The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 14, Issue 2, Article 5, pp. 1-14, 2017

EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTI-POINT BOUNDARY CONDITIONS

N. ADJEROUD

Received 4 January, 2017; accepted 6 July, 2017; published 8 August, 2017.

KHENCHELA UNIVERSITY, DEPARTMENT OF MATHEMATICS, KHENCHELA, 40000, ALGERIA.,
adjnac@gmail.com

ABSTRACT. This paper is devoted to the existence results of positive solutions for a nonlinear fractional differential equations with multi-point boundary conditions. By means of the Schauder fixed point theorem, some results on the existence are obtained.

Key words and phrases: Fractional differential equations; Caputo's fractional derivative; Fixed point theorem; Multi-point.

2000 *Mathematics Subject Classification.* Primary 34A08, 34B10, 34B15.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Analysis of fractional differential equations have gained considerable importance and continuous attention during the past three decades, due mainly to their varied applications in many fields [3, 5, 8, 10]. The differential equations involving Riemann-Liouville and Caputo operators of fractional orders arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, polymer rheology, economic, control theory, signal and image processing, biophysics, etc. [4, 6, 7, 9]. Boundary-value problems for fractional differential equations have been discussed in [1, 2, 12, 13]. Zhang [14] investigated the existence and multiplicity of positive solutions for the problem.

$$\begin{cases} {}^{c}D^{\alpha}u\left(t\right)=f\left(t,u\left(t\right)\right), & 0\leq t\leq 1, 0<\alpha\leq 2,\\ u\left(0\right)+u'\left(0\right)=0, u\left(1\right)+u'\left(1\right)=0. \end{cases}$$

X .Su, S. Zhang [11] have investigated the existence, uniqueness and continuous dependence of solutions for the problem

$$\begin{cases} {}^{c}D^{\alpha}_{0+}u\left(t\right)=f\left(t,u\left(t\right),{}^{c}D^{\beta}_{0+}u\left(t\right)\right), & 0\leq t\leq 1\\ a_1u\left(0\right)-a_2u'\left(0\right)=A,\ b_1u\left(1\right)+b_2u'\left(1\right)=B, \end{cases}$$

where $0<\beta\leq 1, 1<\alpha\leq 2$, $a_i, b_i\geq 0, i=1,2; a_2b_1+a_1b_2>0, f:[0,1]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous.

C. Cheng, Z. Feng, Y. Su [2] have investigated the existence and uniqueness for the problem

$$D^{\alpha}_{0+}u\left(t\right)+f\left(t,u\left(t\right),u'\left(t\right)\right)=0,\ \ t\in\left(0,1\right), n-1<\alpha\leq n,$$

$$u^{(i)}\left(0\right)=0,\ \ i=0,1,2,...,n-2,$$

$$\left[D^{\beta}_{0+}u\left(t\right)\right]_{t=1}=0,\quad 2\leq\beta\leq n-2$$

where $n>4\left(n\in\mathbb{N}\right)$, and $f:[0,1]\times[0,\infty)\times(-\infty,\infty)\longrightarrow[0,\infty)$ satisfies the Carathéodory type conditions.

Motivated by the previous results, in this paper,we study the existence of positive solutions for the fractional differential equation with $m-$point boundary conditions

$$\begin{cases} {}^{c}D^{\alpha}_{0+}u\left(t\right)+f\left(t,u\left(t\right),{}^{c}D^{\beta}_{0+}u\left(t\right)\right)=0, & 0\leq t\leq 1,\\ u\left(0\right)=u\left(1\right)=\sum_{i=1}^{m-2}a_iu\left(\eta_i\right), \end{cases} \tag{1.1}$$

where ${}^{c}D^{\alpha}_{0+}$ is the standard Caputo derivative, and $1<\alpha\leq 2, 0<\beta<1; \alpha,\beta$ a real numbers, $a_i\geq 0$ for $i=1,2,...,m-2$ and ; $\eta_i: 0<\eta_1<\eta_2<...<\eta_{m-2}<1$ with $\sum_{i=1}^{m-2}a_i<1$, $f:[0,1]\times[0,\infty)\times\mathbb{R}\longrightarrow[0,\infty)$ is continuous and $f\left(t,0,0\right)>0.$

The organisation of this paper is as follows. In Section 2, we introduce some notations and definitions of fractional calculus [5, 8] and present preliminary results needed in our proof later. In section 3 we discuss the existence of solutions for the problem (1.1) .

2. PRELIMINARIES

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, Γ is the gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt$.

Definition 2.2. For an at least n-times continuously differentiable function $f : (0, +\infty) \longrightarrow \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$$^{C}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds,\ n-1 < \alpha < n,\ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Remark 2.1. Under natural condition on the function $f(x)$ if $\alpha = n$, the Caputo's derivative of order α is the usual derivative of order n. We have the following properties:

$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t)$, $^{c}D^{\alpha} I^{\alpha} f(t) = f(t)$, for $\alpha, \beta > 0$, $f \in L^1(0,1)$ and $I^{\alpha} : C[0,1] \to C[0,1]$, $\alpha > 0$.

Remark 2.2. For $\alpha = n$, the Caputo's fractional derivative of order α becomes the conventional $n-$th derivative. The Caputo's fractional derivative is defined in [1] as follows:
$^{C}D_{0^+}^{\alpha} f(t) = D_{0^+}^{\alpha} \left(f(t) - \sum_{p=0}^{n-1} \frac{f^{(p)}(0^+)}{p!} t^p \right)$, provided that the right-side derivative exists, we have in particular $^{C}D_{0^+}^{\alpha} k = 0$ for any constant $k \in \mathbb{R}, \alpha > 0$. We have the following properties: $^{c}D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t)$ if $\alpha > 0, f(t) \in C[0,1]$ and $I_{0^+}^{\alpha}\ {}^{C}D_{0^+}^{\alpha} f(t) = f(t) - f(0), 0 \leq \alpha < 1, f(t) \in C[0,1]$.

Lemma 2.1. *The general solution of the equation*

$$^{c}D^{\alpha} u(t) = 0,$$

is given by

$$u(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n-1$ *and* $n = [\alpha] + 1$.

Lemma 2.2. *Suppose that* $u \in C[0,1] \bigcap L^1[0,1]$ *with a derivative of order* n *that belong to* $C[0,1] \bigcap L^1[0,1]$. *Then*

$$I^{\alpha}\ {}^{c}D^{\alpha} u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n-1$ *and* $n = [\alpha] + 1$.

Lemma 2.3. *Let* $h \in C([0,1], \mathbb{R})$ *be a function, then the problem*

$$\begin{cases} ^{c}D_{0^+}^{\alpha} u(t) + h(t) = 0, \quad 0 \leq t \leq 1, \\ u(0) = u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \end{cases} \tag{2.1}$$

has a unique solution given by

$$u(t) = \int_0^1 G(t,s) h(s)\, ds + \frac{\sum_{i=1}^{m-2} a_i \int_0^1 G(\eta_i, s) h(s)\, ds}{1 - \sum_{i=1}^{m-2} a_i}, \tag{2.2}$$

where

$$G(x,y)=\begin{cases}\frac{x(1-y)^{\alpha-1}-(x-y)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le y\le x\le 1,\\ \frac{x(1-y)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le x\le y\le 1.\end{cases}$$

Proof. From Lemma 2.2 we have

$$I_{0^+}^{\alpha}h(t)=u(t)+c_0+c_1 t,$$

then

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds+c_0+c_1 t. \tag{2.3}$$

By the boundary conditions, we obtain

$$c_0=\frac{\sum_{i=1}^{m-2}a_i\eta_i I^{\alpha}h(1)-\sum_{i=1}^{m-2}a_i I^{\alpha}h(\eta_i)}{1-\sum_{i=1}^{m-2}a_i},$$

and

$$c_1=\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds.$$

Substituting c_0 and c_1 into (2.3) we obtain solution (2.2)

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds+\frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds\\
&\quad+\frac{\sum_{i=1}^{m-2}a_i\eta_i\int_0^1\frac{(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2}a_i}-\frac{\sum_{i=1}^{m-2}a_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2}a_i}\\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds+\int_0^t\frac{t(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds\\
&\quad+\int_t^1\frac{t(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\eta_i}\frac{\eta_i(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2}a_i}\\
&\quad+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\eta_i}\frac{-(\eta_i-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2}a_i}+\frac{\sum_{i=1}^{m-2}a_i\int_{\eta_i}^1\frac{\eta_i(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2}a_i}\\
&= \int_0^t\frac{\left(-(t-s)^{\alpha-1}+t(1-s)^{\alpha-1}\right)h(s)}{\Gamma(\alpha)}ds\\
&\quad+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\eta_i}\frac{\left(-(\eta_i-s)^{\alpha-1}+\eta_i(1-s)^{\alpha-1}\right)h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2}a_i}\\
&\quad+\frac{\sum_{i=1}^{m-2}a_i\int_{\eta_i}^1\frac{\eta_i(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2}a_i}+\int_t^1\frac{t(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds\\
&= \int_0^1 G(t,s)h(s)\,ds+\frac{\sum_{i=1}^{m-2}a_i\int_0^1 G(\eta_i,s)h(s)\,ds}{1-\sum_{i=1}^{m-2}a_i}.
\end{aligned}$$

∎

Lemma 2.4. *Let $a_i\ge 0$ for $i=1,2,...,m-2$ and assume that $\sum_{i=1}^{m-2}a_i<1$. If $h\in C[0,1]$ and $h(t)\ge 0$, then the unique solution of the problem (2.1) satisfies*

$$u(t)\ge 0.$$

Proof. From the boundary conditions, we have

$$\begin{aligned} u(1) &= u(0) = \frac{\sum_{i=1}^{m-2} a_i\eta_i \int_0^1 \frac{(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2} a_i} - \frac{\sum_{i=1}^{m-2} a_i \int_O^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2} a_i} \\ &= \frac{\sum_{i=1}^{m-2} a_i\eta_i \int_{\eta_i}^1 \frac{(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2} a_i} + \frac{\sum_{i=1}^{m-2} a_i\eta_i \int_0^{\eta_i} \frac{(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2} a_i} \\ &\quad - \frac{\sum_{i=1}^{m-2} a_i \int_O^{\eta_i} \frac{(\eta_i-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2} a_i} \\ &= \frac{\sum_{i=1}^{m-2} a_i\eta_i \int_{\eta_i}^1 \frac{(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2} a_i} + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\eta_i} \frac{\eta_i(1-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds}{1-\sum_{i=1}^{m-2} a_i} \\ &= \frac{\sum_{i=1}^{m-2} a_i \int_0^{\eta_i} \frac{\left(\eta_i(1-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}\right)}{\Gamma(\alpha)}h(s)\,ds}{1-\sum_{i=1}^{m-2} a_i} + \frac{\sum_{i=1}^{m-2} a_i \int_{\eta_i}^1 \frac{\eta_i(1-s)^{\alpha-1}h(s)}{\Gamma(\alpha)}ds}{1-\sum_{i=1}^{m-2} a_i} \\ &= \frac{\sum_{i=1}^{m-2} a_i \int_0^1 G(\eta_i, s)\,h(s)\,ds}{1-\sum_{i=1}^{m-2} a_i} \geq 0. \end{aligned}$$

This completes the proof. ∎

Lemma 2.5. *Let* $a_i \geq 0$ *for* $i = 1, 2, ..., m-2$ *and assume that* $\sum_{i=1}^{m-2} a_i > 1$. *If* $h \in C[0,1]$ *and* $h(t) \geq 0$, *then the problem* (2.1) *has no positive solutions.*

Proof. We claim the contrary, and suppose that the problem (2.1) has a positive solution u, then we have $u(\eta_i) \geq 0$, $i = 1, 2, ..., m-2$. From (2.2) we have

$$\begin{aligned} u(\eta_i) &= \int_0^1 G(\eta_i, s)\,h(s)\,ds + \frac{\sum_{i=1}^{m-2} a_i \int_0^1 G(\eta_i, s)\,h(s)\,ds}{1-\sum_{i=1}^{m-2} a_i} \\ &= \int_0^1 G(\eta_i, s)\,h(s)\,ds - \frac{\sum_{i=1}^{m-2} a_i \int_0^1 G(\eta_i, s)\,h(s)\,ds}{\sum_{i=1}^{m-2} a_i - 1} > 0, \end{aligned}$$

we get

$$\left(\sum_{i=1}^{m-2} a_i - 1\right) \int_0^1 G(\eta_i, s)\,h(s)\,ds > \sum_{i=1}^{m-2} a_i \int_0^1 G(\eta_i, s)\,h(s)\,ds,$$

and

$$-\int_0^1 G(\eta_i, s)\,h(s)\,ds > 0,$$

then

$$\int_0^1 G(\eta_i, s)\,h(s)\,ds < 0.$$

This contradicts the fact that $\int_0^1 G(\eta_i, s)\,h(s)\,ds > 0$, and the claim is proved. If $\sum_{i=1}^{m-2} a_i > 1$, it is clear that $u(0) = u(1)$ is not positive. ∎

Remark 2.3. The Green function

$$G(x, y) = \begin{cases} \frac{x(1-y)^{\alpha-1}-(x-y)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq y \leq x \leq 1, \\ \frac{x(1-y)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq x \leq y \leq 1 \end{cases}$$

is such that $G(x,y) \geq 0$. In fact $G(x,y) = \frac{x(1-y)^{\alpha-1}}{\Gamma(\alpha)} \geq 0$ for all $(x,y) : 0 \leq x \leq y \leq 1, 1 < \alpha < 2$.

For $(x,y) : 0 \leq y \leq x \leq 1$, we have for $0 \leq y \leq 1$, and $x = 1$ that $G(x,y) = G(x,x) = 0$, for $y = 0, 0 \leq x \leq 1$, we have $G(x,y) = G(x,0) = x - x^{\alpha-1} \geq 0$.

For $0 \leq x \leq 1$, $y = x$, we have $G(x,y) = G(x,x) = x(1-x)^{\alpha-1} \geq 0$. It follows that for all $(x,y) : 0 \leq x \leq y \leq 1$, $G(x,y) \geq 0$.

3. THE MAIN RESULTS

In this section, we study the existence of positive solution for the problem (1.1).

Let $C(I)$ be the Banach space of all continuous real-valued functions on $I = [0,1]$. Define the space

$$E = \left\{ u : u \in C(I) \text{ and } {}^cD_{0^+}^{\beta} u \in C(I), 0 < \beta < 1 \right\},$$

endowed with the norm

$$\|u\| = \max \left\{ \|u\|_{\infty}, \left\| {}^cD_{0^+}^{\beta} u \right\|_{\infty} \right\},$$

where

$$\|u\|_{\infty} = \max_{0 \leq t \leq 1} |u(t)|, \quad \left\| {}^cD_{0^+}^{\beta} u \right\|_{\infty} = \max_{0 \leq t \leq 1} \left| {}^cD_{0^+}^{\beta} u(t) \right|,$$

it is known that $(E, \|.\|)$ is a Banach space. Let the following assumption

(H) $1 < \alpha \leq 2$, $0 < \beta < 1$ with α, β a real numbers, $a_i \geq 0$ for $i = 1, 2, ..., m-2$; $\eta_i : 0 < \eta_1 < \eta_2 < ... < \eta_{m-2} < 1$ and $\sum_{i=1}^{m-2} a_i < 1$. $f : [0,1] \times [0,\infty) \times \mathbb{R} \longrightarrow [0,\infty)$ is continuous, and $f(t,0,0) > 0$ for all $t \in [0,1]$.

Lemma 3.1. *Assume that* (H) *holds. Then the problem* (1.1) *is equivalent to the integral equation*

$$u(t) = \int_0^1 G(t,s) f\left(s, u(s), {}^cD_{0^+}^{\beta} u(s)\right) ds + \frac{\sum_{i=1}^{m-2} a_i \int_0^1 G(\eta_i, s) f\left(s, u(s), {}^cD_{0^+}^{\beta} u(s)\right) ds}{1 - \sum_{i=1}^{m-2} a_i}. \tag{3.1}$$

In other words, every solution of (1.1) *is also a solution of* (3.1) *and vice versa.*

Proof. Let $u \in E$ be a solution of (1.1). We denotes the right-hand side of (3.1) by $z(t)$; i.e;

$$\begin{aligned} z(t) &= -I_{0^+}^{\alpha} f\left(t, u(t), {}^cDu_{0^+}^{\beta}(t)\right) + tI_{0^+}^{\alpha} f\left(1, u(1), {}^cD_{0^+}^{\beta} u(1)\right) \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i \eta_i}{1 - \sum_{i=1}^{m-2} a_i} I_{0^+}^{\alpha} f\left(1, u(1), {}^cD_{0^+}^{\beta} u(1)\right) \\ &\quad - \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} I_{0^+}^{\alpha} f\left(\eta_i, u(\eta_i), {}^cD_{0^+}^{\beta}(\eta_i)\right), \end{aligned}$$

and

$$z'(t) = -I_{0^+}^{\alpha-1} f\left(t, u(t), {}^cD_{0^+}^{\beta} u(t)\right) + I_{0^+}^{\alpha} f\left(1, u(1), {}^cD_{0^+}^{\beta} u(1)\right).$$

From Remarks 2.1 and 2.2, we get

$$\begin{aligned}
{}^{C}D_{0^+}^{\alpha} z(t) &= D_{0^+}^{\alpha}\left(z(t) - z\left(0^+\right) - z'\left(0^+\right)t\right) \\
&= D_{0^+}^{\alpha}\Bigg(z(t) - \frac{\sum_{i=1}^{m-2} a_i\eta_i}{1-\sum_{i=1}^{m-2} a_i} I_{0^+}^{\alpha} f\left(1, u(1), {}^{c}D_{0^+}^{\beta} u(1)\right) \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} I_{0^+}^{\alpha} f\left(\eta_i, u(\eta_i), {}^{c}D_{0^+}^{\beta} u(\eta_i)\right) \\
&\quad - tI_{0^+}^{\alpha} f\left(1, u(1), {}^{c}D_{0^+}^{\beta} u(1)\right)\Bigg) \\
&= D_{0^+}^{\alpha}\left(-I_{0^+}^{\alpha} f\left(t, u(t), {}^{c}D_{0^+}^{\beta} u(t)\right)\right) = -f\left(t, u(t), {}^{c}D_{0^+}^{\beta} u(t)\right),
\end{aligned}$$

then

$$ {}^{C}D_{0^+}^{\alpha} u(t) = -f\left(t, u(t), {}^{c}D_{0^+}^{\beta} u(t)\right). $$

One can verify that $u(0) = u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i)$
$= \frac{\sum_{i=1}^{m-2} a_i\eta_i}{1-\sum_{i=1}^{m-2} a_i} I_{0^+}^{\alpha} f\left(1, u(1), {}^{c}D_{0^+}^{\beta} u(1)\right) - \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i} I_{0^+}^{\alpha} f\left(\eta_i, u(\eta_i), {}^{c}D_{0^+}^{\beta} u(\eta_i)\right).$
Thus, u is a solution of (1.1). Conversely, let $u \in E$ be a solution of (1.1), applying the method used to prove Lemma 2.3, it follows that u is a solution of (3.1), which completes the proof. ∎

From Lemma 2.3, we conclude that the solution of (1.1) coincides with the fixed point of the operator $T : E \longrightarrow E$ defined by

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s) f\left(s, u(s), {}^{c}D_{0^+}^{\beta} u(s)\right) ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^1 G(\eta_i, s) f\left(s, u(s), {}^{c}D_{0^+}^{\beta} u(s)\right) ds}{1-\sum_{i=1}^{m-2} a_i}.
\end{aligned}$$

Note that

$$\begin{aligned}
(Tu)(t) &= -I^{\alpha} f\left(t, u(t), {}^{c}D^{\beta} u(t)\right) \\
&\quad + tI^{\alpha} f\left(1, u(1), {}^{c}D^{\beta} u(1)\right) \\
&\quad - \frac{\sum_{i=1}^{m-2} a_i I^{\alpha} f\left(\eta_i, u(\eta_i), {}^{c}D^{\beta} u(\eta_i)\right) ds}{1-\sum_{i=1}^{m-2} a_i} \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i\eta_i I^{\alpha} f\left(1, u(1), {}^{c}D^{\beta} u(1)\right) ds}{1-\sum_{i=1}^{m-2} a_i},
\end{aligned}$$

$$\begin{aligned}
(Tu)'(t) &= -I^{\alpha-1} f\left(t, u(t), {}^{c}D^{\beta} u(t)\right) \\
&\quad + I^{\alpha} f\left(1, u(1), {}^{c}D^{\beta} u(1)\right),
\end{aligned}$$

and

$$\begin{aligned}
{}^{c}D^{\beta}(Tu)(t) &= I^{1-\beta}(Tu)'(t) \\
&= -I^{\alpha-\beta} f\left(t, u(t), {}^{c}D^{\beta} u(t)\right) \\
&\quad + I^{\alpha-\beta+1} f\left(1, u(1), {}^{c}D^{\beta} u(1)\right).
\end{aligned}$$

Then we see that $(Tu)(t), {}^{c}D^{\beta}(Tu)(t) \in C(I)$.

We give the main result of this section.

Theorem 3.2. *(Schauder's fixed point). Let U be a closed bounded convex subset in a Banach space E, assume that $T : U \longrightarrow U$ is a continuous mapping such that TU is relatively compact subset of U. Then T has a fixed point in U.*

Theorem 3.3. *Assume that (H) holds. If there exist a constant K, such that*

$$\max\left\{\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}, \frac{1}{\Gamma(\alpha)}\left(1+\frac{\sum_{i=1}^{m-2} a_i\eta_i}{1-\sum_{i=1}^{m-2} a_i}\right)\right\} M \leq \frac{K}{2},$$

where $M = \max\{f(t,u,v) : 0 \leq t \leq 1, 0 \leq u(t) \leq K, |v(t)| \leq K\}$ .Then the problem (1.1) has at least one positive solution.

Proof. For any $t \in [0,1]$, we find

$$\begin{aligned}\int_0^1 |G(t,s)|\, ds &\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} ds + \int_0^1 t(1-s)^{\alpha-1} ds\right)\\ &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\alpha}{\alpha}+\frac{t}{\alpha}\right) \leq \frac{2}{\Gamma(\alpha+1)}\\ &\leq \frac{2}{\Gamma(\alpha)},\end{aligned}$$

and

$$\begin{aligned}\int_0^1 |G(\eta_i,s)|\, ds &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{\eta_i^\alpha}{\alpha}+\frac{\eta_i}{\alpha}\right)\\ &\leq \frac{2\eta_i}{\Gamma(\alpha)}.\end{aligned}$$

Define

$$U = \left\{u : u \in E,\ 0 \leq t \leq 1, 0 \leq u(t) \leq K, \left|{}^cD_{0^+}^{\beta} u(t)\right| \leq K\right\}.$$

Then U is a bounded, closed and convex subset of the Banach space E.
We prove that $TU \subset U$. For $u \in U$ we have

$$\begin{aligned}(Tu)(t) &\leq \left(\int_0^1 G(t,s)\, ds + \frac{\sum_{i=1}^{m-2} a_i \int_0^1 G(\eta_i,s)\, ds}{1-\sum_{i=1}^{m-2} a_i}\right) M\\ &\leq \frac{1}{\Gamma(\alpha)}\left(2+\frac{2\sum_{i=1}^{m-2} a_i\eta_i}{1-\sum_{i=1}^{m-2} a_i}\right) M\\ &= \frac{2}{\Gamma(\alpha)}\left(1+\frac{\sum_{i=1}^{m-2} a_i\eta_i}{1-\sum_{i=1}^{m-2} a_i}\right) M\\ &\leq K,\end{aligned}$$

which implies

(3.2) $$0 \leq Tu(t) \leq K, \quad 0 \leq t \leq 1.$$

For $0 < \beta < 1$, we get

$$\begin{aligned}
& {}^{c}D_{0+}^{\beta}\left(Tu\right)(t) \\
= \;& \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(Tu)'(s)\,ds \\
= \;& \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta} \\
& \times\left(\int_0^s \frac{\left((1-\tau)^{\alpha-1}-(\alpha-1)(s-\tau)^{\alpha-2}\right)f\left(\tau,u(\tau),{}^{c}D_{0+}^{\beta}u(\tau)\right)d\tau}{\Gamma(\alpha)}\right. \\
& \left.+\int_s^1 \frac{(1-\tau)^{\alpha-1}f\left(\tau,u(\tau),{}^{c}D_{0+}^{\beta}u(\tau)\right)d\tau}{\Gamma(\alpha)}\right)ds \\
\leq \;& \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta} \\
& \times\left(\int_0^s \frac{(1-\tau)^{\alpha-1}f\left(\tau,u(\tau),{}^{c}D_{0+}^{\beta}u(\tau)\right)d\tau}{\Gamma(\alpha)}\right. \\
& \left.+\int_s^1 \frac{(1-\tau)^{\alpha-1}f\left(\tau,u(\tau),{}^{c}D^{\beta}u(\tau)\right)d\tau}{\Gamma(\alpha)}\right)ds \\
= \;& \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\int_0^1 \frac{(1-\tau)^{\alpha-1}f\left(\tau,u(\tau),{}^{c}D_{0+}^{\beta}u(\tau)\right)d\tau}{\Gamma(\alpha)} \\
\leq \;& \frac{M}{\Gamma(2-\beta)\Gamma(\alpha+1)} \leq \frac{M}{\Gamma(2-\beta)\Gamma(\alpha)} \\
\leq \;& \frac{2M}{\Gamma(2-\beta)\Gamma(\alpha)} \leq K,
\end{aligned}$$

also we have

$$\begin{aligned}
{}^{c}D_{0+}^{\beta}\left(Tu\right)(t) &= \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(Tu)'(s)\,ds \\
&\geq \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta} \\
&\quad\times\left(-\int_0^s \frac{(\alpha-1)(s-\tau)^{\alpha-2}f\left(\tau,u(\tau),{}^{c}D_{0+}^{\beta}u(\tau)\right)d\tau}{\Gamma(\alpha)}\right)ds \\
&\geq \frac{-1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta} \\
&\quad\times\left(\int_0^1 \frac{(\alpha-1)(s-\tau)^{\alpha-2}f\left(\tau,u(\tau),{}^{c}D_{0+}^{\beta}u(\tau)\right)d\tau}{\Gamma(\alpha)}\right)ds.
\end{aligned}$$

Thus

$$\begin{aligned} {}^{c}D_{0^{+}}^{\beta}\left(Tu\right)\left(t\right) &\geq \frac{-M}{\Gamma\left(1-\beta\right)\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{-\beta}ds=\frac{-M}{\Gamma\left(2-\beta\right)\Gamma\left(\alpha\right)} \\ &\geq \frac{-2M}{\Gamma\left(2-\beta\right)\Gamma\left(\alpha\right)}\geq -K. \end{aligned}$$

It follows that

$$-K\leq^{c}D^{\beta}\left(Tu\right)\left(t\right)\leq +K. \tag{3.3}$$

Therefore, from (3.2) and (3.3), we have $Tu\in U$ which implies that $TU\subset U.$

We show that $T:U\longrightarrow U$ is continuous. In fact, for $u_{n}\in U$, $n=1,2,...$ and $u\in U$ such that $\lim_{n\longrightarrow\infty}\left\Vert u_{n}-u\right\Vert_{E}=0$ uniformly on $[0,1]$, where

$$\left\Vert u_{n}-u\right\Vert=\max\left\{\left\Vert u_{n}-u\right\Vert_{\infty},\left\Vert {}^{c}D_{0^{+}}^{\beta}u_{n}-{}^{c}D_{0^{+}}^{\beta}u\right\Vert_{\infty}\right\},$$

we have $\lim_{n\longrightarrow\infty}\left\Vert u_{n}-u\right\Vert_{\infty}=0$ and $\lim_{n\longrightarrow\infty}\left\Vert {}^{c}D_{0^{+}}^{\beta}u_{n}-{}^{c}D_{0^{+}}^{\beta}u\right\Vert_{\infty}=0,$ which implies that

$$\lim_{n\longrightarrow\infty}u_{n}\left(t\right)=u\left(t\right),\ \lim_{n\longrightarrow\infty}{}^{c}D_{0^{+}}^{\beta}u_{n}\left(t\right)={}^{c}D_{0^{+}}^{\beta}u\left(t\right),$$

therefore

$$\lim_{n\longrightarrow\infty}f\left(t,u_{n}\left(t\right),^{c}D_{0^{+}}^{\beta}u_{n}\left(t\right)\right)=f\left(t,u\left(t\right),^{c}D_{0^{+}}^{\beta}u\left(t\right)\right),\ \ t\in[0,1],$$

which gives

$$\begin{aligned} &\left|\left(Tu_{n}\right)\left(t\right)-\left(Tu\right)\left(t\right)\right| \\ =&\ \left|\int_{0}^{1}G\left(t,s\right)\left[f\left(s,u_{n}\left(s\right),^{c}D_{0^{+}}^{\beta}u_{n}\left(s\right)\right)-f\left(s,u\left(s\right),^{c}D_{0^{+}}^{\beta}u\left(s\right)\right)\right]ds\right. \\ &+\frac{\sum_{i=1}^{m-2}a_{i}\int_{0}^{1}G\left(\eta_{i},s\right)\left[f\left(s,u_{n}\left(s\right),^{c}D_{0^{+}}^{\beta}u_{n}\left(s\right)\right)\right.}{1-\sum_{i=1}^{m-2}a_{i}} \\ &\frac{\left.\left.-\ f\left(s,u\left(s\right),^{c}D_{0^{+}}^{\beta}u\left(s\right)\right)\right]ds\right|}{1-\sum_{i=1}^{m-2}a_{i}}, \\ \leq&\ \int_{0}^{1}\left|G\left(t,s\right)\right|\left|f\left(s,u_{n}\left(s\right),^{c}D_{0^{+}}^{\beta}u_{n}\left(s\right)\right)-f\left(s,u\left(s\right),^{c}D_{0^{+}}^{\beta}u\left(s\right)\right)\right|ds \\ &+\frac{\sum_{i=1}^{m-2}a_{i}\int_{0}^{1}\left|G\left(\eta_{i},s\right)\right|\left|f\left(s,u_{n}\left(s\right),^{c}D_{0^{+}}^{\beta}u_{n}\left(s\right)\right)-f\left(s,u\left(s\right),^{c}D_{0^{+}}^{\beta}u\left(s\right)\right)\right|ds}{1-\sum_{i=1}^{m-2}a_{i}}, \end{aligned}$$

then

$$\left|\left(Tu_{n}\right)\left(t\right)-\left(Tu\right)\left(t\right)\right|\longrightarrow 0,\ \text{as } n\longrightarrow\infty, \tag{3.4}$$

and

$$\begin{aligned}
&\left|{}^{c}D_{0^+}^{\beta}\left(Tu_n\right)(t)-{}^{c}D_{0^+}^{\beta}\left(Tu\right)(t)\right| \\
=\ &\left|\frac{1}{\Gamma\left(1-\beta\right)}\int_0^t\left(t-s\right)^{-\beta}\left(Tu_n\right)'(s)\,ds\right. \\
&\left.-\frac{1}{\Gamma\left(1-\beta\right)}\int_0^t\left(t-s\right)^{-\beta}\left(Tu\right)'(s)\,ds\right| \\
=\ &\frac{1}{\Gamma\left(1-\beta\right)}\left|\int_0^t\left(t-s\right)^{-\beta}\right. \\
&\left.\times\left(\int_0^1 G'_s\left(s,\tau\right)\left(f\left(\tau,u_n\left(\tau\right),{}^{c}D_{0^+}^{\beta}u_n\left(\tau\right)\right)-f\left(\tau,u\left(\tau\right),{}^{c}D_{0^+}^{\beta}u\left(\tau\right)\right)\right)d\tau\right)ds\right| \\
\leq\ &\frac{1}{\Gamma\left(1-\beta\right)}\int_0^t\left(t-s\right)^{-\beta} \\
&\times\left(\int_0^1\left|G'_s\left(s,\tau\right)\right|\left|f\left(\tau,u_n\left(\tau\right),{}^{c}D_{0^+}^{\beta}u_n\left(\tau\right)\right)-f\left(\tau,u\left(\tau\right),{}^{c}D_{0^+}^{\beta}u\left(\tau\right)\right)\right|d\tau\right)ds\Bigg|,
\end{aligned}$$

then

(3.5) $$\left|{}^{c}D_{0^+}^{\beta}\left(Tu_n\right)(t)-{}^{c}D_{0^+}^{\beta}\left(Tu\right)(t)\right|\longrightarrow 0,\ \text{as } n\longrightarrow\infty.$$

By (3.4) and (3.5) , we have

$$\left\|Tu_n-Tu\right\|=\max\left\{\left\|Tu_n-Tu\right\|_\infty,\left\|{}^{c}D_{0^+}^{\beta}Tu_n-{}^{c}D_{0^+}^{\beta}Tu\right\|_\infty\right\}\longrightarrow 0,\ \text{as } n\longrightarrow\infty,$$

which means that T is continuous.

We prove that TU is an equicontinuous set. Take $t_1,t_2\in[0,1],t_1<t_2$, and let $M=\max\left\{f\left(t,u\left(t\right),{}^{c}D_{0^+}^{\beta}u\left(t\right)\right):0\leq t\leq 1,0\leq u\left(t\right)\leq K,\left|{}^{c}D_{0^+}^{\beta}u\left(t\right)\right|\leq K\right\}$.

Then we have

$$\begin{aligned}
\left|\left(Tu\right)(t_1)-\left(Tu\right)(t_2)\right| &= \left|\int_0^1\left(G\left(t_1,s\right)-G\left(t_2,s\right)\right)f\left(s,u\left(s\right),{}^{c}D_{0^+}^{\beta}u\left(s\right)\right)ds\right| \\
&\leq \left(\int_0^{t_1}\left|\left(G\left(t_1,s\right)-G\left(t_2,s\right)\right)\right|ds\right. \\
&\quad+\int_{t_1}^{t_2}\left|\left(G\left(t_1,s\right)-G\left(t_2,s\right)\right)\right|ds \\
&\quad\left.+\int_{t_2}^{1}\left|\left(G\left(t_1,s\right)-G\left(t_2,s\right)\right)\right|ds\right)M,
\end{aligned}$$

which means

$$\begin{aligned}
\left|(Tu)(t_1)-(Tu)(t_2)\right| &\leq \left(\left[\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right.\right.\\
&\quad\left.+\frac{(t_2-t_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)}\right]ds\\
&\quad+\int_{t_1}^{t_2}\left[\frac{(t_2-t_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\right]ds\\
&\quad\left.+\int_{t_2}^{1}\frac{(t_2-t_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right)M\\
&= \left(\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.\\
&\quad\left.+(t_2-t_1)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right)M\\
&= \frac{M}{\Gamma(\alpha+1)}\left(t_2^\alpha-t_1^\alpha+t_2-t_1\right),
\end{aligned}$$

which is independent of u, using the fact that the function $t_2^\alpha - t_1^\alpha$ is uniformly continuous on $[0,1]$ then

$$\left\|(Tu)(t_2)-(Tu)(t_1)\right\|_\infty \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1,$$

and

$$\begin{aligned}
&\left|{}^cD_{0^+}^\beta(Tu)(t_1)-{}^cD_{0^+}^\beta(Tu)(t_2)\right|\\
=\;&\frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_1}(t_1-s)^{-\beta}\left(\int_0^1 G_s'(s,\tau)f\left(\tau,u(\tau),{}^cD_{0^+}^\beta u(\tau)\right)d\tau\right)ds\right.\\
&\left.-\int_0^{t_2}(t_2-s)^{-\beta}\left(\int_0^1 G_s'(s,\tau)f\left(\tau,u(\tau),{}^cD_{0^+}^\beta u(\tau)\right)d\tau\right)ds\right|\\
\leq\;&\frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_1}(t_1-s)^{-\beta}\left(\int_0^1 G_s'(s,\tau)f\left(\tau,u(\tau),{}^cD_{0^+}^\beta u(\tau)\right)d\tau\right)ds\right.\\
&\left.-\int_0^{t_1}(t_2-s)^{-\beta}\left(\int_0^1 G_s'(s,\tau)f\left(\tau,u(\tau),{}^cD_{0^+}^\beta u(\tau)\right)d\tau\right)ds\right|\\
&+\frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_1}(t_2-s)^{-\beta}\left(\int_0^1 G_s'(s,\tau)f\left(\tau,u(\tau),{}^cD_{0^+}^\beta u(\tau)\right)d\tau\right)ds\right.\\
&\left.-\int_0^{t_2}(t_2-s)^{-\beta}\left(\int_0^1 G_s'(s,\tau)f\left(\tau,u(\tau),{}^cD_{0^+}^\beta u(\tau)\right)d\tau\right)ds\right|.
\end{aligned}$$

We get

$$\begin{aligned}&\left|{}^{c}D_{0^{+}}^{\beta}\left(Tu\right)\left(t_{1}\right)-{}^{c}D_{0^{+}}^{\beta}\left(Tu\right)\left(t_{2}\right)\right|\\ \leq\ & \frac{2M}{\Gamma\left(1-\beta\right)\Gamma\left(\alpha\right)}\int_{0}^{t_{1}}\left(\left(t_{1}-s\right)^{-\beta}-\left(t_{2}-s\right)^{-\beta}\right)ds\\ &+\frac{2M}{\Gamma\left(1-\beta\right)\Gamma\left(\alpha\right)}\int_{t_{1}}^{t_{2}}\left(t_{2}-s\right)^{-\beta}ds\\ \leq\ & \frac{2M}{\Gamma\left(2-\beta\right)\Gamma\left(\alpha\right)}\left(t_{2}^{1-\beta}-t_{1}^{1-\beta}+2\left(t_{2}-t_{1}\right)^{1-\beta}\right)\end{aligned}$$

since the function $t_{2}^{1-\beta}-t_{1}^{1-\beta}$ is uniformly continuous on $[0,1]$, it follows that

$$\left\|{}^{c}D_{0^{+}}^{\beta}\left(Tu\right)\left(t_{2}\right)-{}^{c}D_{0^{+}}^{\beta}\left(Tu\right)\left(t_{1}\right)\right\|_{\infty}\longrightarrow 0 \text{ as } t_{2}\longrightarrow t_{1}.$$

Consequently

$$\begin{aligned}&\left\|\left(Tu\right)\left(t_{2}\right)-\left(Tu\right)\left(t_{1}\right)\right\|\\ =\ & \max\left\{\left\|\left(Tu\right)\left(t_{2}\right)-\left(Tu\right)\left(t_{1}\right)\right\|_{\infty},\left\|{}^{c}D_{0^{+}}^{\beta}\left(Tu\right)\left(t_{2}\right)-{}^{c}D_{0^{+}}^{\beta}\left(Tu\right)\left(t_{1}\right)\right\|_{\infty}\right\}\longrightarrow 0,\end{aligned}$$

as $t_{2}\longrightarrow t_{1}$.We conclude that TU is an equicontinuous set. Obviously it is uniformly bounded since $TU\subset U$. By the Arzela-Ascoli theorem the operator $T:U\longrightarrow U$ is completely continuous. The Schauder fixed-point theorem asserts the existence of positif solution in U for the problem (1.1) and the theorem is proved. ∎

Example 3.1. *Consider the four-point boundary-value problem*

$$\left\{\begin{array}{l}{}^{c}D_{0^{+}}^{\frac{3}{2}}u\left(t\right)+\frac{1}{4}\left(1+t^{2}\right)\left(1+\sqrt{u\left(t\right)}\right)\\ +\frac{\left|{}^{c}D_{0^{+}}^{\frac{1}{2}}u(t)\right|}{4(1+t^{2})}+\min\left\{t;1-\frac{t}{2}\right\}=0,\ \ 0\leq t\leq 1,\\ u\left(0\right)=u\left(1\right)=\sum_{i=1}^{2}a_{i}u\left(\eta_{i}\right),.\end{array}\right. \tag{3.6}$$

We choose $\alpha=\frac{3}{2}$ $,\beta=\frac{1}{2};\eta_{1}=\frac{1}{4},\eta_{2}=\frac{1}{3};a_{1}=\frac{1}{2},a_{2}=\frac{1}{5}$. *Then* $1-\left(a_{1}+a_{2}\right)=\frac{3}{10}<1$, $\Gamma\left(\frac{3}{2}\right)=0.886,\Gamma\left(2-\frac{1}{2}\right)=0.886$ *and*$\frac{1}{2}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}=0.191$. *Set* $f\left(t,u,v\right)=\frac{1}{4}\left(1+t^{2}\right)\left(1+\sqrt{u}\right)+\frac{|v|}{4(1+t^{2})}+\min\left\{t;1-\frac{t}{2}\right\}$ *and take* $K=12$. *A direct computation shows that*

$M=\max\left\{f\left(t,u,v\right):0\leq t\leq 1,0\leq u\left(t\right)\leq 12,\left|v\left(t\right)\right|\leq 12\right\}=4.295,$

$\frac{1}{\Gamma(\alpha)\Gamma(2-\beta)}=\frac{1}{0.886^{2}}=1.273\,,\ \frac{1}{\Gamma(\alpha)}\left(1+\frac{\sum_{i=1}^{2}a_{i}\eta_{i}}{1-\sum_{i=1}^{2}a_{i}}\right)=\frac{1}{0.886}\times 0.636=0.717.$

Then we have

$\max\left\{1.273,0.717\right\}\times 4.295=1.273\times 4.295=5.467<\frac{K}{2}=6,$

i.e;

$$\max\left\{\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(2-\beta\right)},\frac{1}{\Gamma\left(\alpha\right)}\left(1+\frac{\sum_{i=1}^{2}a_{i}\eta_{i}}{1-\sum_{i=1}^{2}a_{i}}\right)\right\}M=5.467<\frac{K}{2}=6.$$

From Theorem 3.3, the problem (3.6) has at least one positive solution.

REFERENCES

[1] W. CHEN, Y. WANG, Existence of solutions of boundary-value problem of a class of nonlinear fractional differential equations, *Adv. Differ. Equ.*, 2014, 174 (2014).

[2] C. CHENG, Z. FENG, Y. SU, Positive solutions of fractional equations with derivative terms, *Electronic Journal of Differential Equations,* (2012), No. 215, pp. 1-27.

[3] K. DIETHELM, N. J. FORD, analysis of fractional differential equations, *J. Math. Anal. Appl.*, **265** (2002), pp. 229-248.

[4] W. G. GLOCKLE, T. F. NONNENMACHER, A fractional calculus approach to self similar protein dynamics, *Biophys. J.*, **68** (1995), pp. 46-53.

[5] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Dractional Differential Equations*, Elsevier B.V, Amsterdam, 2006.

[6] V. LAKSHMIKANTHAM, S. LEELA, J. VASUNDHARA DEVI, *Theory of Dractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge (2009).

[7] R. METZLER, W. SCHICK, H. G. KILIAN, T. F. NONNENMACHER, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.*, **103** (1995), pp. 7180-7186.

[8] K. S. MILLER, B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York (1993).

[9] S. P. NASHOLM, S. HOLM, Linking multiple relaxation, power-law, attenuation, and fractional wave equations, *J. Acoust. Soc. Amer.*, **130** (2011), pp. 3038-3045.

[10] I. PODLUBNY, Fractional differential equations, *Mathematics in Science and Engineering*, Vol 198, Academic press, New York/London/Toronto, 1998.

[11] X. SU, S. ZHANG, Solutions to boundary-value problems for nonlinear differential equations of fractional order, *Electronic Journal of Differential Equations,* (2009), No. 26, pp. 1-15.

[12] C. WANG, W. LIU, C. REN, Existence of solutions for multi-point nonlinear differential equations of fractional orders with integral boundary conditions, *Electronic Journal of Differential Equations*, (2012), No. 55, pp. 1-10.

[13] X. XU, D. JIANG, C. YUAN, Multiple positive solutions for the boundary problem of a nonlinear fractional differential equation, *Nonlinear Analysis*, **71** (2009), pp. 4676-4688.

[14] S. Q. ZHANG, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electronic Journal of Differential Equations*, **36** (2006), pp. 1-12.