WEAKLY COMPACT COMPOSITION OPERATORS ON REAL LIPSCHITZ SPACES OF COMPLEX-VALUED FUNCTIONS ON COMPACT METRIC SPACES WITH LIPSCHITZ INVOLUTIONS

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ABSTRACT. We first show that a bounded linear operator $T$ on a real Banach space $E$ is weakly compact if and only if the complex linear operator $T'$ on the complex Banach space $E_C$ is weakly compact, where $E_C$ is a suitable complexification of $E$ and $T'$ is the complex linear operator on $E_C$ associated with $T$. Next we show that every weakly compact composition operator on real Lipschitz spaces of complex-valued functions on compact metric spaces with Lipschitz involutions is compact.

Key words and phrases: Compact operator; Composition operator; Lipschitz involution; Weakly compact operator.

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1. Introduction and Preliminaries

The Symbol $\mathbb{K}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. Let $E$ and $F$ be Banach spaces over $\mathbb{K}$. We denote by $B_{\mathbb{K}}(E,F)$ the Banach space over $\mathbb{K}$ consisting of all bounded linear operators from $E$ into $F$ with the operator norm $\| \cdot \|_{op}$. We write $B_{\mathbb{K}}(E)$ instead of $B_{\mathbb{K}}(E,E)$. Let us recall that $T \in B_{\mathbb{K}}(E,F)$ is compact (weakly compact, respectively) if the closure of $T(U)$ in $F$ is compact with the norm-topology (weak-topology, respectively), where $U$ is the open unit ball in $E$.

It is known that if $E$, $F$ and $G$ are Banach spaces over $\mathbb{K}$, $S \in B_{\mathbb{K}}(E,F)$ and $T \in B_{\mathbb{K}}(F,G)$, then $T \circ S$ is compact (weakly compact, respectively) whenever $T$ or $S$ is compact (weakly compact, respectively).

Applying the Eberlein-Šmulian theorem [4, Theorem V.6.1] and the definition of weakly compact operators between Banach spaces over $\mathbb{K}$, we obtain the following result.

**Theorem 1.1.** Let $(E, \| \cdot \|)$ and $(F, \| \cdot \|)$ be Banach spaces and $T : E \rightarrow F$ be a linear operator from $E$ into $F$ over $\mathbb{K}$. Then $T$ is weakly compact if and only if for each bounded sequence $\{a_n\}_{n=1}^{\infty}$ in $(E, \| \cdot \|)$ there exist a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ and an element $b$ of $F$ such that $\lim_{k \rightarrow \infty} T a_{n_k} = b$ in $F$ with the weak-topology.

Let $X$ be a nonempty set, $V_{\mathbb{K}}(X)$ be a vector space of $\mathbb{K}$-valued functions on $X$ and $T : V_{\mathbb{K}}(X) \rightarrow V_{\mathbb{K}}(X)$ be a linear operator on $V_{\mathbb{K}}(X)$ over $\mathbb{K}$. If there exists a self-map $\phi : X \rightarrow X$ such that $T f = f \circ \phi$ for all $f \in V_{\mathbb{K}}(X)$, then $T$ is called the composition operator on $V_{\mathbb{K}}(X)$ induced by $\phi$.

Let $X$ be a topological space. We denote by $C_{\mathbb{K}}(X)$ and $C_{\mathbb{K}}(X)^b$ the set of all $\mathbb{K}$-valued continuous and bounded continuous functions on $X$, respectively. Then $C_{\mathbb{K}}(X)$ is a commutative $\mathbb{K}$-algebra over $\mathbb{K}$ with unit $1_X$, the constant function on $X$ with value 1, and $C_{\mathbb{K}}(X)^b$ is a subalgebra of $C_{\mathbb{K}}(X)$ containing $1_X$. Moreover, $C_{\mathbb{K}}(X)^b$ is a unital commutative Banach algebra over $\mathbb{K}$ with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \quad (f \in C_{\mathbb{K}}(X)^b(X)).$$

Clearly, $C_{\mathbb{K}}(X)^b = C_{\mathbb{K}}(X)$ whenever $X$ is compact. We write $C(X)$ and $C^b(X)$ instead of $C_{\mathbb{K}}(X)$ and $C_{\mathbb{K}}(X)^b$, respectively.

Let $(X,d)$ and $(Y,\rho)$ be metric spaces. A map $\phi : X \rightarrow Y$ is called a Lipschitz mapping from $(X,d)$ into $(Y,\rho)$ if there exists a constant $M \geq 0$ such that $\rho(\phi(s),\phi(t)) \leq M d(s,t)$ for all $s,t \in X$. For a map $\phi : X \rightarrow Y$, the Lipschitz constant of $\phi$ is denoted by $p(\phi)$ and defined by

$$p(\phi) = \sup \left\{ \frac{\rho(\phi(s),\phi(t))}{d(s,t)} : s,t \in X, s \neq t \right\}.$$

Clearly a map $\phi : X \rightarrow Y$ is a Lipschitz mapping if and only if $p(\phi) < \infty$. A map $\phi : X \rightarrow Y$ is called a supercontractive mapping from $(X,d)$ into $(Y,\rho)$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(\phi(s),\phi(t))/d(s,t) < \varepsilon$ for all $s,t \in X$ with $0 < d(s,t) < \delta$. It is clear that if $\phi : X \rightarrow Y$ is a supercontractive mapping from $(X,d)$ into $(Y,\rho)$ such that $\phi(X)$ is bounded set in $(Y,\rho)$, then $\phi$ is a Lipschitz mapping.

Let $(X,d)$ be a metric space. A function $f : X \rightarrow \mathbb{K}$ is called a $\mathbb{K}$-valued Lipschitz function (supercontractive function, respectively) on $(X,d)$ if $f$ is a Lipschitz mapping (supercontractive mapping, respectively) from $(X,d)$ into the Euclidean metric space $\mathbb{K}$. For a Lipschitz function $f$ on $(X,d)$, the Lipschitz number of $f$ is denoted by $L_{(X,d)}(f)$ and defined by

$$L_{(X,d)}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x,y \in X, x \neq y \right\}.$$
Let \((X, d)\) be a pointed metric space with the base point \(e \in X\). We denote by \(\text{Lip}_{0,K}(X, d)\) the set of all \(K\)-valued Lipschitz functions \(f\) on \((X, d)\) for which \(f(e) = 0\). Clearly, \(\text{Lip}_{0,K}(X, d)\) is a linear subspace \(C_K(X)\) over \(K\). Moreover, \(\text{Lip}_{0,K}(X, d)\) with the norm \(L_{(X,d)}(\cdot)\) is a Banach space over \(K\). We denote by \(\text{lip}_{0,K}(X, d)\) the set of all \(f \in \text{Lip}_{0,K}(X, d)\) for which \(f\) is a supercontraction function on \((X, d)\). It is easy to see that \(\text{lip}_{0,K}(X, d)\) is a linear subspace of \(\text{Lip}_{0,K}(X, d)\) and it is a closed set in the Banach space \((\text{Lip}_{0,K}(X, d), L_{(X,d)}(\cdot))\). Therefore, \((\text{lip}_{0,K}(X, d), L_{(X,d)}(\cdot))\) is a Banach space over \(K\). Note that if \(\phi : X \rightarrow X\) is a base point-preserving Lipschitz mapping on \((X, d)\), then \(f \circ \phi \in \text{Lip}_{0,K}(X, d)\) (\(f \circ \phi \in \text{lip}_{0,K}(X, d)\), respectively) for all \(f \in \text{Lip}_{0,K}(X, d)\) (\(f \in \text{lip}_{0,K}(X, d)\), respectively). For further general facts about Lipschitz spaces \(\text{Lip}_{0,K}(X, d)\) and little Lipschitz spaces \(\text{lip}_{0,K}(X, d)\), we refer to [14]. We write \(\text{Lip}_0(X, d)\) (\(\text{lip}_0(X, d)\), respectively) instead of \(\text{Lip}_{0,C}(X, d)\) (\(\text{lip}_{0,C}(X, d)\), respectively). Note that there are \(\text{lip}_{0,K}\) spaces containing only the zero function as for instance, \(\text{lip}_{0,K}([0, 1], d)\) whenever \(d\) is the Euclidean metric on \([0, 1]\).

Let \((X, d)\) be a pointed compact metric space. It is said that \(\text{lip}_{0,K}(X, d)\) separates points uniformly on \(X\) if there exists a constant \(\alpha > 1\) such that, for every \(x, y \in X\) there exists \(f \in \text{Lip}_{0,K}(X, d)\) with \(L_{(X,d)}(f) \leq \alpha\) such that \(f(x) = d(x, y)\) and \(f(y) = 0\). For instance, if \(X\) is the middle-thirds Cantor set in \([0, 1]\) and \(d\) is the Euclidean metric on \(X\), the \(\text{lip}_{0,K}(X, d)\) separates points uniformly on \(X\) (See [14], Proposition 3.2.2(a))).

Let \((X, d)\) be a metric space and \(\alpha \in (0, 1]\). We know that the map \(d^\alpha : X \times X \rightarrow \mathbb{R}\) defined by \(d^\alpha(x, y) = (d(x, y))^\alpha\), is a metric on \(X\) and the generated topology on \(X\) by \(d^\alpha\) coincides by the generated topology on \(X\) by \(d\). It is known [14], Proposition 3.2.2(b)] that \(\text{lip}_0(X, d^\alpha)\) separates points uniformly on \(X\) whenever \((X, d)\) is a pointed compact metric space and \(\alpha \in (0, 1]\).

Let \((X, d)\) be a metric space and \(\alpha \in (0, 1]\). We denote by \(\text{Lip}_K(X, d^\alpha)\) the set of all \(K\)-valued bounded Lipschitz functions on \((X, d^\alpha)\). Clearly, \(\text{Lip}_K(X, d^\alpha)\) is a subalgebra of \(C^\alpha(X)\) containing \(1_X\). Moreover, \(\text{Lip}_K(X, d^\alpha)\) is a Banach space under the norm

\[
\|f\|_{\text{Lip}_K(X, d^\alpha)} = \max\{\|f\|_X, L_{(X,d^\alpha)}(f)\} \quad (f \in \text{Lip}_K(X, d^\alpha)).
\]

Let \(\text{lip}_K(X, d^\alpha)\) denote the set of all \(K\)-valued supercontractive bounded functions on \((X, d^\alpha)\). Clearly, \(\text{lip}_K(X, d^\alpha)\) is a subalgebra of \(\text{Lip}_K(X, d^\alpha)\) and it is a closed set in the Banach space \((\text{Lip}_K(X, d^\alpha), \|\cdot\|_{\text{Lip}_K(X, d^\alpha)})\). Hence, \((\text{lip}_K(X, d^\alpha), \|\cdot\|_{\text{Lip}_K(X, d^\alpha)})\) is a Banach space over \(\mathbb{K}\). Moreover, \(\text{Lip}(X, d^\beta) \subseteq \text{lip}(X, d^\alpha)\) whenever \(0 < \alpha < \beta \leq 1\). It is known that \(\text{lip}_K(X, d^1)\) separates the points of \(X\). Lipschitz algebras \(\text{Lip}(X, d^\alpha)\) and little Lipschitz algebras \(\text{lip}(X, d^\alpha)\) were first introduced by Sherbert in [12] and [13].

Komowitz and Scheinberg in [8] characterized compact composition operators on \(\text{Lip}(X, d^\alpha)\) for \(\alpha \in (0, 1]\) and \(\text{lip}(X, d^\alpha)\) for \((0, 1]\) whenever \((X, d)\) is a compact metric space.

Jiménez-Vargas and Villegas-Vallecillos in [7] studied and characterized compact composition operators on \(\text{Lip}_0(X, d)\) whenever \((X, d)\) is a pointed metric space not necessarily compact, and on \(\text{Lip}_0(X, d)\) and \(\text{lip}_0(X, d)\) whenever \((X, d)\) is a metric space not necessarily compact.

Jiménez-Vargas in [6] studied weakly compact composition operators on \(\text{Lip}_{0,K}(X, d)\) and \(\text{lip}_{0,K}(X, d)\) whenever \((X, d)\) is a pointed compact metric space, and on \(\text{Lip}_{0,K}(X, d^\alpha)\) for \(\alpha \in (0, 1]\) and \(\text{lip}_{0,K}(X, d^\alpha)\) for \(\alpha \in (0, 1]\) whenever \((X, d)\) is a compact metric space and obtained the following results.

**Theorem 1.2** (See [6] Theorem 2.3). Let \((X, d)\) be a pointed compact metric space, the map \(\phi : X \rightarrow X\) be a base point-preserving Lipschitz mapping on \((X, d)\) and \(T : \text{lip}_{0,K}(X, d) \rightarrow \text{lip}_{0,K}(X, d)\) be the composition operator on \(\text{lip}_{0,K}(X, d)\) induced by \(\phi\). Suppose that the little Lipschitz space \(\text{lip}_{0,K}(X, d)\) separates points uniformly on \(X\). If \(T\) is weakly compact, then \(T\) is compact.
Theorem 1.3 (See [6 Corollary 2.4]). Let \((X, d)\) be a pointed compact metric space, the map 
\[ \phi : X \longrightarrow X \] 
is a base point-preserving Lipschitz mapping on \((X, d)\) and \(T : \text{Lip}_0(X, d) \longrightarrow \text{Lip}_{0,K}(X, d)\) be the composition operator on \(\text{Lip}_{0,K}(X, d)\) induced by \(\phi\). Suppose that the little Lipschitz space \(\text{lip}_0(X, d)\) separates points uniformly on \(X\). If \(T\) is weakly compact, then \(T\) is compact.

Theorem 1.4 (See [6 Remark 2.1]). Let \((X, d)\) be a compact metric space, \(E = \text{Lip}_K(X, d^\alpha)\) for \(\alpha \in (0, 1)\) or \(E = \text{lip}_K(X, d^\alpha)\) for \(\alpha \in (0, 1)\), \(\phi : X \longrightarrow X\) be a Lipschitz mapping on \((X, d)\) and \(T : E \longrightarrow E\) be the composition operator on \(E\) induced by \(\phi\). If \(T\) is weakly compact, then \(T\) is compact.

Let \(X\) be a topological space. A self-map \(\tau : X \longrightarrow X\) is called a topological involution on \(X\) if \(\tau\) is continuous and \(\tau(\tau(x)) = x\) for all \(x \in X\). Clearly, such \(\tau\) is a homeomorphism from \(X\) onto \(X\).

Let \(X\) be a Hausdorff space and \(\tau\) be a topological involution on \(X\). Then the map \(\tau^* : C^b(X) \longrightarrow C^b(X)\) defined by \(\tau^*(f) = \bar{f} \circ \tau\) is an algebra involution on \(C^b(X)\), which is called the algebra involution on \(C^b(X)\) induced by \(\tau\). We now define
\[ C(X, \tau) = \{ f \in C(X) : \tau^*(f) = f \}, \]
\[ C^b(X, \tau) = \{ f \in C^b(X) : \tau^*(f) = f \}. \]

Then \(C(X, \tau)\) is a real subalgebra of \(C(X)\), \(1_X \in C(X, \tau), i1_X \notin C(X, \tau)\) and \(C(X) = C(X, \tau) \oplus iC(X, \tau)\). Moreover \(C^b(X, \tau)\) is a unital self–adjoint uniformly closed real subalgebra of \(C^b(X)\), \(i1_X \notin C^b(X, \tau)\), \(C^b(X) = C^b(X, \tau) \oplus iC^b(X, \tau)\)
\[ \max \{ \|f\|_X, \|g\|_X \} \leq \|f + ig\|_X \leq 2 \max \{ \|f\|_X, \|g\|_X \} \]
for all \(f, g \in C^b(X, \tau)\). Clearly, \(C^b(X, \tau) = C(X, \tau)\) if \(X\) is compact.

Real Banach algebra \((C(X, \tau), \| \cdot \|_X)\) was defined explicitly by Kulkarni and Limaye in [9], where \((X, d)\) is a compact Hausdorff space and \(\tau\) is a topological involution on \(X\). For further general facts about \(C(X, \tau)\) and certain real subalgebras, we refer to [10].

Let \((X, d)\) be a metric space. A self-map \(\tau : X \longrightarrow X\) is called a Lipschitz involution on \((X, d)\) if \(\tau(\tau(x)) = x\) for all \(x \in X\) and \(\tau\) is a Lipschitz mapping on \((X, d)\).

Note that if \(\tau\) is a Lipschitz involution on \((X, d)\), then \(\tau\) is a topological involution on \((X, d)\) and \(1 \leq p(\tau) < \infty\).

Let \((X, d)\) be a pointed metric space and \(\tau\) be a base point-preserving Lipschitz involution on \((X, d)\). Then \(\tau^*(\text{Lip}_0(X, d)) = \text{Lip}_0(X, d)\) and \(\tau^*(\text{lip}_0(X, d)) = \text{lip}_0(X, d)\). We now define
\[ \text{Lip}_0(X, d, \tau) = \{ f \in \text{Lip}_0(X, d) : \tau^*(f) = f \}, \]
\[ \text{lip}_0(X, d, \tau) = \{ f \in \text{lip}_0(X, d) : \tau^*(f) = f \}. \]

In fact \(\text{Lip}_0(X, d, \tau) = \text{Lip}_0(X, d) \cap C(X, \tau)\) and \(\text{lip}_0(X, d, \tau) = \text{lip}_0(X, d) \cap C(X, \tau)\). The following result is a modification of [2 Theorem 1.3].

Theorem 1.5. Let \((X, d)\) be a pointed metric space and \(\tau\) be a base point-preserving Lipschitz involution on \((X, d)\). Suppose that \(A = \text{Lip}_0(X, d, \tau)\) and \(B = \text{Lip}_0(X, d)\), or, \(A = \text{lip}_0(X, d, \tau)\) and \(B = \text{lip}_0(X, d)\). Then:

(i) \(A\) is a self-adjoint real subspace of \(C^b(X, \tau)\) and \(B\), \(1_X \notin A\) and \(i1_X \notin A\).

(ii) \(B = A \oplus iA\).

(iii) For all \(f, g \in A\) we have
\[ \max\{L_{(X,d)}(f), L_{(X,d)}(g)\} \leq p(\tau)L_{(X,d)}(f + ig) \]
\[ \leq 2p(\tau)\max\{L_{(X,d)}(f), L_{(X,d)}(g)\} \].
(iv) $A$ is closed in $(B, L_{(X,d)}(\cdot))$ and so $(A, L_{(X,d)}(\cdot))$ is a real Banach space.

(v) $f \circ \phi \in A$ for all $f \in A$ if $\phi : X \to X$ is a base point–preserving Lipschitz mapping on $(X, d)$ with $\phi \circ \phi = \phi \circ \phi$.

(vi) If $\tau$ is the identity map on $X$, then $\text{Lip}_0(X,d,\tau) = \text{Lip}_{0,R}(X,d)$ and $\text{lip}_0(X,d,\tau) = \text{lip}_{0,R}(X,d)$.

Let $(X, d)$ be a metric space and the map $\tau : X \to X$ be a Lipschitz involution on $(X, d)$. Then $\tau^*(\text{Lip}(X,d^\alpha)) = \text{Lip}(X,d^\alpha)$ and $\tau^*(\text{lip}(X,d^\alpha)) = \text{lip}(X,d^\alpha)$ for $\alpha \in (0, 1]$. We now define

$$\text{Lip}(X, d^\alpha, \tau) = \{f \in \text{Lip}(X,d^\alpha) : \tau^*(f) = f\},$$

$$\text{lip}(X, d^\alpha, \tau) = \{f \in \text{lip}(X,d^\alpha) : \tau^*(f) = f\}.$$

The following result is a modification of [2, Theorem 1.2].

**Theorem 1.6.** Let $(X, d)$ be a metric space and $\tau$ be a Lipschitz involution on $(X, d)$. Suppose that $\alpha \in (0, 1]$ and $A = \text{Lip}(X,d^\alpha,\tau)$ and $B = \text{Lip}(X,d^\alpha)$, or, $A = \text{lip}(X,d^\alpha,\tau)$ and $B = \text{lip}(X,d^\alpha)$. Then:

(i) $A$ is a real subalgebra of $C^b(X,\tau)$ and $B$, $1_X \in A$, $i1_X \notin A$.

(ii) $B = A \oplus iA$.

(iii) For all $f, g \in A$ we have

$$\max\{\|f\|_{L_{(X,d^\alpha)}},\|g\|_{L_{(X,d^\alpha)}}\} \leq (p(\tau))^\alpha \|f + ig\|_{L_{(X,d^\alpha)}},$$

$$\leq 2 (p(\tau))^\alpha \max\{\|f\|_{L_{(X,d^\alpha)}},\|g\|_{L_{(X,d^\alpha)}}\}.$$

(iv) $A$ is closed in $(B, \|\cdot\|_{L_{(X,d^\alpha)}})$ and so $(A, \|\cdot\|_{L_{(X,d^\alpha)}})$ is a real Banach space.

(v) $f \circ \phi \in A$ for all $f \in A$ if $\phi : X \to X$ is a Lipschitz mapping on $(X, d)$ with $\phi \circ \phi = \phi \circ \phi$.

(vi) If $\tau$ is the identity map on $X$, then $\text{Lip}(X,d^\alpha,\tau) = \text{Lip}_{R}(X,d^\alpha)$ and $\text{lip}(X,d^\alpha,\tau) = \text{lip}_{R}(X,d^\alpha)$.

Real Lipschitz algebras $\text{Lip}(X,d^\alpha,\tau)$ and real little Lipschitz algebras $\text{lip}(X,d^\alpha,\tau)$ were first introduced in [1], whenever $(X,d)$ is a compact metric space. In this case, Ebadian and Ostadbashi characterized compact composition operators on these algebras in [5]. Compact composition operators on $\text{Lip}_0(X,d,\tau), \text{Lip}(X, d, \tau)$ and $\text{lip}(X, d, \tau)$ characterized in [2].

In Section 2 we first show that a bounded linear operator $T$ on a real Banach space $E$ is weakly compact if and only if the complex linear operator $T'$ on the complex Banach space $E_C$ is weakly compact, where $E_C$ is a suitable complication of $E$ and $T'$ is the complex linear operator on $E_C$ associated with $T$. Next we show that if $T$ is a weakly compact composition operator on real Lipschitz spaces of complex-valued Lipschitz functions $\text{Lip}_0(X,d,\tau)$ and $\text{lip}_0(X,d,\tau)$ on pointed compact metric space $(X,d)$ with Lipschitz involution $\tau$ or on real Lipschitz space of complex-valued Lipschitz functions $\text{Lip}(X,d^\alpha,\tau)$ and $\text{lip}(X,d^\alpha,\tau)$ on compact metric spaces $(X,d)$ with Lipschitz involution $\tau$ for $\alpha \in (0, 1)$, then $T$ is compact under certain conditions. Finally, we show that the class of weakly compact composition operators on real Lipschitz spaces of complex-valued Lipschitz functions on compact metric spaces with Lipschitz involutions is larger than the class of weakly compact composition operators on complex Lipschitz spaces of complex-valued functions on compact metric spaces.

**2. Results**

Let $E$ be a real vector space. A complex vector space $E_C$ is called a complexification of $E$ if there exists an injective real linear map $J : E \to E_C$ such that $E_C = J(E) \oplus iJ(E)$. Clearly,
Let $E \times E$ with addition and scalar multiplication defined by

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \quad (a_1, b_1, a_2, b_2 \in E)$$

$$(\alpha + i\beta)(a, b) = (\alpha a - \beta b, \beta a + \alpha b) \quad (\alpha, \beta \in \mathbb{R}, a, b \in E),$$

is a complexification of $E$ under the injective real linear map $J : E \to E \times E$ defined by $J(a) = (a, 0), a \in X$.

Let $(E, \| \cdot \|)$ be a real Banach space. By a similar proof of [3, Proposition I.13.3], one can show that there is a norm $\| \cdot \|$ on $E \times E$ with $\| (a, 0) \| = \| a \|$ for all $a \in E$ such that

$$\max\{\| a \|, \| b \|\} \leq \| (a, b) \| \leq 2 \max\{\| a \|, \| b \|\}$$

for all $a, b \in E$. Clearly, $(E \times E, \| \cdot \|)$ is a complex Banach space.

**Definition 2.1.** Let $E$ be a real linear space and $E_C$ be a complexification of $E$ under an injective real linear map $J : E \to E_C$. Suppose that $T : E \to E$ is a real linear operator on $E$ and the linear map $T' : E_C \to E_C$ defines by

$$T'(J(a) + iJ(b)) = J(T(a)) + iJ(T(b)) \quad (a, b \in E).$$

Clearly, $T'$ is a complex linear operator on $E_C$. We say that $T'$ is the complex linear operator on $E_C$ associated with $T$.

For further general facts about the complexifications of real Banach spaces, we refer to [II].

The following result is a modification of [II, Theorem 2.1] and we use it in the sequel.

**Theorem 2.1.** Let $(E, \| \cdot \|)$ be a real Banach space and $E_C$ be a complexification of $E$ under an injective real linear map $J : E \to E_C$. Suppose that $\| \cdot \|$ is a norm on $E_C$ with $\| J(a) \| = \| a \|$ for all $a \in E$ and there exist positive constants $k_1$ and $k_2$ such that

$$\max\{\| a \|, \| b \|\} \leq k_1 \| J(a) + iJ(b) \| \leq k_2 \max\{\| a \|, \| b \|\}$$

for all $a, b \in E$. Let $T : E \to E$ be a bounded real linear operator on $E$ and $T' : E_C \to E_C$ be the complex linear operator on $E_C$ associated with $T$. Then the following statements hold.

(i) $T'$ is bounded and $\| T' \|_{op} \leq k_1 k_2 \| T \|_{op}$

(ii) $T'$ is compact if and only if $T$ is compact.

For a Banach space $E$ over $\mathbb{K}$, we denote by $E^*$ the dual space of $E$.

The following lemma is a modification [II, Theorem 7] and its proof is straightforward. We will use this lemma in the next theorem.

**Lemma 2.2.** Let $(E, \| \cdot \|)$ be a real linear Banach space and $E_C$ be a complexification of $E$ under an injective real linear map $J : E \to E_C$. Suppose that $\| \cdot \|$ is a norm on $E_C$ with $\| J(a) \| = \| a \|$ for all $a \in E$ and there exist positive constant $k_1$ and $k_2$ such that

$$\max\{\| a \|, \| b \|\} \leq k_1 \| J(a) + iJ(b) \| \leq k_2 \max\{\| a \|, \| b \|\}$$

for all $a, b \in E$.

(i) If for $\lambda, \mu \in E^*$ the map $\lambda \circ \mu : E_C \to \mathbb{C}$ defines by

$$(\lambda \circ \mu)(J(a) + iJ(b)) = \lambda(a) - \mu(b) + i(\mu(a) + \lambda(b)) \quad (a, b \in E),$$

then $\lambda \circ \mu \in (E_C)^*$ and

$$\| \lambda \circ \mu \|_{op} \leq 2k_1 (\| \lambda \|_{op} + \| \mu \|_{op}).$$
(ii) Let $|| \cdot ||$ be a norm on $E^* \times E^*$, as a complexification of $E^*$, with $||(\lambda, 0)|| = ||\lambda||_{op}$ for all $\lambda \in E^*$ and

$$\max\{||\lambda||_{op}, ||\mu||_{op}\} \leq ||(\lambda, \mu)|| \leq 2\max\{||\lambda||_{op}, ||\mu||_{op}\}$$

for all $\lambda, \mu \in E^*$. Then the map $\Psi : E^* \times E^* \longrightarrow (E_C)^*$ defined by

$$\Psi(\lambda, \mu) = \lambda \circ \mu \quad (\lambda, \mu \in E^*),$$

is a bijective complex linear operator and continuous from the complex Banach space $(E^* \times E^*, || \cdot ||_{op})$ onto the complex Banach space $((E_C)^*, || \cdot ||_{op})$. Moreover, $\Psi^{-1}$ is a bounded linear operator from $((E_C)^*, || \cdot ||_{op})$ to $(E^* \times E^*, || \cdot ||_{op})$.

**Theorem 2.3.** Let $(E, || \cdot ||)$ be a real linear Banach space and $E_C$ be a Complexification of $E$ under an injective real linear map $J : E \longrightarrow E_C$. Suppose that $|| \cdot ||$ is a norm on $E_C$ with $||J(a)|| = ||a||$ for all $a \in E$ and there exist positive constants $k_1$ and $k_2$ such that

$$\max\{||a||, ||b||\} \leq k_1 ||J(a) + iJ(b)|| \leq k_2 \max\{||a||, ||b||\}$$

for all $a, b \in E$. Let $T : E \longrightarrow E$ be a bounded real linear operator on $E$ and $T' : E_C \longrightarrow E_C$ be the complex linear operator on $E_C$ associated with $T$. Then $T$ is a weakly compact operator on the real Banach space $(E, || \cdot ||)$ if and only if $T'$ is weakly compact operator on the complex Banach space $(E_C, || \cdot ||)$.

**Proof.** Let $|| \cdot ||$ be a norm on $E^* \times E^*$, as a complexification of $E^*$, with $||(\lambda, 0)|| = ||\lambda||_{op}$ for all $\lambda \in E^*$ and

$$\max\{||\lambda||_{op}, ||\mu||_{op}\} \leq ||(\lambda, \mu)|| \leq 2\max\{||\lambda||_{op}, ||\mu||_{op}\}$$

for all $\lambda, \mu \in E^*$. Define the map $\Psi : E^* \times E^* \longrightarrow (E_C)^*$ by

$$\Psi(\lambda, \mu) = \lambda \circ \mu \quad (\lambda, \mu \in E^*),$$

where $\lambda \circ \mu \in (E_C)^*$ defines by

$$(\lambda \circ \mu)(J(a) + iJ(b)) = (\lambda(a) - \mu(b)) + i(\mu(a) + \lambda(b)), \quad (a, b \in E).$$

By Lemma 2.2 $\Psi$ is a bijection complex linear operator and a homeomorphism from the complex Banach space $(E^* \times E^*, || \cdot ||_{op})$ onto the complex Banach space $((E_C)^*, || \cdot ||_{op})$.

We first assume that $T$ is weakly compact. To prove the weakly compactness of $T'$, let $\{c_n\}_{n=1}^\infty$ be a bounded sequence in $(E_C, || \cdot ||)$. For each $n \in \mathbb{N}$ there exists $(a_n, b_n) \in E \times E$ such that $c_n = J(a_n) + iJ(b_n)$. It is clear that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are bounded seques in $(E, || \cdot ||)$. Since $T$ is a weakly compact linear operator on $(E, || \cdot ||)$, by Theorem 1.1 there exist strictly increasing functions $q, r : \mathbb{N} \longrightarrow \mathbb{N}$ and elements $a, b \in E$ such that

$$\lim_{k \to \infty} Ta_{q(k)} = a \quad \text{(in } E \text{ with the weak-topology)},$$

$$\lim_{k \to \infty} Tb_{r(k)} = b \quad \text{(in } E \text{ with the weak-topology)}.$$

For each $k \in \mathbb{N}$, set $n_k = r(q(k))$. Then $\{a_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{a_n\}_{n=1}^\infty$, $\{b_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{b_n\}_{n=1}^\infty$,

$$\lim_{k \to \infty} Ta_{n_k} = a \quad \text{(in } E \text{ with the weak-topology)},$$

and,

$$\lim_{k \to \infty} Tb_{n_k} = b \quad \text{(in } E \text{ with the weak-topology)}.$$

Let $\Lambda \in (E_C)^*$. Then there exist $\lambda, \mu \in E^*$ such that

$$\Lambda = \Psi(\lambda, \mu) = \lambda \circ \mu.$$
Now we have
\begin{equation}
\lim_{k \to \infty} \lambda(a_{nk}) = \lambda(a),
\end{equation}
and,
\begin{equation}
\lim_{k \to \infty} \mu(b_{nk}) = \mu(b),
\end{equation}
by (2.1) and (2.2), respectively. From (2.4), (2.5) and (2.3) we get
\begin{equation}
\lim_{k \to \infty} \Lambda(T^c C_{n_k}) = \Lambda(J(a) + iJ(b)).
\end{equation}
Since \((E_C)^*\) separates the point of \(E_C\) and (2.6) holds for each \(\Lambda \in (E_C)^*\), we conclude that
\begin{equation}
\lim_{k \to \infty} T^c(C_{n_k}) = J(a) + iJ(b) \quad \text{(in } E_C \text{ with the weak-topology)}.
\end{equation}
This implies that \(T^c\) is weakly compact operator on \((E_C, \| \cdot \|)\) by Theorem [1,1].

We now assume that \(T^c\) is a weakly compact operator on \((E_C, \| \cdot \|)\). To prove the weakly compactness of \(T\) on \((E, \| \cdot \|)\), let \(\{a_n\}_{n=1}^{\infty}\) be a bounded sequence in \((E, \| \cdot \|)\). It is clear that \(\{J(a_n)\}_{n=1}^{\infty}\) is a bounded sequence in \((E_C, \| \cdot \|)\). Since \(T^c : E_C \to E_C\) is a weakly compact linear operator on \(E_C\), by Theorem [1,1] there exist a subsequence \(\{a_{nk}\}_{k=1}^{\infty}\) of \(\{a_n\}_{k=1}^{\infty}\) and an element \(c \in E_C\) such that
\begin{equation}
\lim_{k \to \infty} T^c(J(a_{nk})) = c \quad \text{(in } E_C \text{ with the weak-topology)}.
\end{equation}
Since \(c \in E_C\), there exists \((a, b) \in E \times E\) such that
\begin{equation}
c = J(a) + iJ(b).
\end{equation}
We claim that
\begin{equation}
\lim_{k \to \infty} T(a_{nk}) = a \quad \text{(in } E \text{ with the weak-topology)}.
\end{equation}
Let \(\lambda \in E^*\). Set \(\Lambda = \Psi(\lambda, 0)\). Then \(\Lambda \in (E_C)^*\). Hence, by (2.8) we have
\begin{equation}
\lim_{k \to \infty} \Lambda(T^c(J(a_{nk}))) = \Lambda(c).
\end{equation}
From (2.10) and (2.9), we get
\begin{equation}
\lim_{k \to \infty} (\lambda \circ 0)(J(Ta_{nk})) = (\lambda \circ 0)(J(a) + iJ(b))
\end{equation}
and so
\begin{equation}
\lim_{k \to \infty} \lambda(Ta_{nk}) = \lambda(a).
\end{equation}
Since \(E^*\) separates the points of \(E\) and (2.11) holds for each \(\lambda \in E^*\), we deduce that
\begin{equation}
\lim_{k \to \infty} T(a_{nk}) = a \quad \text{(in } E \text{ with the weak-topology)}.
\end{equation}
Therefore, \(T\) is weakly compact by Theorem [1,1].

**Theorem 2.4.** Let \((X, d)\) be a pointed compact metric space, \(\tau\) be a base point-preserving Lipschitz involution on \((X, d)\) and \(A = \text{Lip}_0(X, d, \tau)\) or \(A = \text{lip}_0(X, d, \tau)\). Suppose that the complex little Lipschitz space \(\text{lip}_0(X, d)\) separates points uniformly on \(X\). Let \(\phi : X \to X\) be a base point-preserving Lipschitz mapping on \((X, d)\) with \(\tau \circ \phi = \phi \circ \tau\) and \(T : A \to A\) be the composition operator on \(A\) induced by \(\phi\). If \(T\) is weakly compact, then \(T\) is compact.
Proof. We assume that $A_C = \text{Lip}_0(X, d)$ if $A = \text{Lip}_0(X, d, \tau)$ and $A_C = \text{lip}_0(X, d, \tau)$ if $A = \text{Lip}_0(X, d)$. By Theorem 1.5, $A_C$ is a complexification of $A$ under the injective real linear map $J : A \rightarrow A_C$ defined by $J(f) = f (f \in A), (A, \| \|_{L(x,d)})$ is a real Banach space and $L(x,d)(\cdot)$ is a norm on the complex vector space $A_C$ with $L(x,d)(J(f)) = L(x,d)(f)$ for all $f \in A$ and

$$\max\{L(x,d)(f), L(x,d)(g)\} \leq p(\tau)L(x,d)(J(f) + iJ(g)) \leq 2p(\tau)\max\{L(x,d)(f), L(x,d)(g)\}$$

for all $f, g \in A$. Suppose that $T$ is weakly compact. Let $T' : A_C \rightarrow A_C$ be the complex linear operator on $A_C$ associated with $T$. Then $T'$ is weakly compact by Theorem 2.3. It is easy to see that $T'$ is the composition operator on $A_C$ induced by $\phi$. Since $\text{lip}_0(X, d)$ separates points uniformly on $X$, we deduce that $T'$ is compact by Theorem 1.2. This implies that $T : A \rightarrow A$ is compact by Theorem 2.1.

By part (vi) of Theorem 1.5, it is clear that Theorem 2.4 extends [6, Theorem 2.3] and [6, Corollary 2.4] whenever $K = \mathbb{R}$.

Theorem 2.5. Let $(X, d)$ be a compact metric space, $\tau$ be a Lipschitz involution on $(X, d)$, $\alpha \in (0, 1)$ and $A = \text{Lip}(X, d^\alpha, \tau)$ or $A = \text{lip}(X, d^\alpha, \tau)$. Let $\phi : X \rightarrow X$ be a Lipschitz mapping on $(X, d)$ with $\tau \circ \phi = \phi \circ \tau$ and $T : A \rightarrow A$ be the composition operator on $A$ induced by $\phi$. If $T$ is weakly compact, then $T$ is compact.

Proof. We assume that $A_C = \text{Lip}(X, d^\alpha)$ if $A = \text{Lip}(X, d^\alpha, \tau)$ and $A_C = \text{lip}(X, d^\alpha)$ if $A = \text{lip}(X, d^\alpha, \tau)$. By Theorem 1.6, $A_C$ is a complexification of $A$ under the injective real linear map $J : A \rightarrow A_C$ defined by $J(f) = f (f \in A), (A, \| \|_{L(x,d^\alpha)})$ is a real Banach space and $\| \|_{L(x,d^\alpha)}$ is a norm on the complex vector space $A_C$ with $\| J(f) \|_{L(x,d^\alpha)} = \| f \|_{L(x,d^\alpha)}$ for all $f \in A$ and

$$\max\{\| f \|_{L(x,d^\alpha)}, \| g \|_{L(x,d^\alpha)}\} \leq (p(\tau))^\alpha \| J(f) + (J(g)) \|_{L(x,d^\alpha)} \leq 2(p(\tau))^\alpha \max\{\| f \|_{L(x,d^\alpha)}, \| g \|_{L(x,d^\alpha)}\}$$

for all $f, g \in A$. Suppose that $T$ is weakly compact. Let $T' : A_C \rightarrow A_C$ be the complex linear operator on $A_C$ associated with $T$. Then $T'$ is weakly compact by Theorem 2.3. It is easy to see that $T'$ is the composition operator on $A_C$ induced by $\phi$. By Theorem 1.4, $T'$ is compact. This implies that $T$ is compact by Theorem 2.1.

By part (vi) of Theorem 1.6, it is clear that Theorem 2.5 extends Theorem 1.4 in the case $K = \mathbb{R}$.

Now, we show that the class of weakly compact composition operators on real Lipschitz spaces of complex-valued functions on compact metric spaces with Lipschitz involutions is larger than the class of complex linear operators on complex Lipschitz spaces of complex-valued functions on compact metric spaces.

Theorem 2.6. Let $(X, d)$ be a compact metric space, $B = \text{Lip}(X, d^\alpha)$ for $\alpha \in (0, 1]$, or $B$ be the complexification of $B$ induced by the Lipschitz mapping $\phi$ on $(X, d)$. Let $Y = X \times \{0, 1\}, \rho$ be the metric on $Y$ defined by $\rho((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), |y_1 - y_2|\}$ and $\tau : Y \rightarrow Y$ be the Lipschitz involution on $(Y, \rho)$ defined by $\tau(x, 0) = (x, 1), \tau(x, 1) = (x, 0) (x \in X)$.

Suppose that $A = \text{Lip}(Y, \rho^\alpha, \tau)$ or $B = \text{Lip}(X, d^\alpha)$, and $A = \text{lip}(Y, \rho^\alpha, \tau)$ if $B = \text{lip}(X, d^\alpha)$. Let $\phi : X \rightarrow Y$ be the self-map of $Y$ defined by $\psi(x, 0) = (\phi(x), 0), \psi(x, 1) = (\phi(x), 1) (x \in X)$. 

Then the following statements hold.

(i) \( \psi \) is a Lipschitz involution on \((Y, \rho)\) and \( \psi \circ \tau = \tau \circ \psi \).

(ii) If \( S : A \rightarrow A \) is the composition endomorphism of \( A \) induced by \( \psi \), then \( S \) is weakly compact if and only if \( T \) is weakly compact.

Proof. Clearly, (i) holds. We prove (ii) in the case \( B = \text{Lip}(X, d^\alpha) \) and \( A = \text{Lip}(Y, \rho^\alpha, \tau) \) for \( \alpha \in (0, 1) \). Define the map \( \Lambda : B \rightarrow A \) by

\[
(\Lambda f)(x, 0) = f(x) \quad (f \in B, x \in X),
\]

\[
(\Lambda f)(x, 1) = \overline{f(x)} \quad (f \in B, x \in X).
\]

Then \( \Lambda \) is an injective bounded real linear operator from \((B, \| \cdot \|_{\text{Lip}(X, d^\alpha)})\), regarded as a real Banach algebra, onto \((A, \| \cdot \|_{\text{Lip}(Y, \rho^\alpha)})\). By open mapping theorem for real Banach spaces, \( \Lambda^{-1} \) is a bounded linear operator from \((A, \| \cdot \|_{\text{Lip}(Y, \rho^\alpha)})\) into \((B, \| \cdot \|_{\text{Lip}(X, d^\alpha)})\). We can easily show that \( \Lambda \circ T \circ \Lambda^{-1} = S \). Therefore, \( S \) is weakly compact if only if \( T \) is weakly compact.

To prove (ii) in the case \( B = \text{lip}(X, d^\alpha) \) and \( A = \text{lip}(Y, \rho^\alpha, \tau) \) for \( \alpha \in (0, 1) \), it is sufficient that we apply \( \Gamma = \Lambda |_{\text{lip}(X, d^\alpha)} \) instead of \( \Lambda \).

According to Theorem 2.5, we deduce that Theorem 2.6 extends Theorem 1.4 whenever \( K = \mathbb{C} \).

REFERENCES


