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MAPPED CHEBYSHEV SPECTRAL METHODS FOR SOLVING SECOND KIND INTEGRAL EQUATIONS ON THE REAL LINE

AHMED GUECHI AND AZEDINE RAHMOUNE

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BORDJ BOU ARRÉRIDJ, EL ANASSER, 34030, BBA, ALGERIA.

a.guechi2017@gmail.com

a.rahmoune@univ-bba.dz

ABSTRACT. In this paper we investigate the utility of mappings to solve numerically an important class of integral equations on the real line. The main idea is to map the infinite interval to a finite one and use Chebyshev spectral-collocation method to solve the mapped integral equation in the finite interval. Numerical examples are presented to illustrate the accuracy of the method.

Key words and phrases: Integral equations, Chebyshev polynomials, Infinite interval, Spectral method.

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1. INTRODUCTION

Integral equations appear in many fields of scientific and engineering, such as the radiative transfer [10, 22], acoustic resonance scattering [9], population dynamics [6, 14], electromagnetic and elastic waves [12], fluid dynamics [23]. So it is necessary to solve them over bounded or unbounded domains. In recent years spectral methods for bounded domains have been found to be a powerful tool for the solution of differential and integral equations due to their bigger accuracy when compare to standard methods. The rate of convergence of spectral approximations depends only on the smoothness of the solution, yielding the ability to achieve high precision with a small number of data (see, e.g., [4, 5, 11, 20, 21]).

Spectral methods in unbounded domains have also been used and discussed by many authors with different approaches, see, e.g., [2, 3, 16, 18, 19, 20]. In general, they are essentially classified into four approaches: Domain truncation, Approximation by Laguerre or Hermite polynomials/functions, Approximation by non classical orthogonal systems, or by mapped orthogonal systems, and by Mapping of the unbounded domain to a bounded one. We should note here that, the domain truncation approach is only a viable option for problems with rapidly(exponentially) decaying solutions. However, with proper choices of mapping or scaling parameters, the other three approaches can all be effectively applied to a variety of problems with rapid or slow decaying solutions.

It's been more than thirty years since Grosch and Orszag [13] showed that differential equations on a semi-infinite interval can be solved very effectively by mapping the interval into $[-1, 1]$ using an algebraic function for the map. Boyd [2] generalized their technique and provided in [4, Ch. 17, p. 338] an excellent extensive review on general properties and practical implementations for many of these approaches. The most frequently used mappings are algebraic, exponential and logarithmic given by the following formulas, in which the constant $s > 0$ sets the length scale of the mappings :

- Algebraic map $\quad x = sz(1-z^2)^{-1/2}, \qquad z = \dfrac{x}{\sqrt{x^2+s^2}}.$
- Logarithmic map $\quad x = s\tanh^{-1} z = \dfrac{s}{2}\ln\dfrac{1+z}{1-z}, \qquad z = \tanh(s^{-1}x).$
- Exponential map $\quad x = \sinh sz = s^{-1}\ln(z+\sqrt{z^2+1}),$

where $x \in (-\infty, +\infty)$. The name of these families of maps are chosen by how rapidly x increases with $z \to \pm 1$.

In this paper we are concerned with the numerical solution of integral equations on the real line $(-\infty, +\infty)$. The essential idea in our approach is to map the given infinite interval to $[-1, 1]$, and use Chebyshev spectral-collocation method to solve the mapped integral equation. The main advantage of this technique is that it can be implemented and analyzed using standard procedures and approximation results.

2. CHEBYSHEV POLYNOMIALS

Chebyshev polynomials of the first kind of degree n are defined by

$$T_n(z) = \cos(n\cos^{-1} z), \quad n = 0, 1, 2\cdots \tag{2.1}$$

Also they are derived by the following recursive formula

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \quad n \geq 1, \tag{2.2}$$

with $T_0(z) = 1$ and $T_1(z) = z$.

It is well known that Chebyshev polynomials are an orthogonal system on the segment $[-1,1]$ with respect to the weight function $\omega(z)=(1-z^2)^{-1/2}$. Namely,

$$\int_{-1}^{1}\frac{T_n(z)T_m(z)}{\sqrt{1-z^2}}=\frac{\alpha_n\pi}{2}\delta_{nm} \tag{2.3}$$

where $\alpha_0=2$ and $\alpha_n=1$ for $n\geq 1$.

The Chebyshev expansion of a function $u\in L^2_\omega(-1,1)$ is

$$u(z)=\sum_{i=0}^{\infty}\hat{u}_iT_i(z),\quad \hat{u}_i=\frac{2}{\pi\alpha_i}\int_{-1}^{1}u(z)T_i(z)\omega(z)dz. \tag{2.4}$$

The polynomial $T_{n+1}(z)$ of degree $n+1$ has $n+1$ different simple zeros in $[-1,1]$ at

$$z_i=\cos\left(\frac{2i+1}{2n+2}\pi\right),\quad i=0,1,2\cdots n. \tag{2.5}$$

Theorem 2.1. *Given $f\in\mathcal{C}^{2n}[-1,1]$ and let $z_0,z_1,...,z_n$ be the $n+1$ zeros of $T_{n+1}(z)$. Then*

$$\int_{-1}^{1}\frac{f(z)}{\sqrt{1-z^2}}dz\approx\frac{\pi}{n+1}\sum_{i=0}^{n}f(z_i) \tag{2.6}$$

3. SOLUTION METHODS

Consider integral equations of the form

$$\varphi(x)-\int_{-\infty}^{+\infty}k(x,t)\varphi(t)dt=f(x) \tag{3.1}$$

where the functions $k(x,t)$ and $f(x)$ are given, and $\varphi(x)$ is an unknown function to be determined. The regular part of (3.1) is assumed to be exists in the Riemann sense. Which is considered becomes

$$\lim_{L\to\infty}\int_{-L}^{L}k(x,t)\varphi(t)dt \tag{3.2}$$

We first use Logarithmic map. Then (3.1) becomes

$$\varphi(x)-\int_{-1}^{1}k(x,s\tanh^{-1}y)\varphi(s\tanh^{-1}y)\frac{s}{1-y^2}dy=f(x) \tag{3.3}$$

If we pose $x=s\tanh^{-1}z$, then

$$\varphi(s\tanh^{-1}z)-\int_{-1}^{1}k(s\tanh^{-1}z,s\tanh^{-1}y)\varphi(s\tanh^{-1}y)\frac{s}{1-y^2}dy=f(s\tanh^{-1}z) \tag{3.4}$$

and let $u(z)=\varphi(s\tanh^{-1}z)$, so we obtain

$$u(z)-\int_{-1}^{1}k(s\tanh^{-1}z,s\tanh^{-1}y)u(y)\frac{s}{(1-y^2)}dy=f(s\tanh^{-1}z) \tag{3.5}$$

By setting

$$M(z,y)=\frac{k(s\tanh^{-1}z,s\tanh^{-1}y)}{\sqrt{1-y^2}}$$

$$\omega(y)=\frac{1}{\sqrt{1-y^2}}\quad\text{and}\quad F(z)=f(s\tanh^{-1}z)$$

equation (3.5) can be written as follows

$$u(z) - s\int_{-1}^{1} M(z,y)u(y)\omega(y)dy = F(z) \tag{3.6}$$

In the same manner using algebraic map, we also obtain an integral equation of the form (3.6), where

$$M(z,y) = \frac{k(sz(1-z^2)^{-1/2}, sy(1-y^2)^{-1/2})}{1-y^2},$$

and

$$F(z) = f(sz(1-z^2)^{-1/2}),$$

Therefore, in both cases, the problem is to solve approximately the mapped integral equation (3.6) which can be written in the operator form

$$(I - s\mathcal{K})u = F \tag{3.7}$$

To do this, we assume that the operator $\mathcal{K}$ is compact on the space $L^2_\omega(-1,1)$, and let $\mathbb{P}_n$ the $n+1$-dimensional subspace spanned by the Chebyshev polynomials $T_0, \cdots, T_n$. Let $\mathcal{P}_n : L^2_\omega(-1,1) \longrightarrow \mathbb{P}_n$ be a bounded projection operator. Our motivation is to approximate (3.6) by attempting to solve the problem

$$(I - s\mathcal{P}_n\mathcal{K})u_n = \mathcal{P}_n F, \quad u_n \in \mathbb{P}_n \tag{3.8}$$

Using Gauss-Chebyshev quadrature formula given by theorem (2.1) to approximate the integral part of the equation (3.6) gives

$$\tilde{u}(z) - \frac{s\pi}{n+1}\sum_{i=0}^{n} M(z,y_i)\tilde{u}(y_i) = F(z) \tag{3.9}$$

Now, forcing this semi-discrete equation to be almost exact in the sense that the residual

$$r_n(z) = \tilde{u}(z) - \frac{s\pi}{n+1}\sum_{i=0}^{n} M(z,y_i)\tilde{u}(y_i) - F(z) \tag{3.10}$$

is zero at collocation points $z_j, j = 0, ..., n$. Thus, the condition $r(z_j) = 0$, for j from 0 to n, lead to the following system of linear equations

$$\tilde{u}(z_j) - \frac{s\pi}{n+1}\sum_{i=0}^{n} M(z_j,y_i)\tilde{u}(y_i) = F(z_j) \tag{3.11}$$

Therefore, the Chebyshev polynomial approximation is given by

$$\mathcal{P}_n u(z) = \sum_{i=0}^{n} c_i T_i(z) \tag{3.12}$$

where

$$c_0 \approx \frac{1}{n+1}\sum_{i=0}^{n} \tilde{u}(z_i) \tag{3.13}$$

$$c_j \approx \frac{2}{n+1}\sum_{i=0}^{n} \tilde{u}(z_j)T_j(z_i), \quad j = 1, \ldots, n \tag{3.14}$$

Consequently, the approximate solution of equation (3.1) in the real line is given by

$$\varphi_n(x) = \sum_{i=0}^{n} c_i T_i\left(\tanh(s^{-1}x)\right), \tag{3.15}$$

using Logarithmic map, and

$$\varphi_n(x) = \sum_{i=0}^{n} c_i T_i\left(x/\sqrt{x^2+s^2}\right), \tag{3.16}$$

in the case of algebraic map.

3.1. **Spectral convergence analysis.** In this subsection, a convergence analysis for the numerical schemes for the mapped integral equation (3.6) will be provided. The goal is to show that the rate of convergence depends on the regularity properties of the corresponding exact solution. Also, we note that the following results are based on framework in [1, 5].

Lemma 3.1 (See [5], p. 296)**.** *Let* $u \in H_\omega^m(-1,1)$ *with* $m \geqslant 1$. *The truncation error* $u - \mathcal{P}_n u$, *satisfies the inequality*

$$\|u - \mathcal{P}_n u\|_{L^2_\omega(-1,1)} \leqslant C n^{-m} |u|_{H_\omega^{m,n}(-1,1)} \tag{3.17}$$

where,

$$|u|_{H_\omega^{m,n}(-1,1)} = \left(\sum_{k=\min(m,n+1)}^{m} \|u^{(k)}\|^2_{L^2_\omega(-1,1)} \right)^{1/2}. \tag{3.18}$$

Theorem 3.2 (See [1], p. 55)**.** *Assume* $\mathcal{K} : L^2_\omega(-1,1) \longrightarrow L^2_\omega(-1,1)$ *is bounded, and assume* $I - s\mathcal{K}$ *is one-to-one and onto operator. Further assume*

$$\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| \to 0 \textit{ as } n \to \infty \tag{3.19}$$

Then for all sufficiently large n, *the operator* $(I - s\mathcal{P}_n\mathcal{K})^{-1}$ *exists as a bounded operator. Moreover, it is uniformly bounded :*

$$\sup_n \|(I - s\mathcal{P}_n\mathcal{K})^{-1}\| \leqslant M \tag{3.20}$$

For the solution of (3.7) and (3.8),

$$\|u - u_n\| \leqslant s^{-1} M \|u - \mathcal{P}_n u\| \tag{3.21}$$

Lemma 3.3. *Assume* $\mathcal{K} : L^2_\omega(-1,1) \longrightarrow L^2_\omega(-1,1)$ *is a compact operator and assume* $\mathcal{P}_n u \to u$ *for all sufficiently large* n. *Then,*

$$\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| \to 0 \tag{3.22}$$

Proof. From the definition of operator norm,

$$\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| = \sup_{\|u\|\leqslant 1} \|\mathcal{K}u - \mathcal{P}_n\mathcal{K}u\| = \sup_{v\in\mathcal{K}(U)} \|v - \mathcal{P}_n v\|$$

with $\mathcal{K}(U) = \{\mathcal{K}u \quad | \quad \|u\| \leqslant 1\}$. Since the set $\mathcal{K}(U)$ is compact. Therefore, by the preceding lemma,

$$\sup_{v\in\mathcal{K}(U)} \|v - \mathcal{P}_n v\| \to 0$$

for all sufficiently large n.

4. ILLUSTRATIVE EXAMPLES

Example 4.1. Consider Fredholm integral equation (3.1) with

$$k(x,t) = xt\sin(t), \quad f(x) = e^{-x^2} - \frac{\sqrt{\pi}}{2e^{1/4}}x.$$

Whose exact solution is $\varphi(x) = e^{-x^2}$, which is a smooth function and decay exponentially at infinity. In table 4.1, we give Maximum absolute errors at 1000 selected equally spaced points on the interval $[-5, 5]$ using logarithmic and algebraic maps against various n and different scaling factor s. We observe that the error decay exponentially and we remark also that for a fixed integer n, the numerical results with $s = 2.5$ are better than other choices of scaling factor either with logarithmic or algebraic map.

Example 4.2. Consider Fredholm integral equation (3.1) with

$$k(x,t) = xt\cos(t), \quad f(x) = \frac{1}{1+x^2}.$$

The corresponding exact solution is $\varphi(x) = \dfrac{1}{1+x^2}$, which decays algebraically at infinity. In table 4.2, we give Maximum absolute errors at 1000 selected equally spaced points on the interval $[-500, 500]$ using algebraic map against various n and different scaling factor s. We observe that $s = 1$ gives better numerical results than other choices, contrarily to the first example where the choice of $s = 2.5$ is better.

Table 4.1: Maximum absolute error at 1000 equally spaced points on the interval $[-5, 5]$ for example 1

		s				
Mapping	n	1	2.5	5.0	7.5	10
Logarithmic	20	1.13E-03	6.54E-07	2.74E-03	1.16E-01	–
	30	1.10E-04	2.91E-09	2.26E-06	3.46E-03	4.38E-02
	40	2.30E-05	2.42E-11	1.92E-10	8.81E-05	3.82E-03
	80	2.36E-07	7.21E-14	6.62E-13	1.72E-12	4.04E-09
Algebraic	20	1.13E-03	5.11E-05	2.23E-03	1.19E-01	–
	30	1.06E-04	8.96E-07	3.50E-06	3.19E-03	4.35E-02
	40	1.48E-05	2.27E-08	1.43E-08	5.89E-05	3.66E-03
	80	1.48E-08	4.93E-11	1.45E-09	1.92E-09	1.45E-09

Table 4.2: Maximum absolute error at 1000 equally spaced points on the interval $[-500, 500]$ for example 2 using algebraic map.

	s				
n	0.5	1	2.5	5	7.5
10	5.49E-03	1.49E-15	8.49E-03	7.95E-02	1.89E-01
20	2.26E-05	2.81E-15	1.50E-04	1.01E-02	4.84E-02
30	9.29E-08	3.00E-15	2.03E-06	1.42E-03	1.11E-02
40	3.83E-10	3.75E-15	2.73E-08	1.64E-04	3.76E-03
80	9.83E-15	1.34E-14	4.78E-14	7.24E-08	1.20E-05

5. CONCLUSION

In this paper, Fredholm integral equation of the second kind on the real-line is solved by Mapped Chebyshev spectral methods. Convergence of the presented method is analyzed and tested. The proposed method provides a good efficiency for smooth solutions decaying exponentially or algebraically at infinity as shown in Tables 1-2. The scaling factor s offers great flexibility to improve the numerical resolution. However, the best choice of s is one that gives a good adjustment of the collocation points.

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