COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF BI-UNIVALENT SAKAGUCHI TYPE FUNCTIONS BY USING FABER POLYNOMIAL

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ABSTRACT. In this work, considering a general subclass of bi-univalent Sakaguchi type functions, we determine estimates for the general Taylor-Maclaurin coefficients of the functions in these classes. For this purpose, we use the Faber polynomial expansions. In certain cases, our estimates improve some of those existing coefficient bounds.

Key words and phrases: Analytic univalent function; Faber polynomial; Coefficient estimate; Bi-univalent function; Sakaguchi type function.

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1. Introduction

Let \( \mathcal{A} \) denote the class of all functions of the form

\[
(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \). We also denote by \( \mathcal{S} \) the class of all functions in the normalized analytic function class \( \mathcal{A} \) which are univalent in \( \mathbb{U} \).

It is well known that every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), which is defined by \( f^{-1}(f(z)) = z \). For the analytic bi-univalent functions given by (1.1) using Faber polynomial expansions \([17, 20, 21]\), Hamidi and Jahangiri \([17]\) considered the class of analytic bi-close-to-convex functions.

Motivated by the works of Serap Bulut we defined the following classes.

For Definition 2.1.

2. The Class \( M_{\Sigma}(\alpha, \rho, \mu, t) \)

**Definition 2.1.** For \( 0 \leq \mu \leq \rho \leq 1, |t| \leq 1 \) and \( t \neq 1 \), a function \( f \in \Sigma \) given by (1.1) is said to be in class \( M_{\Sigma}(\alpha, \rho, \mu, t) \) if the following conditions are satisfied

\[
(2.1) \quad \operatorname{Re} \left\{ \frac{(1-t)[\rho \mu z^3 f'''(z) + (2 \rho \mu - \rho - \mu) z^2 f''(z) + z f'(z)]}{\rho \mu z^2 [f''(z) - t f''(tz)] + (\rho - \mu) z [f'(z) - t f'(tz)] + (1 - \rho + \mu) [f(z) - f(tz)]} \right\} > \alpha
\]

and
\begin{equation}
(2.2) \quad \text{Re} \left\{ \frac{(1-t)[\rho zw^2g''(w) + (2\rho w + \rho - \mu)w^2g''(w) + wg'(w)]}{\rho zw''(w) - tw'g'(tw) + (\rho - \mu)wg'(w) - twg(tw)} \right\} > \alpha
\end{equation}
where \(0 \leq \alpha < 1\) and \(z, w \in \mathbb{U}\) and \(g = f^{-1}\) is defined by (1.2).

**Definition 2.2.** For \(0 \leq \rho \leq 1, |t| \leq 1\) and \(t \neq 1\), a function \(f \in \Sigma\) given by (1.1) is said to be in class \(N_\Sigma(\alpha, \rho, t)\) if the following conditions are satisfied
\begin{equation}
(2.3) \quad \text{Re} \left\{ \frac{(1-t)[\rho z^2f''(z) + zf'(z)]}{pz[f''(z) - tf'(tz)] + (1-\rho)[f(z) - f(tz)]} \right\} > \alpha
\end{equation}
and
\begin{equation}
(2.4) \quad \text{Re} \left\{ \frac{(1-t)[\rho w^2g''(w) + wg'(w)]}{\rho w[g'(w) - twg(tw)] + (1-\rho)[g(w) - gw(tw)]} \right\} > \alpha
\end{equation}
where \(0 \leq \alpha < 1\) and \(z, w \in \mathbb{U}\) and \(g = f^{-1}\) is defined by (1.2).

**Remark 2.1.** For \(t = 0\) and \(\mu = 0\), we get the well-known class \(N_\Sigma(\alpha, \rho)\) of bi-starlike functions or order \(\alpha\). This class consists of functions \(f \in \Sigma\) satisfying \(\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha\) and \(\text{Re} \left\{ \frac{wg'(w)}{g(w)} \right\} > \alpha\) where \(0 \leq \alpha < 1\) and \(z, w \in \mathbb{U}\) and \(g = f^{-1}\) is defined by (1.2).

### 3. Coefficient Estimates

Using the Faber polynomial expansion of functions \(f \in \mathcal{A}\) of the form (1.1), the coefficients of its inverse map \(g = f^{-1}\) may be expressed as [1]
\begin{equation}
(3.1) \quad g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n
\end{equation}
where
\begin{align}
K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n+1)!} a_2^{-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{-3} a_3
+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{-4} a_4 + \frac{(-n)!}{2(-n+2)!(n-5)!} a_2^{-5} a_5 + (-n+2)a_3^2 \\
&\quad+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{-6} a_6 + (2(-n+5)a_3 a_4) + \sum_{j \geq 7} a_2^{-j} v_j
\end{align}
such that \(v_j\) with \(7 \leq j \leq n\) is a homogenous polynomial in the variables \(a_2, a_3, \ldots, a_n\) [3]. In particular, the first three terms of \(K_{n-1}^{-n}\) are
\begin{equation}
(3.3) \quad K_{n-1}^{-2} = -2a_2, \quad K_{n-1}^{-3} = 3(2a_2^2 - a_3), \quad K_{n-1}^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).
\end{equation}
In general, for any \(p \in \mathbb{N} := \{1, 2, 3, \ldots\}\), an expansion of \(K_{n-1}^{-n}\) is as, [1],
\begin{equation}
(3.4) \quad K_{n-1}^{-p} = p a_2 + \frac{p(p-1)}{2} D_2^n + \frac{p!}{(p-3)!} D_3^n + \cdots + \frac{p!}{(p-n)!} D_n^n,
\end{equation}
where \(D_n^p = D_n^p(a_2, a_3, \ldots, a_n)\), and by [31],
\begin{align}
D_n^m(a_1, a_2, \ldots, a_n) &= \sum_{i_1 + \cdots + i_n = m} \frac{m!}{i_1! \cdots i_n!} a_1^{i_1} \cdots a_n^{i_n} \quad \text{while} \quad a_1 = 1, \quad \text{and the sum is taken over all non-negative integers} \quad i_1, \ldots, i_n \quad \text{satisfying} \quad i_1 + i_2 + \cdots + i_n = m, \quad i_1 + 2i_2 + \cdots + ni_n = n, \quad \text{it is clear that} \quad D_n^m(a_1, a_2, \ldots, a_n) = a_1^n.
\end{align}
Consequently, for functions $f \in N_{\Sigma}(\alpha, \rho, \mu, t)$ of the form (1.1), we can write:

$$\left\{ \frac{(1-t)[\rho^2 z^3 f''(z) + (2\mu + \rho - \mu)z^2 f''(z) + z f'(z)]}{\rho z^2[f''(z) - t^2 f''(tz)] + (\rho - \mu)z[f'(z) - tf'(tz)] + (1-\rho + \mu)[f(z) - f(tz)]} \right\}$$

$$= 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \ldots, a_n)z^{n-1},$$

(3.5)

which is the Faber polynomial of degree $(n - 1)$ where

$$F_1 = \left\{ \frac{2[\rho(1-t)]}{1+t(\rho-\mu)} \right\} a_2,$$

$$F_2 = \left\{ \frac{[6\rho(2-t) + 3(\rho - \mu) + 3] - [1 + (\rho - \mu)(3t - 1) + 6\rho t^2]u_3}{1+t(\rho-\mu)} \right\} a_3$$

$$- \{4[\rho(1-t)]/[\rho(3 - 2t) + 2 + (\rho - \mu)] + 3u_2[1 + (\rho - \mu)(2t - 1) + 2\rho t^2]$$

$$[\rho(1 - 2t) + 1] + 2u_3[1 + (\rho - \mu)(3t - 1) + 6\rho t^2][-2\rho t - \rho + \mu + 1$$

$$- u_2[1 + (\rho - \mu)(2t - 1) + 2\rho t^2]]} [1 + t(\rho-\mu)^2]$$

$$+ \{4[\rho(1-t) + 1] / [\rho(1+t) + (\rho - \mu)] - 2[1 + (\rho - \mu)(2t - 1) + 2\rho t^2]$$

$$[\rho(1 - 2t) + 1] + 2u_3[1 + (\rho - \mu)(3t - 1) + 6\rho t^2][-2\rho t - \rho + \mu + 1$$

$$- u_2[1 + (\rho - \mu)(2t - 1) + 2\rho t^2]]} [1 + t(\rho-\mu)^2]$$

$$+ [1 + (\rho - \mu)(2t - 1) + 2\rho t^2]u_2] \frac{a_2 a_3}{[1 + t(\rho-\mu)]^2}, \text{etc.}$$

If $f \in N_{\Sigma}(\alpha, \rho, t)$ of the form (1.1), we can write

$$\frac{(1-t)[\rho z^2 f''(z) + z f'(z)]}{\rho z[f'(z) - tf'(tz)] + (1-\rho)[f(z) - f(tz)]} = 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \ldots, a_n)z^{n-1},$$

(3.6)

which is the Faber polynomial of degree $(n - 1)$ where

$$F_1 = \frac{2 - u_2(2\rho t - \rho + 1)}{(1 + \rho t)} a_2,$$

$$F_2 = \frac{3\rho + 3 - u_3(3\rho t - \rho + 1)}{(1 + \rho t)} a_3 - \frac{[2 - u_2(2\rho t - \rho + 1)][2\rho + u_2(2\rho t - \rho + 1)]}{(1 + \rho t)^2} a_2,$$

$$F_3 = \frac{8\rho + 4 - u_4(4\rho t - 4 + 1)}{(1 + \rho t)} a_4 + \frac{[2 + u_2(2\rho t - \rho + 1)][2 - u_2(2\rho t - \rho + 1)]}{(1 + \rho t)^3} a_2^3$$

$$- \left\{ \frac{[2 - u_2(2\rho t - \rho + 1)][3\rho + u_3(3\rho t - \rho + 1)]}{(1 + \rho t)^2} a_2 a_3, \text{etc.} \right\}$$

where $u_1 = 1, u_2 = 1 + t, u_3 = 1 + t + t^2$ and $u_4 = 1 + t + t^2 + t^3.$

Our first theorem introduces an upper bound for the coefficients $|a_n|$ of bi-univalent Sakaguchi type functions in the class $N_{\Sigma}(\alpha, \rho, \mu, t).$
Theorem 3.1. For $0 \leq \mu \leq \rho \leq 1$, $|t| \leq 1$, $t \neq 1$, let the function $f \in M_\Sigma(\alpha, \rho, \mu, t)$ be given by (1.1). If $a_k = 0$ ($2 \leq k \leq n - 1$), then

\[
|a_n| \leq \frac{2(1 - \alpha)[1 + t(\rho - \mu)]}{n[(n - 1)(n - 1 - t)\rho\mu + (n - 2)(\rho - \mu) + 1] - [1 + (\rho - \mu)(nt - 1) + n(n - 1)\rho\mu t^2]u_n}
\]

Proof. For the function $f \in M_\Sigma(\alpha, \rho, \mu, t)$ of the form (1.1), we have the expansion (3.5) and for the inverse map $g = f^{-1}$, considering (1.2), we obtain

\[
(1 - t)[\rho\mu w^3 g'''(w) + (2\rho\mu + \rho - \mu)w^2 g''(w) + wg'(w)]
\]

\[
\rho \mu w^2[g''(w) - t^2 g''(tw)] + (\rho - \mu)w[g'(w) - tg'(tw)] + (1 - \rho + \mu)[g(w) - g(tw)]
\]

(3.7)

with

\[
A_n = \frac{1}{n}K_{n-1}^{-1}(a_2, a_3, \ldots, a_n).
\]

On the other hand, since $f \in M_\Sigma(\alpha, \rho, \mu, t)$ and $g = f^{-1} \in M_\Sigma(\alpha, \rho, \mu, t)$, by definition, there exist two positive real-part functions

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in A \quad \text{and} \quad q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in A,
\]

where $Re\{p(z)\} > 0$ and $Re\{q(w)\} > 0$ in $U$ so that

\[
(1 - t)[\rho\mu z^3 f'''(z) + (2\rho\mu + \rho - \mu)z^2 f''(z) + zf'(z)]
\]

\[
\rho \mu z^2[f''(z) - t^2 f''(tz)] + (\rho - \mu)z[f'(z) - tf'(tz)] + (1 - \rho + \mu)[f(z) - f(tz)]
\]

(3.9)

\[
= 1 + (1 - \alpha)\sum_{n=1}^{\infty} K_{n-1}^1(c_1, c_2, \ldots, c_n)z^n
\]

and

\[
(1 - t)[\rho\mu w^3 g'''(w) + (2\rho\mu + \rho - \mu)w^2 g''(w) + wg'(w)]
\]

\[
\rho \mu w^2[g''(w) - t^2 g''(tw)] + (\rho - \mu)w[g'(w) - tg'(tw)] + (1 - \rho + \mu)[g(w) - g(tw)]
\]

\[
= 1 + (1 - \alpha)\sum_{n=1}^{\infty} K_{n-1}^1(d_1, d_2, \ldots, d_n)w^n
\]

(3.10)

Note that, by the Caratheodory lemma (e.g. [12]), $|c_n| \leq 2$ and $|d_n| \leq 2$ ($n \in \mathbb{N}$).

Comparing the corresponding coefficients of (3.5) and (3.9), for any $n \geq 2$, yields

\[
F_{n-1}(a_2, a_3, \ldots, a_n) = (1 - \alpha)K_{n-1}^1(c_1, c_2, \ldots, c_{n-1})
\]

(3.11)

and similarly, from (3.7) and (3.10) we find

\[
F_{n-1}(A_2, A_3, \ldots, A_n) = (1 - \alpha)K_{n-1}^1(d_1, d_2, \ldots, d_{n-1})
\]

(3.12)

Note that for $a_k = 0$ ($2 \leq k \leq n - 1$), we have $A_n = -a_n$ and so

\[
\frac{n[(n - 1)(n - 1 - t)\rho\mu + (n - 2)\rho\mu + 1] + [1 + (\rho - \mu)(nt - 1) + n(n - 1)\rho\mu t^2]u_n}{1 + (\rho - \mu)}a_n = (1 - \alpha)c_{n-1},
\]
where
\[
\frac{n[(n-1)(n-1-\ell)\mu+(n-2)\mu+1]+[1+(\rho-\mu)(nt-1)+n(n-1)\rho t^2]u_n}{1+t(\rho-\mu)} a_n = (1 - \alpha) d_{n-1}.
\]

Taking the absolute values of the above equalities, we obtain
\[
|a_n| = \frac{(1 - \alpha)|a_{n-1}|[1 + t(\rho - \mu)]}{n[(n-1)(n-1-\ell)\mu+(n-2)\mu+1]+[1+(\rho-\mu)(nt-1)+n(n-1)\rho t^2]u_n}
\]

which completes the proof of Theorem 3.1.

The following corollary is an immediate consequence of the above theorem.

**Corollary 3.2.** (see [20]) For \(0 \leq \rho \leq 1\), \(|t| \leq 1\), \(t \neq 1\) and \(0 \leq \alpha < 1\), let the function \(f \in M_\Sigma(\alpha, \rho, t)\) be given by (1.1). If \(a_k = 0\) (\(2 \leq k \leq n - 1\)), then
\[
|a_n| \leq \frac{2(1 - \alpha)(1 + \rho t)}{n[(n-2)(\rho + 1) - u_n[n\rho t - \rho + 1]]} \quad (n \geq 4).
\]

where \(u_n = 1 + t + t^2 + \cdots + t^{n-1}\)

**Theorem 3.3.** For \(0 \leq \mu \leq \rho \leq 1\), \(|t| \leq 1\), \(t \neq 1\) and \(0 \leq \alpha < 1\), let the function \(f \in M_\Sigma(\alpha, \rho, \mu, t)\) be given by (1.1). Then one has the following
\[
|a_2| \leq \begin{cases} 
\sqrt{\frac{2(1-\alpha)[1+t(\rho-\mu)]^2}{B}}, & 0 \leq \alpha < \frac{A}{2B}
\end{cases}
\]

(3.13)

\[
|a_3| \leq \begin{cases} 
\min \left\{ \begin{array}{l}
\frac{4(1-\alpha)^2[1+t(\rho-\mu)]^2}{2[\rho t(1-t)+1-1+(\rho-\mu)(2t-1)+2\rho t^2]u_2]^2} + \frac{2(1-\alpha)[1+t(\rho-\mu)]}{[2\rho t(2-t)+(\rho-\mu)+1-1+(\rho-\mu)(3t-1)+6\rho t^2]u_3]}, \\
\frac{2(1-\alpha)[1+t(\rho-\mu)]}{3[2\rho t(2-t)+(\rho-\mu)+1-1+(\rho-\mu)(3t-1)+6\rho t^2]u_3]}, & 0 \leq \rho \leq \mu < 1
\end{array} \right.
\end{cases}
\]

(3.14)

\[
|a_3| = \frac{C}{2(1-\alpha)[1+t(\rho-\mu)]^2} \leq \frac{3[2\rho t(2-t)+(\rho-\mu)+1-1+(\rho-\mu)(3t-1)+6\rho t^2]u_3}{2(1-\alpha)[1+t(\rho-\mu)]^2}
\]

where
\[
A = \left\{ \begin{array}{l}
2(1+(\rho-\mu)[3t-1]+(\rho-\mu)3t-2\rho t(3t^2-2t+3)+\rho t(2\rho t^2+2t-3) \\
+(4-6t)]+1+(\rho-\mu)(2t-1)+2\rho t^2]u_2[4\rho t(1+t)+(\rho-\mu)] \\
+1+(\rho-\mu)(2t-1)+2\rho t^2]u_2-2[1+(\rho-\mu)(3t-1)+6\rho t^2]u_3
\end{array} \right\}
\]
Also from (3.16) and (3.18), respectively, we get
\begin{equation}
\frac{\{2[\rho \mu(t-1) + 1] - [1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2\} a_2}{1 + t(\rho - \mu)} = (1 - \alpha)c_2
\end{equation}
\begin{equation}
\frac{-\{2[\rho \mu(t-1) + 1] - [1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2\} a_2}{1 + t(\rho - \mu)} = (1 - \alpha)d_2
\end{equation}
\begin{equation}
\frac{\{3[2\rho \mu(2t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho \mu t^2] u_3\} - \{4[\rho \mu(1t) + 1] + (\rho - \mu)] + 2[1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2\}}{[2\rho \mu + \rho - \mu - 1] + [1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2^2} = (1 - \alpha)c_1
\end{equation}
\begin{equation}
\frac{\{3[2\rho \mu(2t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho \mu t^2] u_3\} - \{4[\rho \mu(1t) + 1] + (\rho - \mu)] + 2[1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2\}}{[2\rho \mu + \rho - \mu - 1] + [1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2^2} = (1 - \alpha)d_1
\end{equation}
\begin{equation}
\frac{\{3[2\rho \mu(2t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho \mu t^2] u_3\} - \{4[\rho \mu(1t) + 1] + (\rho - \mu)] + 2[1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2\}}{[2\rho \mu + \rho - \mu - 1] + [1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2^2} = (1 - \alpha)c_1
\end{equation}
\begin{equation}
\frac{\{3[2\rho \mu(2t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho \mu t^2] u_3\} - \{4[\rho \mu(1t) + 1] + (\rho - \mu)] + 2[1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2\}}{[2\rho \mu + \rho - \mu - 1] + [1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2^2} = (1 - \alpha)d_1
\end{equation}
\begin{equation}
\frac{\{3[2\rho \mu(2t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho \mu t^2] u_3\} - \{4[\rho \mu(1t) + 1] + (\rho - \mu)] + 2[1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2\}}{[2\rho \mu + \rho - \mu - 1] + [1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2^2} = (1 - \alpha)c_1
\end{equation}
\begin{equation}
\frac{\{3[2\rho \mu(2t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho \mu t^2] u_3\} - \{4[\rho \mu(1t) + 1] + (\rho - \mu)] + 2[1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2\}}{[2\rho \mu + \rho - \mu - 1] + [1 + (\rho - \mu)(2t - 1) + 2\rho \mu t^2] u_2^2} = (1 - \alpha)d_1
\end{equation}

Also from (3.16) and (3.18), we obtain
\begin{equation}
2B \frac{a_2^2}{1 + t(\rho - \mu)^2} = (1 - \alpha)(c_2 + d_2)
\end{equation}

Using the Caratheodory lemma, we get
\begin{equation}
|a_2| \leq \sqrt{\frac{2(1 - \alpha)[1 + t(\rho - \mu)^2]}{B}}
\end{equation}

and combining this with inequality (3.19), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (3.13).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (3.18) from (3.16), we thus get
\begin{equation}
\{3[2\rho \mu(2t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho \mu t^2] u_3\} \frac{-2a_2^2 + 2a_3}{1 + t(\rho - \mu)} = (1 - \alpha)(c_2 - d_2)
\end{equation}
Upon substituting the value of $a_2^2$ from (3.15) into (3.21), it follows that

\[
a_3 = a_2^2 + \frac{(1 - \alpha)(c_2 - d_2)[1 + t(\rho - \mu)]}{2\{2[\rho\mu(1 - t) + 1] - [1 + (\rho - \mu)(2t - 1) + 2\rho\mu t^2]u_2\}}.
\]

Upon substituting the value of $a_2^2$ from (3.15) into (3.21), it follows that

\[
a_3 = \frac{(1 - \alpha)^2 c_2^2[1 + t(\rho - \mu)]}{\{2[\rho\mu(1 - t) + 1] - [1 + (\rho - \mu)(2t - 1) + 2\rho\mu t^2]u_2\}^2} + \frac{2[32\rho\mu(2 - t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho\mu t^2]u_3\}}.
\]

We thus find (by the Carathéodory lemma) that

\[
|a_3| \leq \frac{4(1 - \alpha)^2[1 + t(\rho - \mu)]^2}{\{2[\rho\mu(1 - t) + 1] - [1 + (\rho - \mu)(2t - 1) + 2\rho\mu t^2]u_2\}^2} + \frac{2(1 - \alpha)[1 + t(\rho - \mu)]}{\{32\rho\mu(2 - t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho\mu t^2]u_3\}}.
\]

On the other hand, upon substituting the value of $a_2^2$ from (3.20) into (3.21), it follows that

\[
a_3 = \frac{(1 - \alpha)(1 + t(\rho - \mu))}{2B[32\rho\mu(2 - t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho\mu t^2]u_3\}}.
\]

Consequently (by the Carathéodory lemma), we have

\[
|a_3| \leq \frac{2(1 - \alpha)[1 + t(\rho - \mu)]^2}{B}.
\]

Combining (3.22) and (3.23), we get the desired estimate on the coefficient $|a_3|$ as asserted in (3.14). Finally, from (3.18), we deduce (by the Carathéodory lemma) that

\[
|a_3| = \frac{2(1 + (\rho - \mu)(3t - 1) + 6\rho\mu t^2]u_3[1 + t(\rho - \mu)]^2}{A}
\]

where

\[
A = \left\{ \begin{array}{l}
2[1 + (\rho - \mu)(3t - 1) + 6\rho\mu t^2]u_3[1 + t(\rho - \mu)]^2
\end{array} \right\}
\]

and

\[
B = \left\{ \begin{array}{l}
[1 + t(\rho - \mu)]\{32\rho\mu(2 - t) + (\rho - \mu) + 1] - [1 + (\rho - \mu)(3t - 1) + 6\rho\mu t^2]u_3\}
\end{array} \right\}
\]

This evidently completes the proof of Theorem 3.3.
Corollary 3.4. (see [20]) For $0 \leq \rho \leq 1$, $|t| \leq 1$, $t \neq 1$ and $0 \leq \alpha < 1$, let the function $f \in N_2(\alpha, \lambda, t)$ be given by (1.1). Then one has the following

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)(1+\rho t)^2}{E}} & 0 \leq \alpha < \frac{D}{2E} \\ \frac{2(1-\alpha)(1+\rho t)}{2-u_2(2\rho t-\rho+1)} & \frac{D}{2E} \leq \alpha < 1 \end{cases}$$

$$|a_3| \leq \begin{cases} \frac{2(1-\alpha)(1+\rho t)^2}{E}, & 0 \leq \rho < 1 \\ \frac{2(1-\alpha)(1+\rho t)}{3\rho + 3 - u_3(3\rho t - \rho + 1)}, & \rho = 1 \end{cases}$$

$$|a_3 - \frac{2E}{2(1-\alpha)(1+\rho t)} \frac{a_2^2}{[3\rho + 3 - u_3(3\rho t - \rho + 1)](1 + \rho t)}| \leq \frac{2E}{2(1-\alpha)(1+\rho t)} \frac{a_2^2}{[3\rho + 3 - u_3(3\rho t - \rho + 1)]}$$

where

$$D = 2[3\rho + 3 - u_3(3\rho t - \rho + 1)](1 + \rho t) - [4\rho + 2 - 2u_2(2\rho t - \rho + 1)]\rho]^2$$

and

$$E = [3\rho + 3 - u_3(3\rho t - \rho + 1)](1 + \rho t) - [2 - 2u_2(2\rho t - \rho + 1)]\rho]^2$$

REFERENCES


