|\int_0^1 f\left(\frac{x}{s}\right)\frac{ds}{s}\right|^p dx\right)^{1/p} \\
&\le \left(\frac{1}{(b-a)^\lambda}\right)^{1/p}\left(\int_0^1 \left(\int_a^b \left|f\left(\frac{x}{s}\right)\right|^p dx\right)^{1/p}\frac{ds}{s}\right) \\
&= \int_0^1 \left(\frac{1}{\left(\frac{b}{s}-\frac{a}{s}\right)^\lambda}\int_{a/s}^{b/s} |f(x)^p dx\right)^{1/p}\frac{ds}{s^{1-\frac{1-\lambda}{p}}} \\
&\le \|f\|_{L^{p,\lambda}(R^+)}\int_0^1 \frac{ds}{s^{1-\frac{1-\lambda}{p}}} \\
&= \frac{p}{1-\lambda}\|f\|_{L^{p,\lambda}(R^+)} = C_{p,\lambda}\|f\|_{L^{p,\lambda}(R^+)}.
\end{aligned}$$

where $C_{p,\lambda} = \frac{p}{1-\lambda}$. Therefore the desired result follows immediately by the definition of $L^{p,\lambda}(R^+)$.

∎

The result that follows is a combination of Theorem 2.1 and Lemma 1.1.

Theorem 2.2. *Let* $1 \le p < \infty$. *Then for the Calderón Operator* S *we have*

$$\|S(f)\|_{L^{p,\lambda}(R^+)} \le C_{p,\lambda}\|f\|_{L^{p,\lambda}(R^+)}$$

for any $f \in L^{p,\lambda}(R^+)$, *where* $C_{p,\lambda} = \frac{p^2}{(1-\lambda)(p+\lambda-1)}$, *i.e.* S *is bounded on* $L^{p,\lambda}(R^+)$.

Now we are going to work with Hardy operator H and its adjoint H^* on the weighted Morrey space $L^{p,\lambda}(w)$.

Theorem 2.3. *For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, if w is nondecreasing and satisfies M_p condition, then the Hardy operator H is bounded on $L^{p,\lambda}(w)$.*

Proof. For any $f \in L^{p,\lambda}(w)$, by Hölder inequality, we get

$$\begin{aligned} |H(f)(x)| &= \left| \frac{1}{x} \int_0^x f(t)dt \right| \\ &\leq \frac{1}{x} \int_0^x |f(t)| w(t)^{1/p} w(t)^{-1/p} dt \\ &\leq \frac{1}{x} \left(\int_0^x |f(t)|^p w(t) dt \right)^{1/p} \left(\int_0^x w(t)^{-p'/p} dt \right)^{1/p'} \\ &= x^{\frac{\lambda}{p}-1} \left(\frac{1}{x^\lambda} \int_0^x |f(t)|^p w(t) dt \right)^{1/p} \left(\int_0^x w(t)^{-p'/p} dt \right)^{1/p'} \\ &\leq C x^{\frac{\lambda}{p}-1} \|f\|_{L^{p,\lambda}(w)} \left(\int_0^x w(t)^{-p'/p} dt \right)^{1/p'}. \end{aligned}$$

With the assumption we made in this theorem about w, we claim that

$$C^{-1} \frac{w(x)}{x^{p-1}} \leq \left(\int_0^x w(t)^{-p'/p} dt \right)^{-p/p'} \leq C \frac{w(x)}{x^{p-1}}.$$

In fact, since w is nondecreasing and satisfies the condition M_p,

$$\begin{aligned} \frac{1}{p-1} \frac{w(x)}{x^{p-1}} &\leq \int_x^\infty \frac{w(t)}{t^p} dt \\ &\leq C \left(\int_0^x w(t)^{-p'/p} dt \right)^{-p/p'} \\ &\leq C \frac{1}{x^{p-1}} \left(\frac{1}{x} \int_0^x w(t)^{-p'/p} dt \right)^{-p/p'} \qquad \left(\text{Note } \frac{p}{p'} = p-1\right) \\ &\leq \frac{C}{x^{p-1}} \left(\frac{1}{x} \int_0^x w(t) dt \right) \qquad \text{Jensen's inequality} \\ &\leq C \frac{w(x)}{x^{p-1}}. \end{aligned}$$

It is sufficient to show that for any bounded interval $I = (a, b],\ 0 < a < b < \infty$,

$$\left(\frac{1}{|I|^\lambda} \int_I |H(f)(x)|^p w(x) dx \right)^{1/p} \leq C \|f\|_{L^{p,\lambda}(w)}$$

Let's start with the following.

$$\begin{aligned}
& \frac{1}{|I|^\lambda}\int_I |H(f)(x)|^p w(x)dx \\
\leq\ & \frac{C}{|I|^\lambda}\|f\|^p_{L^{p,\lambda}(w)}\int_I x^{\lambda-p}\left(\int_0^x w(t)^{-p'/p}dt\right)^{-p/p'} w(x)dx \\
\leq\ & \frac{C}{|I|^\lambda}\|f\|^p_{L^{p,\lambda}(w)}\int_I x^{\lambda-p}\left(\frac{w(x)}{x^{p-1}}\right)^{-1} w(x)dx \\
\leq\ & \frac{C}{|I|^\lambda}\|f\|^p_{L^{p,\lambda}(w)}\int_I x^{\lambda-1}dx \\
=\ & C\|f\|^p_{L^{p,\lambda}(w)}\frac{1}{(b-a)^\lambda}\int_a^b x^{\lambda-1}dx \\
=\ & \frac{C}{\lambda}\|f\|^p_{L^{p,\lambda}(w)}\frac{b^\lambda-a^\lambda}{(b-a)^\lambda} \leq C\|f\|^p_{L^{p,\lambda}(w)}.
\end{aligned}$$

Here we recall that for $a>0$, $b>0$, and $0<\lambda<1$, we have $b^\lambda - a^\lambda \leq (b-a)^\lambda$, that is, $\frac{b^\lambda-a^\lambda}{(b-a)^\lambda} \leq 1$.

Therefore

$$\|H(f)\|_{L^{p,\lambda}(w)} \leq C\|f\|_{L^{p,\lambda}(w)}.$$

∎

In the discussion that follows we'll have the next theorem which is about the adjoint Hardy operator H^* on the weighted Morrey spaces.

Theorem 2.4. *For $1<p<\infty$, if w is nondecreasing, then the Hardy operator H^* is bounded on $L^{p,\lambda}(w)$, that is, for any $f \in L^{p,\lambda}(w)$*

$$\|H^*(f)\|_{L^{p,\lambda}(w)} \leq C\|f\|_{L^{p,\lambda}(w)}.$$

Proof. For any bounded interval $I=(a,b],\ 0<a<b<\infty$, consider

$$\begin{aligned}
& \left[\frac{1}{|I|^\lambda}\int_I |H^*(f)(x)|^p w(x)dx\right]^{1/p} \\
=\ & \left[\frac{1}{|I|^\lambda}\int_I \left|\int_0^1 f\left(\frac{x}{s}\right)\frac{ds}{s}\right|^p w(x)dx\right]^{1/p} \\
\leq\ & \frac{1}{|I|^{\lambda/p}}\int_0^1 \left[\int_I \left|f\left(\frac{x}{s}\right)\right|^p w(x)dx\right]^{1/p}\frac{ds}{s} \\
=\ & \frac{1}{|I|^{\lambda/p}}\int_0^1 \left[\int_{\frac{1}{s}I} |f(t)|^p w(st)dt\right]^{1/p}\frac{ds}{s} \qquad \left(\frac{x}{s}=t\right) \\
\leq\ & \int_0^1 \left[\frac{1}{|I|^{\lambda/p}}\int_{\frac{1}{s}I} |f(t)|^p w(t)dt\right]^{1/p}\frac{ds}{s^{1-\frac{1}{p}}} \qquad (w(st)\leq w(t))
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left[\frac{1}{\left|\frac{1}{s}I\right|^\lambda} \int_{\frac{1}{s}I} |f(t)|^p w(t)dt \right]^{1/p} \frac{ds}{s^{1-\frac{1}{p}+\frac{\lambda}{p}}} \\
&\leq \|f\|_{L^{p,\lambda}(w)} \int_0^1 \frac{ds}{s^{1-\frac{1-\lambda}{p}}} \\
&= \frac{p}{1-\lambda} \|f\|_{L^{p,\lambda}(w)} = C\|f\|_{L^{p,\lambda}(w)},
\end{aligned}
$$

where $C = \frac{p}{1-\lambda}$. By the definition of Morrey spaces, we have completed the proof.
∎

With Theorem 2.3 and 2.4 we know that S is bounded on the weighted Morrey space.

Theorem 2.5. *For $1 < p < \infty$, if w is nondecreasing, then the Hardy operator S is bounded on $L^{p,\lambda}(w)$, that is, for any $f \in L^{p,\lambda}(w)$*

$$
\|S(f)\|_{L^{p,\lambda}(w)} \leq C\|f\|_{L^{p,\lambda}(w)},
$$

where C is dependent on p and λ only.

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