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SPLIT EQUALITY FOR MONOTONE INCLUSION PROBLEM AND FIXED POINT PROBLEM IN REAL BANACH SPACES

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ABSTRACT. In this paper a new iterative algorithm for approximating a common solution of split equality monotone inclusion problem and split equality fixed point problem is introduced. Using our algorithm, we state and prove a strong convergence theorem for approximating an element in the intersection of the set of solutions of a split equality monotone inclusion problem and the set of solutions of a split equality fixed point problem for right Bregman strongly nonexpansive mappings in the setting of p-uniformly convex Banach spaces which are also uniformly smooth. We also give some applications.

Key words and phrases: Monotone inclusion problem; Fixed point problem; Right Bregman strongly nonexpansive mapping; Maximal monotone mapping; Resolvent operators.

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1. INTRODUCTION

Let E be a real Banach space and $f : E \to \mathbb{R}$ be a convex function, then the domain of f is defined by

$$\text{dom}f := \{x \in E : f(x) < +\infty\}.$$

The function f is said to be proper if dom$f \neq \emptyset$.
Let $x \in$ int dom f, then for any $y \in E$, we define the right-hand derivative of f at x by

$$f^o(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{1.1}$$

The function f is said to be Gâteaux differentiable at x if the limit in (1.1) exists as $t \to 0$ for each y. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ int dom f.
Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) := \sup \Big\{ \frac{1}{2}(||x + y|| + ||x - y||) - 1 : ||x|| \leq 1, ||y|| \leq t \Big\}. \tag{1.2}$$

It is generally known that E is uniformly smooth if and only if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$$

and for any $q > 1$, E is said to be q-uniformly smooth if there exists $C_q > 0$ such that $\rho_E(t) \leq C_q t^q$ for any $t > 0$.
The duality mapping $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\}.$$

The duality mapping J_p^E is said to be weak-to-weak continuous if $x_n \rightharpoonup x \implies \langle J_p^E x_n, y \rangle \to \langle J_p^E x, y \rangle$ holds for any $y \in E$. We note that l_p $(p > 1)$ spaces have this property, but L_p $(p > 2)$ does not posses this property.
Let dim $E \geq 2$, then the modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\{1 - ||\frac{x + y}{2}|| : ||x|| = ||y|| = 1; \epsilon = ||x - y||\}.$$

E is said to be uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and E is p-uniformly convex if there exists $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p \epsilon^p$ for any $\epsilon \in (0, 2]$.

Remark 1.1. [1] . It is generally known that E is p-uniformly convex and uniformly smooth if and only if E^* is q-uniformly smooth and uniformly convex. In this case, the duality map J_p is one-to-one, single valued and satisfies $J_p = (J_q^*)^{-1}$ where J_q^* is the duality map of E^*.

Definition 1.1. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable convex function, the Bregman distance with respect to f is defined as:

$$\Delta_f(x, y) := f(y) - f(x) - \langle f'(x), y - x \rangle, \; \forall x, y \in E.$$

The duality mapping J_p is the derivative of the function $f_p(x) = (\frac{1}{p})||x||^p$. Given that $f = f_p$, the Bregman distance with respect to f_p now becomes

$$\begin{aligned} \Delta_p(x, y) &= \frac{1}{p} (||y||^p - ||x||^p) + \langle J_p x, x - y \rangle \\ &= \frac{1}{q} (||x||^p - ||y||^p) - \langle J_p x - J_p y, y \rangle. \end{aligned} \tag{1.3}$$

We note that the Bregman distance is not symmetric, therefore it is not a metric but it posses the following properties

$$\Delta_p(x,y) = \Delta_p(x,z) + \Delta_p(z,y) + \langle z-y, J_px - J_py\rangle,\ \forall x,y,z \in E. \tag{1.4}$$

$$\Delta_p(x,y) + \Delta_p(y,x) = \langle x-y, J_px - J_py\rangle,\ \forall x,y \in E. \tag{1.5}$$

For any p-uniformly convex Banach space E, the metric and Bregman distance have the following relation:

$$k||x-y||^p \leq \Delta_p(x,y) \leq \langle x-y, J_px - J_py\rangle,$$

where $k > 0$ is a fixed number.

Definition 1.2. Let C be a nonempty, closed and convex subset of int dom$f$, where int dom$f$ denotes the interior domain of f. Let $T : C \to C$ be any mapping, a point $p \in C$ is called a fixed point of T if $Tp = p$. While $p \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of fixed points of T and asymptotic fixed points of T are denoted by $F(T)$ and $\hat{F}(T)$ respectively. It is clear that $F(\hat{T}) \subset F(T)$.
A mapping $T : C \to C$ is said to be

(i) *right Bregman firmly nonexpansive* if

$$\langle J_p^E(Tx) - J_p^E(Ty), Tx - Ty\rangle \leq \langle J_p^E(Tx) - J_p^E(Ty), x - y\rangle,\ \forall x,y \in C,$$

equivalently,

$$\Delta_p(Tx,Ty) + \Delta_p(Ty,Tx) + \Delta_p(x,Tx) + \Delta_p(y,Ty) \leq \Delta_p(x,Ty) + \Delta_p(y,Tx),$$

(ii) *right Bregman strongly nonexpansive* (see [14]) with respect to a nonempty $\hat{F}(T)$ if

$$\Delta_p(Tx,y) \leq \Delta_p(x,y),\ \forall x \in C,\ y \in \hat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $y \in \hat{F}(T)$ and

$$\lim_{n\to\infty} (\Delta_p(x_n,y) - \Delta_p(Tx_n,y)) = 0,$$

it follows that

$$\lim_{n\to\infty} \Delta_p(x_n,Tx_n) = 0.$$

Remark 1.2. [14]. Every right Bregman firmly nonexpansive mapping is right Bregman strongly nonexpansive mapping with respect to $F(T) = \hat{F}(T)$.

Let E^* be the dual space of a Banach space E. A mapping $M : E \to 2^{E^*}$ is called monotone if

$$\langle \xi - \eta, x - y\rangle \geq 0\ \forall x,y \in E,\ \xi \in M(x), \eta \in M(y). \tag{1.6}$$

A monotone mapping M is said to be maximal if the graph of M denoted by $G(M)$ is not properly contained in the graph of any other monotone mapping. We recall that for any set-valued operator M, the graph of M is defined by

$$G(T) = \{(x,y) : y \in M(x)\}. \tag{1.7}$$

Let E be a $p-$uniformly convex Banach space. The resolvent of M is the operator $\text{Res}_p^{\lambda M} : E \to 2^E$ defined by

$$\text{Res}_p^{\lambda M} := \left(J_p^E + \lambda M\right)^{-1} \circ J_p^E,\ \lambda > 0. \tag{1.8}$$

The resolvent operator $\text{Res}_p^{\lambda M}$ is a Bregman firmly nonexpansive operator. Furthermore, $0 \in M(x)$ if and only if $x = \text{Res}_p^{\lambda M}(x)$ (see e.g., [24], for more details).

The Bregman projection $\prod_C$ is defined by

$$\prod_C x = \arg\min_{y\in C} \Delta_p(x,y),\ \forall\, x\in E, \tag{1.9}$$

which is a unique minimizer of the Bregman distance.
Let C and Q be nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 respectively. Let $A: E_1 \to E_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) is to find a point

$$x\in C \text{ such that } Ax\in Q. \tag{1.10}$$

The SFP was first introduced in 1994 by Censor and Elvin [5] in finite-dimensional spaces. The SFP is known to have wide applications in many fields, such as phases retrieval, medical image reconstruction, signal processing and radiation therapy treatment (for example, see [3, 4, 5, 6, 20, 21, 22] and the references therein).
Byrne [2] applied the forward-backward method, a type of projected gradient method to present the so-called CQ-iterative procedure for approximating a solution of (1.10) in Hilbert spaces, which is defined by

$$x_{n+1} = P_C(I-\gamma A^*(I-P_Q)A)x_n,\ n\in\mathbb{N}, \tag{1.11}$$

where $\gamma\in(0,\frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A. The approximation of solutions of problem (1.10) has also been studied by numerous authors in both finite and infinite dimensional Hilbert spaces (see for examples, [2, 6, 8, 16, 17, 18, 28, 29, 34, 35, 36, 38, 39]).
The SFP has been extended from the setting of Hilbert spaces to more general Banach spaces by many authors. Schopfer *et al.* [26] introduced and studied the following algorithm (which is a generalization of algorithm (1.11)) for solving the SFP (1.10) in p-uniformly smooth Banach spaces: For any $x_0\in E_1$ and $n\geq 0$,

$$x_{n+1} = \prod_C J^*\left[J_p^{E_1}(x_n) - tA^*J_p^{E_2}(Ax_n - P_Q(Ax_n))\right], \tag{1.12}$$

where A^* is the adjoint of a bounded linear operator A, t is any positive real number, P_Q is the metric projection onto Q and C, Q are nonempty, closed and convex subsets of E_1, E_2 respectively. They obtained weak convergence result under the assumption that the duality mapping of E is sequentially weak-to-weak continuous.
Wang [33] modified Algorithm (1.12) and obtained strong convergence result for the following Multiple-Sets Split Feasibility Problem (MSSFP): Find

$$x\in \cap_{i=1}^r C_i \text{ such that } Ax\in\cap_{j=1+r}^{r+s} Q_j, \tag{1.13}$$

where $r,\ s$ are two given integers, $C_i,\ i=1,2,3,\dots,r$ are closed convex subsets of E_1 and $Q_j,\ j=r+1,\dots,r+s$ are closed convex subsets of E_2. He introduced the following algorithm: For any $x_0\in E_1$, define $\{x_n\}$ by

$$\begin{cases} y_n = T_n x_n; \\ D_n = \{u\in E_1 : \Delta_p(y_n,u)\leq \Delta_p(x_n,u)\}; \\ E_n = \{u\in E_1 : \langle x_n - u, J_p^{E_1}x_0 - J_p^{E_1}x_n\rangle \geq 0\}; \\ x_{n+1} = \prod_{D_n\cap E_n}(x_0); \end{cases} \tag{1.14}$$

where T_n is defined for each $n\in\mathbb{N}$ by

$$T_n(x) = \begin{cases} \prod_{C_{i(n)}}(x); & 1\leq \mathrm{i}(n)\leq r; \\ J_q^{E_1^*}\left[J_p^{E_1}x - t_nA^*J_p^{E_2}(I-P_{Q_{i(n)}})Ax\right]; & r+1\leq \mathrm{i}(n)\leq r+s; \end{cases} \tag{1.15}$$

where $\mathrm{i} : \mathbb{N} \to I$ is the cyclic control mapping

$$\mathrm{i}(n) = n \bmod (r+s) + 1$$

and t_n satisfies

$$0 < t \le t_n \le \left(\frac{q}{C_q||A||^p}\right)^{\frac{1}{q-1}}. \tag{1.16}$$

Very recently, Shehu *et al.* [28] introduced and studied the following iterative algorithm for approximating a common solution of SFP and fixed point problems for right Bregman strongly nonexpansive mappings in $p-$uniformly convex Banach spaces which are also uniformly smooth: For a fixed $u \in C$, $u_0 \in E_1$, define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ recursively by

$$\begin{cases} x_n = \prod_C J_q^{E_1^*}\left[J_p^{E_1}(u_n) - t_n A^* J_p^{E_2}(I - P_Q)Au_n\right]; \\ u_{n+1} = \prod_C J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)\right]; \end{cases} \tag{1.17}$$

where $0 < t \le t_n \le k < \left(\frac{q}{C_q||A||^p}\right)^{\frac{1}{q-1}}$. They established strong convergence of algorithm (1.17) under some suitable conditions.

An important generalization of the SFP is the Split Equality Fixed Point Problem (SEFPP) which is to find

$$x \in C := F(T),\ y \in Q := F(S) \text{ such that } Ax = By, \tag{1.18}$$

where $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are two bounded linear operators, C and Q are two nonempty, closed and convex subsets of E_1 and E_2 respectively, $F(T)$ and $F(S)$ denotes the sets of fixed points of the operators T and S defined on E_1 and E_2 respectively, E_1, E_2 and E_3 are real Banach spaces. Note that if $E_2 = E_3$ and $B = I$ (where I is the identity map on E_2) in (1.18), then problem (1.18) reduces to problem (1.10). Moudafi and Al-Shemas [19] introduced the SEFPP in Hilbert spaces. They proposed the following algorithm for solving the SEFPP:

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = S(y_n + \gamma_n B^*(Ax_n - By_n)),\ \forall n \ge 1; \end{cases} \tag{1.19}$$

where $T : H_1 \to H_1,\ S : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3,\ B : H_2 \to H_3$ are two bounded linear operators, A^*, B^* are the adjoints of A, B respectively, $\{\gamma_n\} \subset \left(\epsilon, \frac{2}{\lambda_{A^*A} + \lambda_{B^*B}} - \epsilon\right)$, λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B respectively. Furthermore, they established the weak convergence result for problem (1.18) using algorithm (1.19). Since then, many authors have studied the SEFPP in Hilbert spaces for mappings more general than the firmly quasi-nonexpansive mappings. (for example, see [10, 11, 12, 31] and the references therein). In this paper, we study the following problem: Let E_1, E_2 and E_3 be $p-$uniformly convex Banach spaces which are also uniformly smooth and $A : E_1 \to E_3$, $B : E_2 \to E_3$ be bounded linear operators. Let $M_i : E_1 \to 2^{E_1^*},\ N_i : E_2 \to 2^{E_2^*}$ $i = 1, 2, ..., m$ be multivalued maximal monotone mappings and $T : E_1 \to E_1$, $S : E_2 \to E_2$ be right Bregman strongly nonexpansive mappings: Find $\bar{x} \in F(T)$ and $\bar{y} \in F(S)$ such that

$$0 \in M_i(\bar{x}), \tag{1.20}$$

$$0 \in N_i(\bar{y}) \text{ and } A\bar{x} = B\bar{y}. \tag{1.21}$$

Furthermore, motivated by the recent work of Shehu *et al.* [28] we propose a new iterative algorithm and using the algorithm, we state and prove a strong convergence result for the approximation of a solution of problem (1.20)-(1.21).

2. PRELIMINARY

We state some known and useful results which will be needed in the proof of our main theorem. In what follows, we shall denote the solution set of problem (1.20)-(1.21) by Γ defined by $\Gamma := \{(\bar{x}, \bar{y}) \in F(T) \times F(S)$ such that $0 \in \cap_{i=1}^m M_i(\bar{x}),\ 0 \in \cap_{i=1}^m N_i(\bar{y})$ and $A\bar{x} = B\bar{y}\}$.

Lemma 2.1. [9] *Let E be a Banach space and $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$||x-y||^q \le ||x||^q - q\langle J_p^E(x), y\rangle + C_q||y||^q. \tag{2.1}$$

From [1] and [7], we make use of the function $V_p : E^* \times E \to [0, +\infty)$ which is defined by

$$V_p(x,y) := \frac{1}{q}||x||^q - \langle x, y\rangle + \frac{1}{p}||y||^p,\ \forall x \in E^*, y \in E. \tag{2.2}$$

It then follows that V_p is nonnegative and $V_p(x,y) = \Delta_p(J_q^{E^*}(x), y)$ for all $x \in E^*$ and $y \in E$. Also, by the subdifferential inequality, we have

$$V_p(x^*, x) + \langle y^*, J_p^{E^*}(x^*) - x\rangle \le V_p(x^*+y^*, x),\ \forall x \in E,\ x^*, y^* \in E^* \tag{2.3}$$ (see [28] for more details).

Furthermore, we have that $f^* = f_p^*$ is a proper weak* lower semi-continuous and convex function since $f = f_p$ is a proper lower semi-continuous and convex function (see [23]). Thus for all $z \in E$, we have

$$\Delta_p\left(J_q^{E^*}\left(\sum_{i=1}^N t_i J_p^E(x_i)\right), z\right) = V_p\left(\sum_{i=1}^N t_i J_p^E(x_i), z\right) \le \sum_{i=1}^N t_i \Delta_p(x_i, z), \tag{2.4}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$ (see [27, 32, 28] for more details).

Lemma 2.2. [35]. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n,\ n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\Sigma_{n=0}^\infty \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ or $\Sigma_{n=0}^\infty |\delta_n\gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. [13]. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} < a_{n_j+1}\ \forall j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}.$$

In fact, $m_k = \max\{i \le k : a_i < a_{i+1}\}$.

Lemma 2.4. [15] *Let H be a Hilbert space, $f : H \to H$ a contraction with coefficient $0 < \alpha < 1$, and A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \bar{\gamma}/\alpha$,*

$$\langle x-y, (A-\gamma f)x - (A-\gamma f)y\rangle \ge (\bar{\gamma} - \gamma\alpha)\|x-y\|^2,\ \ x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

3. MAIN RESULT

Theorem 3.1. *Let* E_1, E_2 *and* E_3 *be three* p*-uniformly convex Banach space which are also uniformly smooth and* $A : E_1 \to E_3,\ B : E_2 \to E_3$ *be two bounded linear operators. Let* $M_i : E_1 \to 2^{E_1^*},\ N_i : E_2 \to 2^{E_2^*}\ i = 1, 2, ..., m$ *be multivalued maximal monotone mappings and* $T : E_1 \to E_1$, $S : E_2 \to E_2$ *be right Bregman strongly nonexpansive mappings such that* $F(T) = \hat{F}(T)$ *and* $F(S) = \hat{F}(S)$*. Suppose that* $\Gamma \neq \emptyset$ *and* $\{\alpha_n\}$, $\{\beta_n\}$*, are sequences in* $(0, 1)$*. Let* $u, x_0 \in E_1$ *and* $v, y_0 \in E_2$ *be arbitrary and the sequence* $\{(x_n, y_n)\}$ *be generated by*

$$(3.1)\qquad \begin{cases} v_n = \Psi^m J_q^{E_2}\left[J_q^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)\right];\\ u_n = \Phi^m J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right];\\ w_n = J_q^{E_2^*}\left[(1-\beta_n)J_p^{E_2}(v_n) + \beta_n J_p^{E_2} S v_n\right];\\ z_n = J_q^{E_1^*}\left[(1-\beta_n)J_p^{E_1}(u_n) + \beta_n J_p^{E_1} T u_n\right];\\ y_{n+1} = J_q^{E_2^*}\left[\alpha_n J_p^{E_2}(v) + (1-\alpha_n)J_p^{E_2} w_n\right];\\ x_{n+1} = J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1} z_n\right],\ n \geq 1, \end{cases}$$

where $\Phi^m = Res_p^{\lambda M_m} \circ Res_p^{\lambda M_{m-1}} \circ \cdots \circ Res_p^{\lambda M_1}$, $\Phi^0 = I$ *and* $\Psi^m = Res_p^{\lambda N_m} \circ Res_p^{\lambda N_{m-1}} \circ \cdots \circ Res_p^{\lambda N_1}$, $\Psi^0 = I$*, with conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < t \leq t_n \leq k \leq \left(\frac{q}{2C_q||A||^q}\right)^{\frac{1}{q-1}}$, $0 < t \leq t_n \leq k \leq \left(\frac{q}{2D_q||B||^q}\right)^{\frac{1}{q-1}}$,
(iv) $\beta_n \in (a, b)$ *for some* $a, b \in (0, 1)$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. Let $(x^*, y^*) \in \Gamma$, $a_n = J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right]$ and $b_n = J_q^{E_2^*}\left[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)\right]$. Then, using (1.8), (3.1) and Lemma (2.1), we have

$$\begin{aligned}\Delta_p(u_n,x^*) &= \Delta_p(\Phi^m J_q^{E_1^*}\left[J_p^{E_1}(x_n)-t_nA^*J_p^{E_3}(Ax_n-By_n)\right],x^*)\\ &\le \Delta_p(\Phi^{m-1} J_q^{E_1^*}\left[J_p^{E_1}(x_n)-t_nA^*J_p^{E_3}(Ax_n-By_n)\right],x^*)\\ &\vdots\\ &\le \Delta_p(J_q^{E_1^*}\left[J_p^{E_1}(x_n)-t_nA^*J_p^{E_3}(Ax_n-By_n)\right],x^*)\\ &= \frac{1}{q}||J_p^{E_1}(x_n)-t_nA^*J_p^{E_3}(Ax_n-By_n)||^q-\langle J_p^{E_1}(x_n),x^*\rangle\\ &\quad +t_n\langle Ax^*,J_p^{E_3}(Ax_n-By_n)\rangle+\frac{1}{p}||x^*||^p\\ &\le \frac{1}{q}||J_p^{E_1}(x_n)||^q-t_n\langle Ax_n,J_p^{E_3}(Ax_n-By_n)\rangle+\frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n-By_n)||^q\\ &\quad -\langle J_p^{E_1}(x_n),x^*\rangle+\frac{1}{p}||x^*||^p+t_n\langle Ax^*,J_P^{E_3}(Ax_n-By_n)\rangle\\ &= \frac{1}{q}||x_n||^p-\langle J_p^{E_1}(x_n),x^*\rangle+\frac{1}{p}||x^*||^p+t_n\langle Ax^*-Ax_n,J_p^{E_3}(Ax_n-By_n)\rangle\\ &\quad +\frac{C_q(t_n||A||)^q}{q}||(Ax_n-By_n)||^p\\ &= \Delta_p(x_n,x^*)+t_n\langle J_p^{E_3}(Ax_n-By_n),Ax^*-Ax_n\rangle+\frac{C_q(t_n||A||)^q}{q}||Ax_n-By_n||^p. \quad (3.2)\end{aligned}$$

Similarly, from (3.1) and Lemma (2.1), we have

$$\begin{aligned}\Delta_p(v_n,y^*) &\le \Delta_p(y_n,y^*)-t_n\langle J_p^{E_3}(Ax_n-By_n),By^*-By_n\rangle\\ &\quad +\frac{D_q(t_n||B||)^q}{q}||Ax_n-By_n||^p. \quad (3.3)\end{aligned}$$

Adding (3.2) and (3.3) and using the fact that $Ax^*=By^*$, we have

$$\begin{aligned}\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*) &\le \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)-t_n\langle J_p^{E_3}(Ax_n-By_n),Ax_n-By_n\rangle\\ &\quad +\frac{C_q(t_n||A||)^q}{q}||Ax_n-By_n||^p+\frac{D_q(t_n||B||)^q}{q}||Ax_n-By_n||^p\\ &= \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\\ &\quad -\left[t_n-\left(\frac{C_q(t_n||A||)^q}{q}+\frac{D_q(t_n||B||)^q}{q}\right)\right]||Ax_n-By_n||^p. \quad (3.4)\end{aligned}$$

Using condition (iii) in (3.4), we have

$$\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*) \le \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*). \quad (3.5)$$

From (3.1) and (2.4), we have

$$\begin{aligned}\Delta_p(z_n,x^*) &= \Delta_p\left(J_q^{E_1^*}\left[(1-\beta_n)J_p^{E_1^*}u_n+\beta_nJ_p^{E_1^*}Tu_n\right],x^*\right)\\ &\le (1-\beta_n)\Delta_p(u_n,x^*)+\beta_n\Delta_p(Tu_n,x^*)\\ &\le (1-\beta_n)\Delta_p(u_n,x^*)+\beta_n\Delta_p(Tu_n,x^*)\\ &\le \Delta_p(u_n,x^*). \quad (3.6)\end{aligned}$$

Similarly, we have

$$\Delta_p(w_n,y^*)\le\Delta_p(v_n,y^*). \quad (3.7)$$

From (3.1) and (3.6) and (2.4), we have

$$\begin{aligned}\Delta_p(x_{n+1},x^*) &= \Delta_p\left(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}z_n\right],x^*\right)\\ &\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(z_n,x^*)\\ &\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(u_n,x^*)\end{aligned}$$

(3.8)

Similarly, from (3.1) and (3.7), we have

$$\Delta_p(y_{n+1},y^*) \le \alpha_n\Delta_p(v,y^*)+(1-\alpha_n)\Delta_p(v_n,y^*). \tag{3.9}$$

Adding (3.8) and (3.9) and using (3.5), we have

$$\begin{aligned}\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*) &\le \alpha_n\left[\Delta_p(u,x^*)+\Delta_p(v,y^*)\right]+(1-\alpha_n)\left[\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\right]\\ &\le \alpha_n\left[\Delta_p(u,x^*)+\Delta_p(v,y^*)\right]+(1-\alpha_n)\left[\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\right]\\ &\le \max\{\Delta_p(u,x^*)+\Delta_p(v,y^*),\ \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\}\\ &\ \ \vdots\\ &\le \max\{\Delta_p(u,x^*)+\Delta_p(v,y^*),\ \Delta_p(x_0,x^*)+\Delta_p(y_0,y^*)\}.\end{aligned} \tag{3.10}$$

Therefore, $\{\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\}$ is bounded and consequently, $\{\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{Ax_n\}$ and $\{By_n\}$ are all bounded.
Also, from (3.1) and inequality (2.3) with $y^*=-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(x^*))$, we obtain

$$\begin{aligned}\Delta_p(x_{n+1},x^*) &= \Delta_p\left(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}(z_n)\right],x^*\right)\\ &= V_p\left(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}(z_n),x^*\right)\\ &\le V_p\left(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}(z_n)-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(x^*)),x^*\right)\\ &\quad -\langle-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(x^*)),J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}(z_n)\right]-x^*\rangle\\ &= V_p\left(\alpha_n J_p^{E_1}(x^*)+(1-\alpha_n)J_p^{E_1}(z_n),x^*\right)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\ &= \Delta_p\left(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(x^*)+(1-\alpha_n)J_p^{E_1}(z_n)\right],x^*\right)\\ &\quad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\ &\le \alpha_n\Delta_p(x^*,x^*)+(1-\alpha_n)\Delta_p(z_n,x^*)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\ &\le (1-\alpha_n)\Delta_p(z_n,x^*)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\ &\le (1-\alpha_n)\Delta_p(u_n,x^*)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle.\end{aligned} \tag{3.11}$$

Similarly, we have

$$\Delta_p(y_{n+1},y^*) \le (1-\alpha_n)\Delta_p(v_n,y^*)+\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle. \tag{3.12}$$

Adding (3.11) and (3.12), we have

$$\begin{aligned}\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*) &\le (1-\alpha_n)\left[\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\right]\\ &\quad +\alpha_n\big[\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\ &\quad +\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle\big]\\ &\le (1-\alpha_n)\left[\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\right]\\ &\quad +\alpha_n\big[\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\ &\quad +\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle\big].\end{aligned} \tag{3.13}$$

We now consider two cases to establish the strong convergence of $\{(x_n,y_n)\}$ to $(\bar{x},\bar{y})$.

Case 1. Suppose that $\{\Delta_p(x_n, x^*)+\Delta_p(y_n, y^*)\}$ is monotone non-increasing, then $\{\Delta_p(x_n, x^*)+\Delta_p(y_n, y^*)\}$ is convergent. Thus,

$$\lim_{n\to\infty}\left[(\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*))\right] = 0.$$

From (3.6), (3.7), (3.8) and (3.9), we have

$$\begin{aligned} 0 &\leq (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(w_n, x^*) + \Delta_p(z_n, y^*)) \\ &= (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\ &\quad + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(w_n, x^*) + \Delta_p(z_n, y^*)) \\ &\leq (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\ &\quad +\alpha_n\left(\Delta_p(u, x^*) + \Delta_p(v, y^*)\right) + (1-\alpha_n)\left(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)\right) \\ &\quad - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty, \end{aligned}$$

which implies

$$\lim_{n\to\infty}(\Delta_p(u_n, x^*) - \Delta_p(z_n, x^*)) = \lim_{n\to\infty}(\Delta_p(v_n, y^*) - \Delta_p(w_n, y^*)) = 0. \tag{3.14}$$

Also, from the definition of z_n, we have

$$\begin{aligned} \Delta_p(z_n, x^*) &= \Delta_p\left(J_q^{E_1^*}\left((1-\beta_n)J_p^{E_1}(u_n) + \beta_n J_p^{E_1}(Tu_n)\right), x^*\right) \\ &\leq (1-\beta_n)\Delta_p(u_n, x^*) + \beta_n\Delta_p(Tu_n, x^*) \\ &= \Delta_p(u_n, x^*) + \beta_n\left[\Delta_p(Tu_n, x^*) - \Delta_p(u_n, x^*)\right]. \end{aligned} \tag{3.15}$$

Also, from (3.14), (3.15) and condition (iv) we obtain

$$\beta_n\left(\Delta_p(u_n, x^*) - \Delta_p(Tu_n, x^*)\right) \leq \Delta_p(u_n, x^*) - \Delta_p(z_n, x^*) \to 0, \text{ as } n \to \infty. \tag{3.16}$$

Since $\{\beta_n\}$ is bounded (see condition (iv)), we have

$$\lim_{n\to\infty}(\Delta_p(u_n, x^*) - \Delta_p(Tu_n, x^*)) = 0. \tag{3.17}$$

Similarly, we have

$$\lim_{n\to\infty}(\Delta_p(v_n, y^*) - \Delta_p(Sv_n, y^*)) = 0. \tag{3.18}$$

Since T and S are right Bregman strongly nonexpansive mappings, then from (3.17) and (3.18), we have

$$\lim_{n\to\infty}\Delta_p(Tu_n, u_n) = 0$$

and

$$\lim_{n\to\infty}\Delta_p(Sv_n, v_n) = 0$$

respectively, which implies

$$\lim_{n\to\infty}||Tu_n - u_n|| = 0 \tag{3.19}$$

and

$$\lim_{n\to\infty}||Sv_n - v_n|| = 0. \tag{3.20}$$

From (3.4) and (3.11), we have

$$\begin{aligned}
&\left[t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right] ||Ax_n - By_n||^p \\
\leq\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
=\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
\leq\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +(1-\alpha_n)\left(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)\right) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
& +\alpha_n \left[\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle\right] \\
=\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +\alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle - \Delta_p(u_n, x^*)) \\
& +\alpha_n(\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle - \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty.
\end{aligned}$$

That is,

$$\lim_{n\to\infty} \left[t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right] ||Ax_n - By_n||^p = 0.$$

Since $0 < t\left(1 - \left(\frac{C_q k^{q-1}(||A||)^q}{q} + \frac{D_q k^{q-1}(||B||)^q}{q}\right)\right) \leq \left(t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right)$, we have

(3.21) $$\lim_{n\to\infty} ||Ax_n - By_n||^p = 0.$$

From the definitions of a_n and b_n, we have

$$\begin{aligned}
||J_p^{E_1} a_n - J_p^{E_1} x_n|| &= ||J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n) - J_p^{E_1}(x_n)|| \\
&\leq t_n ||A^*|| ||J_p^{E_3}(Ax_n - By_n)|| \\
&\leq \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}} ||A^*|| ||Ax_n - By_n|| \to 0,\ n \to \infty.
\end{aligned}$$

Since $J_p^{E_1^*}$ is norm to norm uniformly continuous on bounded subsets of E_1^*, we have

(3.22) $$\lim_{n\to\infty} ||a_n - x_n|| = 0.$$

Similarly, we have

(3.23) $$\lim_{n\to\infty} ||b_n - y_n|| = 0.$$

Since $\text{Res}_p^{\lambda M_m}$ is a right Bregman firmly nonexpansive mapping, we have
$\Delta_p\left(\text{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n), x^*\right) + \Delta_p\left(x^*, \text{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n)\right) + \Delta_p(\Phi^{m-1}a_n, \text{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n)) + \Delta_p(x^*, x^*) \leq \Delta_p(\Phi^{m-1}a_n, x^*) + \Delta_p\left(x^*, \text{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n)\right)$,
which implies

$$\Delta_p\left(\text{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n), x^*\right) + \Delta_p\left(\Phi^{m-1}a_n, \text{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n)\right) \leq \Delta_p(\Phi^{m-1}a_n, x^*).$$

That is,

(3.24) $$\Delta_p(\Phi^{m-1}a_n, \Phi^m a_n) \leq \Delta_p(\Phi^{m-1}a_n, x^*) - \Delta_p(\Phi^m a_n, x^*).$$

Similarly, we have

$$\Delta_p(\Psi^N b_n, \Psi^{N-1} b_n) \;\leq\; \Delta_p(\Psi^{N-1} b_n, y^*) - \Delta_p(\Psi^N b_n, y^*). \tag{3.25}$$

Adding (3.24) and (3.25), we have

$$\begin{aligned}
& \Delta_p(\Phi^{m-1} a_n, \Phi^m a_n) + \Delta_p(\Psi^{N-1} b_n, \Psi^N b_n) \\
\leq\; & \Delta_p(\Phi^{m-1} a_n, x^*) + \Delta_p(\Psi^{N-1} b_n, y^*) - \big(\Delta_p(\Phi^m a_n, x^*) + \Delta_p(\Psi^N b_n, y^*)\big) \\
\vdots\; & \\
\leq\; & \Delta_p(a_n, x^*) + \Delta_p(b_n, y^*) - \big(\Delta_p(\Phi^m a_n, x^*) + \Delta_p(\Psi^N b_n, y^*)\big) \qquad (3.26) \\
\leq\; & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - \big(\Delta_p(\Phi^m a_n, x^*) + \Delta_p(\Psi^N b_n, y^*)\big) \\
=\; & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
\leq\; & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (1-\alpha_n)(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
& + \alpha_n \left[\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle\right] \\
=\; & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + \alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle - \Delta_p(u_n, x^*)) \\
& + \alpha_n(\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle - \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \Delta_p(\Phi^{m-1} a_n, \Phi^m a_n) = \lim_{n\to\infty} \Delta_p(\Psi^{N-1} b_n, \Psi^N b_n) = 0. \tag{3.27}$$

By the same argument as (3.24)-(3.26), we have

$$\begin{aligned}
& \Delta_p(\Phi^{m-2} a_n, \Phi^{m-1} a_n) + \Delta_p(\Psi^{N-2} b_n, \Psi^{N-1} b_n) \\
\leq\; & \Delta_p(a_n, x^*) + \Delta_p(b_n, y^*) - \big(\Delta_p(\Phi^{m-1} a_n, x^*) + \Delta_p(\Psi^{N-1} b_n, y^*)\big) \\
\leq\; & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - \big(\Delta_p(\Phi^m a_n, x^*) + \Delta_p(\Psi^N b_n, y^*)\big) \\
=\; & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
\leq\; & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (1-\alpha_n)(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
& + \alpha_n \left[\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle\right] \\
=\; & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + \alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle - \Delta_p(u_n, x^*)) \\
& + \alpha_n(\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle - \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty, \qquad (3.28)
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \Delta_p(\Phi^{m-2} a_n, \Phi^{m-1} a_n) = \lim_{n\to\infty} \Delta_p(\Psi^{N-2} b_n, \Psi^{N-1} b_n) = 0. \tag{3.29}$$

Continuing in the same manner, we have that

$$\lim_{n\to\infty} \Delta_p(\Phi^{m-3} a_n, \Phi^{m-2} a_n) = \cdots = \lim_{n\to\infty} \Delta_p(a_n, \Phi^1 a_n) = 0, \tag{3.30}$$

and

$$\lim_{n\to\infty} \Delta_p(\Psi^{N-3} a_n, \Psi^{N-2} a_n) = \cdots = \lim_{n\to\infty} \Delta_p(a_n, \Phi^1 a_n) = 0. \tag{3.31}$$

From (3.27), (3.29), (3.30) and (3.31), we can conclude that

$$\lim_{n\to\infty} \Delta_p(\Phi^{l-1}a_n, \Phi^l a_n) = 0,\ l = 1, 2, \ldots, m,$$

$$\lim_{n\to\infty} \Delta_p(\Psi^{r-1}b_n, \Psi^r b_n) = 0,\ r = 1, 2, \ldots, N.$$

Which implies

$$\lim_{n\to\infty} ||\Phi^l a_n - \Phi^{l-1}a_n|| = 0,\ l = 1, 2, \ldots, m, \tag{3.32}$$

$$\lim_{n\to\infty} ||\Psi^r b_n - \Psi^{r-1}b_n|| = 0,\ r = 1, 2, \ldots, N. \tag{3.33}$$

Also, we have that

$$\lim_{n\to\infty} \|a_n - \Phi(a_n)\| = 0.$$

Hence,

$$\lim_{n\to\infty} ||a_n - u_n|| \leq \lim_{n\to\infty} \left[||a_n - \Phi^1 a_n|| + ||\Phi^1 a_n - \Phi^2 a_n|| + \cdots + ||\Phi^{m-1}a_n - u_n||\right] \to 0, \text{ as } n \to \infty.$$

That is,

$$\lim_{n\to\infty} ||a_n - u_n|| = 0. \tag{3.34}$$

Similarly, we obtain

$$\lim_{n\to\infty} ||b_n - v_n|| = 0. \tag{3.35}$$

From (3.22) and (3.34), we have

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{3.36}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_n - v_n|| = 0. \tag{3.37}$$

Also we have

$$\begin{aligned} \|x_n - Tx_n\| &= \|x_n - u_n + u_n - Tu_n + Tu_n - Tx_n\| \\ &\leq \|x_n - u_n\| + \|u_n - Tu_n\| + \|Tu_n - Tx_n\| \\ &\leq 2\|u_n - x_n\| + \|u_n - Tu_n\|. \end{aligned}$$

Hence from (3.19) and (3.36), we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.38}$$

Similarly, from (3.20) and (3.37) we obtain

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \tag{3.39}$$

Since $J_p^{E_1}$ and $J_p^{E_2}$ are uniformly continuous on bounded subsets of E_1 and E_2 respectively, we have from (3.32) and (3.33) that

$$\lim_{n\to\infty} ||J_p^{E_1}\Phi^l a_n - J_p^{E_1}\Phi^{l-1}a_n|| = 0,\ l = 1, 2, \ldots, m, \tag{3.40}$$

and

$$\lim_{n\to\infty} ||J_p^{E_2}\Psi^r b_n - J_p^{E_2}\Psi^{r-1}b_n|| = 0,\ r = 1, 2, \ldots, N. \tag{3.41}$$

Since $\{x_n\}$ is bounded in E_1 and E_1 is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $\bar{x}$. By (3.38), we have that $\bar{x} \in F(T)$ since $F(T) = \hat{F}(T)$. Also since $\{y_n\}$

is bounded in E_2 and E_2 is reflexive, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ that converges weakly to $\bar{y}$. By (3.39), we have that $\bar{y} \in F(S)$ since $F(S) = \hat{F}(S)$.
Next, we show that $0 \in M_l(\bar{x})$ and $0 \in M_r(\bar{y})$, for each $l = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, N$.
Let $i \in \{1, 2, 3, ...m\}$. Let $(z, \eta) \in G(M_l)$, then $\eta \in M_l z$., From $\Phi^l a_n = \text{Res}_p^{\lambda M_l}(\Phi^{l-1} a_n)$, we have that

$$J_p^{E_1} \Phi^{l-1} a_n \in \left(J_p^{E_1} + \lambda M_l\right) \Phi^l a_n,$$

which implies

$$\frac{1}{\lambda}\left(J_p^{E_1} \Phi^{l-1} a_n - J_p^{E_1} \Phi^l a_n\right) \in M_l \Phi^l a_n.$$

By the monotonicity of M_l, for each $l = 1, 2, \ldots, m$, we have

$$\langle \eta - \frac{1}{\lambda}\left(J_p^{E_1} \Phi^{l-1} a_n - J_p^{E_1} \Phi^l a_n\right), z - \Phi^l a_n \rangle \geq 0.$$

This implies

$$\langle \eta, z - \Phi^l a_n \rangle \geq \left\langle \frac{1}{\lambda}\left(J_p^{E_1} \Phi^{l-1} a_n - J_p^{E_1} \Phi^l a_n\right), z - \Phi^{l-1} a_n \right\rangle.$$

Since $\{x_n\}$ converges weakly to $\bar{x}$, we have from (3.40) and (3.36) that

$$\langle \eta, z - \bar{x} \rangle \geq 0.$$

Hence, by the maximal monotonicity of M_l, we have that $0 \in M_l(\bar{x})$. Since i was arbitrary, we have $0 \in \cap_{l=1}^m M_l(\bar{x})$.
By similar argument, we obtain that $0 \in \cap_{r=1}^N M_r(\bar{y})$.
We now show that $A\bar{x} = B\bar{y}$.
Since $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators,and $\{x_n\}$ and $\{y_n\}$ converges weakly to $\bar{x}$ and $\bar{y}$, respectively we have that for arbitrary $f \in E_3^*$,

$$f(Ax_n) = (f \circ A)(x_n) \to (f \circ A)(\bar{x}) = f(A\bar{x}).$$

Similarly

$$f(Bx_n) = (f \circ B)(y_n) \to (f \circ B)(\bar{y}) = f(B\bar{y}).$$

This convergence implies that

$$Ax_n - By_n \rightharpoonup A\bar{x} - B\bar{y}.$$

Also, by weakly semi-continuity of the norm, it follows that

$$||A\bar{x} - B\bar{y}|| \leq \liminf_{n\to\infty} ||Ax_n - By_n|| = 0. \tag{3.42}$$

That is, $A\bar{x} = B\bar{y}$. Therefore $(\bar{x}, \bar{y}) \in \Gamma$.
We now show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

$$\Delta_p(z_n, u_n) \leq (1 - \beta_n)\Delta_p(u_n, u_n) + \beta_n \Delta_p(Tu_n, u_n) \to 0, \text{ as } n \to \infty. \tag{3.43}$$

Also, we have

$$\begin{aligned} \Delta_p(x_{n+1}, u_n) &= \Delta_p\left(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + (1 - \alpha_n) J_p^{E_1} z_n\right], u_n\right) \\ &\leq \alpha_n \Delta_p(u, u_n) + (1 - \alpha_n)\Delta_p(z_n, u_n) \to 0, \text{ as } n \to \infty, \end{aligned} \tag{3.44}$$

which implies

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = 0. \tag{3.45}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_{n+1} - v_n|| = 0. \tag{3.46}$$

From (3.36) and (3.45), we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.47}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0. \tag{3.48}$$

From (3.13), we have

$$\begin{aligned} \Delta_p(x_{n+1}, \bar{x}) + \Delta_p(y_{n+1}, \bar{y}) &\leq (1-\alpha_n)\left[\Delta_p(x_n, \bar{x}) + \Delta_p(y_n, \bar{y})\right] \\ &\quad +\alpha_n \left[\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\right]. \end{aligned} \tag{3.49}$$

Using Lemma (2.2) in (3.49), we conclude that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

Case 2: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\Delta_p(x_{n_i}, x^*) + \Delta_p(y_{n_i}, y^*) < \Delta_p(x_{n_i+1}, x^*) + \Delta_p(y_{n_i+1}, y^*)\ \forall\, i \in \mathbb{N}.$$

By Lemma (2.3), we can find a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and for all $k \in \mathbb{N}$, we have

$$\Delta_p(x_{m_k}, x^*) + \Delta_p(y_{m_k}, y^*) \leq \Delta_p(x_{m_k+1}, x^*) + \Delta_p(y_{m_k+1}, y^*)$$

and

$$\Delta_p(x_k, x^*) + \Delta_p(y_k, y^*) \leq \Delta_p(x_{m_k+1}, x^*) + \Delta_p(y_{m_k+1}, y^*). \tag{3.50}$$

Then, by the same arguments as in (3.14), (3.15) and (3.16), we have that

$$\lim_{k\to\infty} ||Tu_{m_k} - u_{m_k}|| = 0 \tag{3.51}$$

and

$$\lim_{k\to\infty} ||Sv_{m_k} - v_{m_k}|| = 0. \tag{3.52}$$

From (3.13), we have

$$\begin{aligned} &\Delta_p(x_{m_k+1}, \bar{x}) + \Delta_p(y_{m_k+1}, \bar{y}) \\ \leq\ &(1-\alpha_{m_k})\left(\Delta_p(x_{m_k}, \bar{x}) + \Delta_p(y_{m_k}, \bar{y})\right) \\ &+\alpha_{m_k}\left(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_k+1} - \bar{x}\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_k+1} - \bar{y}\rangle\right), \end{aligned} \tag{3.53}$$

which implies

$$\begin{aligned} \alpha_{m_k}\left(\Delta_p(x_{m_k}, \bar{x}) + \Delta_p(y_{m_k}, \bar{y})\right) &\leq \left(\Delta_p(x_{m_k}, \bar{x}) + \Delta_p(y_{m_k}, \bar{y})\right) - \left(\Delta_p(x_{m_k+1}, \bar{x}) + \Delta_p(y_{m_k+1}, \bar{y})\right) \\ &\quad +\alpha_{m_k}\big(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_k+1} - \bar{x}\rangle \\ &\quad +\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_k+1} - \bar{y}\rangle\big) \\ &\leq \alpha_{m_k}\big(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_k+1} - \bar{x}\rangle \\ &\quad +\langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_k+1} - \bar{y}\rangle\big). \end{aligned} \tag{3.54}$$

That is

$$\left(\Delta_p(x_{m_k}, \bar{x}) + \Delta_p(y_{m_k}, \bar{y})\right) \leq \left(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_k+1} - \bar{x}\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_k+1} - \bar{y}\rangle\right).$$

Which implies

$$\lim_{k\to\infty}\left(\Delta_p(x_{m_k}, \bar{x}) + \Delta_p(y_{m_k}, \bar{y})\right) = 0. \tag{3.55}$$

From (3.50) and (3.55), we have

$$\Delta_p(x_k, \bar{x}) + \Delta_p(y_k, \bar{y}) \leq \Delta_p(x_{m_k+1}, \bar{x}) + \Delta_p(y_{m_k+1}, \bar{y}) \to 0,\ \text{as } k \to \infty,$$

which implies that $\{(x_k, y_k)\}$ converges strongly to $(\bar{x}, \bar{y})$. Thus, $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$. ∎

Corollary 1. *Let E_1, E_2 and E_3 be three p-uniformly convex real Banach spaces which are also uniformly smooth and $A : E_1 \to E_3,\ B : E_2 \to E_3$ be two bounded linear operators. Let $M : E_1 \to 2^{E_1^*},\ N : E_2 \to 2^{E_2^*}$ be multivalued maximal monotone mappings and $T : E_1 \to E_1$, $S : E_2 \to E_2$ be right Bregman strongly nonexpansive mappings such that $F(T) = \hat{F}(T)$ and $F(S) = \hat{F}(S)$. Suppose that $\Gamma := \{(\bar{x}, \bar{y}) \in F(T) \times F(S)$ such that $0 \in M(\bar{x}),\ 0 \in N(\bar{y})$ and $A\bar{x} = B\bar{y}\} \neq \emptyset$ and $\{\alpha_n\}$, $\{\beta_n\}$, are sequences in $(0, 1)$. Let $u, x_0 \in E_1$ and $v, y_0 \in E_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} v_n = Res_p^{\lambda N} J_q^{E_2}\left[J_q^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)\right]; \\ u_n = Res_p^{\lambda M} J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right]; \\ w_n = J_q^{E_2^*}\left[(1-\beta_n) J_p^{E_2}(v_n) + \beta_n J_p^{E_2} S v_n\right]; \\ z_n = J_q^{E_1^*}\left[(1-\beta_n) J_p^{E_1}(u_n) + \beta_n J_p^{E_1} T u_n\right]; \\ y_{n+1} = J_q^{E_2^*}\left[\alpha_n J_p^{E_2}(v) + (1-\alpha_n) J_p^{E_2} w_n\right]; \\ x_{n+1} = J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1} z_n\right], \end{cases} \tag{3.56}$$

with conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < t \le t_n \le k \le \left(\frac{q}{2C_q||A||^q}\right)^{\frac{1}{q-1}}, 0 < t \le t_n \le k \le \left(\frac{q}{2D_q||B||^q}\right)^{\frac{1}{q-1}}$,
(iv) $\beta_n \in (a, b)$ *for some* $a, b \in (0, 1)$.

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.
Recall that in a real Hilbert space, the duality mapping J_p becomes the identity mapping I. Thus, the resolvent of M now becomes:

$$Res_p^{\lambda M} = (I + \lambda M)^{-1} \circ I = (I + \lambda M)^{-1} =: J_\lambda^M.$$

Also note that if H is a real Hilbert space, then $H = H^*$. Using these facts, we obtain the following corollary in real Hilbert spaces.

Corollary 2. *Let H_1, H_2 and H_3 be three real Hilbert spaces and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be two bounded linear operators. Let $M_i : H_1 \to 2^{H_1},\ N_i : H_2 \to 2^{H_2}\ i = 1, 2, ..., m$ be multivalued maximal monotone mappings and $T : H_1 \to H_1$, $S : H_2 \to H_2$ be strongly nonexpansive mappings. Suppose that $\Gamma \neq \emptyset$ and $\{\alpha_n\}$, $\{\beta_n\}$, are sequences in $(0, 1)$. Let $u, x_0 \in H_1$ and $v, y_0 \in H_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} v_n = J_\lambda^{N_m} \circ J_\lambda^{N_{m-1}} \circ \cdots \circ J_\lambda^{N_1}\left[y_n + t_n B^*(Ax_n - By_n)\right]; \\ u_n = J_\lambda^{M_m} \circ J_\lambda^{M_{m-1}} \circ \cdots \circ J_\lambda^{M_1}\left[x_n - t_n A^*(Ax_n - By_n)\right]; \\ w_n = (1-\beta_n) v_n + \beta_n S v_n; \\ z_n = (1-\beta_n) u_n + \beta_n T u_n; \\ y_{n+1} = \alpha_n v + (1-\alpha_n) w_n; \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \end{cases} \tag{3.57}$$

with conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $t_n \in \Big(\epsilon, \dfrac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\Big), n \in \Omega$ *otherwise,* $t_n = t$ *(*t *being any nonegative value), where the set of indexes* $\Omega = \{n : Ax_n - By_n \neq 0\}$,

(iv) $\beta_n \in (a, b)$ *for some* $a, b \in (0, 1)$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma$.

4. Application

4.1. **Split Equality Variational Inequality Problem and Split Equality Fixed Point Problem.** Let $D : C \subset E \to E^*$ be strongly positive bounded linear mapping with coefficient $\tau > 0$ and $f : C \subset E \to E^*$ be a contraction mapping with coefficient $0 < \alpha < 1$. Then Lemma 2.4 also holds in a more general Banach space, i.e., we have that $(D - \gamma f)$ is a monotone operator in E. Indeed, for all $x, y \in C$ and $0 < \gamma < \frac{\tau}{\alpha}$, we obtain

$$\begin{aligned}
\langle (D-\gamma f)x - (D-\gamma f)y, x-y\rangle &= \langle Ax - Ay - \gamma(fx - fy), x - y\rangle \\
&= \langle Ax - Ay, x - y\rangle - \gamma\langle fx - fy, x - y\rangle \\
&\geq \tau||x-y||^2 - \gamma||fx - fy||||x-y|| \\
&\geq \tau||x-y||^2 - \gamma\alpha||x-y||^2,
\end{aligned}$$

which implies that $\langle (D - \gamma f)x - (D - \gamma f)y, x - y\rangle \geq 0$.

It can also be shown that $(D - \gamma f)$ is a Lipschitzian mapping. Indeed, $\forall x, y \in C$, we have

$$\begin{aligned}
\|(D-\tau f)x - (D-\tau f)y\|_* &= \|Dx - \tau fx - Dy + \tau fy\|_* \\
&\leq \|Dx - Dy\|_* + \tau\|fx - fy\|_* \\
&\leq k\|x-y\| + \tau\alpha\|x-y\| \\
&\leq (k + \alpha\tau)\|x-y\|, \qquad (4.1)
\end{aligned}$$

which implies that $(D - \tau f)$ is Lipschitzian with coefficient $L = k + \alpha\tau$. Hence, $(D - \tau f)$ is a monotone and L-Lipschitz mapping if $0 < \gamma < \frac{\tau}{\alpha}$. Therefore, if we define $M : E \to 2^{E^*}$ by

$$Mx = \begin{cases} N_C x + (D - \tau f)x & if \;\; x \in C, \\ \emptyset & if \;\; x \notin C, \end{cases} \qquad (4.2)$$

where $N_C x$ is the normal cone of C at x, defined by

$$N_C x = \{w \in E^* : \langle w, x - y\rangle \geq 0, \; \forall y \in C\}.$$

Then, M is maximal monotone and $M^{-1}(0) = VIP(C, D - \tau f)$ (see, for example, [[25], Theorem 3]), where $VIP(C, D - \tau f)$ is the solution set of the variational inequality problem: Find $x^* \in C$ such that

$$\langle (D - \tau f)x^*, y - x^*\rangle \geq 0 \;\; \forall y \in C. \qquad (4.3)$$

Let,
$\Gamma^* := \{(\bar{x}, \bar{y}) \in F(T) \times F(S),$ such that $(\bar{x}, \bar{y}) \in (\cap_{l=1}^m VIP(C, D_l - \tau f_l)) \times \big(\cap_{r=1}^N VIP(Q, D_r - \tau f_r)\big)\}$.
Then, we state the following theorem for approximating a common solution of split equality variational inequality problem and split equality fixed point problem, whose proof follows from the proof of Theorem 3.1.

Theorem 4.1. *Let* E_1, E_2 *and* E_3 *be three* p*-uniformly convex real Banach spaces which are also uniformly smooth and* C, Q *be nonempty, closed and convex subset of* E_1, E_2 *respectively. For each* $l = 1, 2, \ldots, m$ *and* $r = 1, 2, \ldots, N$, *let* $D_l : C \to E_1^*$, $D_r : Q \to E_2^*$ *be strongly positive bounded linear mappings with coefficient* $\tau > 0$ *and* $f_l : C \to E_1^*$, $f_r : Q \to E_2^*$ *be contraction mappings with coefficient* $0 < \alpha < 1$ *such that* $0 < \gamma < \frac{\tau}{\alpha}$. *Let* $T : E_1 \to E_1$,

$S : E_2 \to E_2$ be right Bregamn strongly nonexpansive mappings such that $F(T) = \hat{F}(T)$ and $F(S) = \hat{F}(S)$. Let $A : E_1 \to E_3, B : E_2 \to E_3$ be two bounded linear operators and $\Gamma^* \neq \emptyset$. Let the sequence $\{(x_n, y_n)\}$ be generated by Algorithm 3.1, then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma^*$.

Remark 4.1. Our work extend results for split equality monotone inclusion problem from the framework of Hilbert spaces to the more general p-uniformly convex Banach spaces which are also uniformly smooth.

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REFERENCES

[1] Y. I. ALBER, Metric and generalized projection operator in Banach spaces: Properties and applications, *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type,* **178** of Lecture Notes in Pure and Applied Mathematics, New York: Dekker, (1996), pp. 15-50.

[2] C. BYRNE, Iterative oblique projection onto convex sets and the split feasibility problem, *Inverse Problems,* **18**(2002), pp. 441–453.

[3] C. BYRNE, A unified treatment for some iterative algorithms in signal processing and image reconstruction, *Inverse Problems,* **20**(2004), pp. 103-120.

[4] Y. CENSOR, T. BORTFELD, B. MARTIN, A. TROFIMOV, A unified approach for inversion problems in intensity modulated radiation therapy, *Phys. Med. Biol.,* **51**(2006), pp. 2353-2365.

[5] Y. CENSOR and T. ELFVING, A multiprojection algorithm using Bregman projections in product space, *Numer. Algorithms*, **8**(1994), pp. 221-239.

[6] Y. CENSOR, T. ELFVING, T. KOPF, N. BORTFELD, The multiple-sets split feasibility problem and its applications for inverse problems, *Inverse Problems,* **21**(2005), pp. 2071-2084.

[7] Y. CENSOR and A. LENT, An iterative row-action method for interval convex programming, *J. Optim. Theory Appl.,* **34**(1981), pp. 321-353.

[8] Y. CENSOR and A. SEGAL, The split common fixed point problem for directed operators, *Journal of Convex Analsysis,* **2**(2009), pp. 587-600.

[9] C. E. CHIDUME, *Geometric Properties of Banach Spaces and Nonlinear Iterations,* Springer-Verlag London Limited, (2009).

[10] C. E. CHIDUME, P. NDAMBOMVE, A. U. BELLO, The split equality fixed point problem for demi-contractive mappings, *Journal of Nonlinear Analysis and Optimization,* **1**(2015), pp. 61-69.

[11] Y. FANG, L. WANG, X. ZI, Strong and weak convergence theorems for a new split feasibility problem, *International Mathematical Forum,* **33**(2013), pp. 1621-1627.

[12] H. GUO, H. HE, R. CHEN, Convergence theorems for split equality variational inclusion problem and fixed point problem in Hilbert spaces, *Fixed Point Theory and Applications,* (2015).

[13] P. E. MAING$\acute{E}$, Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization, *Set-Valued Anal.,* **16**(2008), pp. 899-912.

[14] V. MART$\acute{I}$N-M$\acute{A}$RQUEZ, S. RIECH, S. SABACH, Right Bregman nonexpansive operators in Banach spaces, *Nonlinear Analysis,* **75**(2012), pp. 5448-5465.

[15] R. MARINO and H. K. XU, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.,* **318**(2006), pp. 43-52.

[16] E. MASAD and S. REICH, A note on the multiple-set split convex feasibility problem in Hilbert spaces, *Journal of Nonlinear and Convex Analysis,* **3**(2007), pp. 367-371.

[17] A. MOUDAFI, A note on the split common fixed-point problem for quasi-nonexpansive operators, *Nonlinear Analysis,* **74**(2011), pp. 4083-4087.

[18] A. MOUDAFI, The split common fixed point problem for demi-contractive mappings, *Inverse Probl.,* **26**(2010).

[19] A. MOUDAFI and E. AL-SHEMAS, Simultaneous iterative methods for split equality problem, *Transactions on Mathematical Programming and Applications*, **2**(2013), pp. 1-11.

[20] F. U. OGBUISI and O. T. MEWOMO, Iterative solution of split variational inclusion problem in real Banach space, *Afrika Mathematika,* (2017), DOI. 10.1007/s13370-016-0450-2.

[21] F. U. OGBUISI and O. T. MEWOMO, Convergence analysis of common solution of certain non-linear problems, *Fixed Point Theory,* (2017), (Accepted, to appear).

[22] F. U. OGBUISI and O. T. MEWOMO, On split generalized mixed equilibrium problem and fixed point problems with no prior knowledge of operator norm, *J. Fixed Point Theory and Appl.,* (2017), DOI. 10.1007/s11784-016-0397-6.

[23] R. P. PHELPS, *Convex Functions, Monotone Operators, and Differentiability*, Springer Verlag, 2nd ed. Berlin, (1993).

[24] S. RIECH and S. SABACH, Two strong convergence theorems for a proximal method in Reflexive Banach spaces, *Numer. Funct. Anal. Optim.,* **31**(2010), pp. 24-44.

[25] R. T. ROCKAFELLAR,On the maximality of sums of nonlinear monotone operators, *Transactions of the American Mathematical Society*, **149** (1970), pp. 75-88.

[26] F. SCHOPFER, T. SCHUSTER, A. K. LOUIS, An iterative regularization method for the solution of the split feasibility problem in Banach spaces, *Inverse Problems,* **24**(5)(2008).

[27] Y. SHEHU, O. S. IYIOLA, C. D. ENYI, Iterative algorithm for split feasibility problems and fixed point problems in Banach spaces, In Press, *Numer. Algor.*, DOI:10.1007/s11075-015-0069-4.

[28] Y. SHEHU, O. T. MEWOMO, F. U. OGBUISI, Further investigation into approximation of a common solution of fixed point problems and split feasibility problems, *Acta. Math. Sci. Ser. B,* Engl. Ed. **36**(3)(2016), pp. 913-930.

[29] Y. SHEHU, O. T. MEWOMO, F. U. OGBUISI, Further investigation into split common fixed point problem for demi-contractive operators, *Acta. Math. Sci. Sinica. (English Series),* **32**(11)(2016), pp. 1357-1376.

[30] Y. SHEHU and F. U. OGBUISI, Approximation of common fixed points of left Bregman strongly nonexpansive mappings and solutions of equilibrium problems, *J. Appl. Anal.,* **21** (2) (2015), pp. 63-77.

[31] Y. SHEHU and F. U. OGBUISI, An iterative method for solving split monotone variational inclusion and fixed point problems, *RACSAM* DOI 10.1007/s13398-015-0245-3.

[32] Y. SHEHU, F. U. OGBUISI, O. S. IYIOLA, Convergence analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces, In Press, *Optimization*, DOI:10.1080/02331934.2015.1039533.

[33] F. WANG, A new algorithm for solving the multiple sets feasibility problem in Banach spaces, *Numerical Functional Anal. Opt.,* **35**(1)(2014), 99-110.

[34] F. WANG and H. K. XU, Cyclic algorithms for split feasibility problems in Hilbert spaces, *Nonlinear Analysis,* **12**(2011), pp. 40105-4111.

[35] H. K. XU, Iterative algorithms for nonlinear operators, *J. London. Math. Soc.*, **2**(2002), pp. 240-256.

[36] H. K. XU, A vaiariable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem, *Inverse Problems*, **22**(6)(2006), pp. 2021-2034.

[37] H. K. XU, Iterative methods for split feasibility problem in infinite-dimensional Hilbert spaces, *Inverse Probl.*, **26**(2010).

[38] Q. YANG, The relaxed CQ algorithm for solving the split feasibility problem, *Inverse Problems*, **4**(2004), pp. 1261-1266.

[39] J. ZHAO and Q. YANG, Several solution methods for the split feasibility problem, *Inverse Problems*, **5**(2005), pp. 1791-1799.