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SOME PROPERTIES OF $k-$QUASI CLASS Q^* OPERATORS

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ABSTRACT. In this paper, we give some results of $k-$quasi class Q^* operators. We proved that if T is an invertible operator and N be an operator such that N commutes with T^*T, then N is $k-$quasi class Q^* if and only if TNT^{-1} is of $k-$quasi class Q^*. With example we proved that exist an operator $k-$quasi class Q^* which is quasi nilpotent but it is not quasi hyponormal.

Key words and phrases: $k-$quasi class Q^* operator; quasi nilpotent; quasi hyponormal.

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1. INTRODUCTION

Let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra of all bounded operators on $\mathcal{H}$ and let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T and by $r(t)$ the spectral radius of operator T which is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is a positive operator, $T \geq O$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. If two operator $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$ are positive operators and if $AB = BA$ then the product AB is also positive operator. The $|T| = (T^*T)^{\frac{1}{2}}$ is a positive operator and we have that $|T|^2 = T^*T$ and $|T^*|^2 = TT^*$.

The operator T is called unitary operator if $T^*T = TT^* = I$. The operator T is normaloid if $r(T) = \|T\|$ and it is quasi nilpotent if $r(T) = 0$.

An operator $T \in \mathcal{L}(\mathcal{H})$, is said to be paranormal [4], if $\|Tx\|^2 \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$. Further, T is said to be quasi hyponormal [3], if $\|T^*Tx\| \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$.

An operator T is called $k-$quasi$-*-$paranormal if $\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\|$, for all $x \in \mathcal{H}$, where k is a natural number, [8].

An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to class Q^* if $T^{*2}T^2 - 2TT^* + I \geq O$ or equivalent if $\|T^*x\|^2 \leq \frac{1}{2}\left(\|T^2x\|^2 + \|x\|^2\right)$, for all $x \in \mathcal{H}$ ([6]).

An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to $k-$quasi class Q^* if

$$\|T^*T^kx\|^2 \leq \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^kx\|^2\right),$$

for all $x \in \mathcal{H}$, where k is a natural number. Equivalently, operator $T \in \mathcal{L}(\mathcal{H})$ belongs to $k-$quasi class Q^* if $T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k \geq O$, where k is a natural number ([5]).

Aluthge in [1] define a transformation $\tilde{T}$ of operator T by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of operator T. $\tilde{T}$ is called Aluthge transformation.

Yamazaki in [7] define the $^*-$Aluthge transformation of operator T. The $*-$Aluthge transformation is defined by $\tilde{T}^{(*)} \overset{def}{=} (\tilde{T^*})^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$.

It is proved that $U^*|T^*|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U^*, U^*|T^*| = |T|U^*, U|T|^{\frac{1}{2}} = |T^*|^{\frac{1}{2}}U, U|T| = |T^*|U$.

2. MAIN RESULTS

In this section we prove some properties of $k-$quasi class Q^* operators.

Theorem 2.1. *Let be T be an invertible operator and N be an operator such that N commutes with T^*T. Then N is $k-$quasi class Q^* if and only if TNT^{-1} is of $k-$quasi class Q^*.*

Proof. Let N be a $k-$quasi class Q^* operator.

$$N^{*k}(N^{*2}N^2 - 2NN^* + I)N^k \geq 0.$$

From this we have that:

$$TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^* \geq 0.$$

Consider,

$$\begin{aligned}
&TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*[TT^*]\\
=&TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^k[T^*T]T^*\\
=&T[T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*\\
=&[TT^*]TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*.
\end{aligned}$$

So, we see that operator TT^* commutes with operator

$$TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*.$$

Then operator $[TT^*]^{-1}$ also commutes with operator

$$TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*.$$

Since the operators $[TT^*]^{-1}$ and $TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*$ are positive and since they commute with each other we have that their product is also positive operator:

$$T^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*[TT^*]^{-1} \geq 0.$$

Since operator N commutes with operator T^*T, we get,

$$(TNT^{-1})^{*k} = (TNT^{-1})^*(TNT^{-1})^* \cdots (TNT^{-1})^*$$

$$= T^{*-1}N^*T^*T^{*-1}N^*T^* \cdots T^{*-1}N^*T^* = T^{*-1}N^{*k}T^* \tag{2.1}$$

$$(TNT^{-1})^k = TNT^{-1}TNT^{-1} \cdots TNT^{-1} = TN^kT^{-1} \tag{2.2}$$

$$(TNT^{-1})^{*2}(TNT^{-1})^2 = TN^{*2}N^2T^{-1} \tag{2.3}$$

$$(TNT^{-1})(TNT^{-1})^* = TNT^{-1}T^{*-1}N^*T^* = TNN^*T^{-1} \tag{2.4}$$

To prove that TNT^{-1} is $k-$quasi class Q^* operator, the equation (2.1), (2.2), (2.3) and (2.4)we substitute in above expression:

$$(TNT^{-1})^{*k}[(TNT^{-1})^{*2}(TNT^{-1})^2 - 2(TNT^{-1})(TNT^{-1})^* + I](TNT^{-1})^k$$

and we have

$$\begin{aligned}
&(TNT^{-1})^{*k}[(TNT^{-1})^{*2}(TNT^{-1})^2 - 2(TNT^{-1})(TNT^{-1})^* + I](TNT^{-1})\\
=&T^{*-1}N^{*k}T^*[TN^{*2}N^2T^{-1} - 2TNN^*T^{-1} + I]TN^kT^{-1}\\
=&T^{*-1}N^{*k}T^*T[N^{*2}N^2 - 2NN^* + I]T^{-1}TN^kT^{-1}\\
=&T^{*-1}N^{*k}T^*T[N^{*2}N^2 - 2NN^* + I]N^kT^{-1}\\
=&T^{*-1}T^*TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^{-1}\\
=&TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^{-1}
\end{aligned}$$

Now we have to prove that that the last expression is positive. From the fact that we prove that

$$TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*[TT^*]^{-1} \geq 0$$

we have that:

$$\begin{aligned}
&\Rightarrow TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^*T^{*-1}T^{-1} \geq 0\\
&\Rightarrow TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^{-1} \geq 0
\end{aligned}$$

Hence, TNT^{-1} is $k-$quasi class Q^* operator.

Conversely, let TNT^{-1} be a $k-$quasi class Q^* operator.

$$(TNT^{-1})^{*k}[(TNT^{-1})^{*2}(TNT^{-1})^2 - 2(TNT^{-1})(TNT^{-1})^* + I](TNT^{-1})^k \geq 0.$$

Then similar as before, after substituting the equation (2.1), (2.2), (2.3) and (2.4)we have:

$$\begin{aligned}&\Rightarrow TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^{-1} \geq 0\\&\Rightarrow T^*TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^{-1}T \geq 0\\&\Rightarrow [T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k \geq 0.\end{aligned}$$

Since operator $[T^*T]$ commutes with operator N and hence commute with operator

$$[T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k.$$

The also the operator $[T^*T]^{-1}$ commutes with operator

$$[T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k.$$

Since the operators $[T^*T]^{-1}$ and $[T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k$ are positive and since they commute with each other we have:

$$[T^*T]^{-1}[T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k \geq 0.$$

Therefore,

$$N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k \geq 0.$$

Hence, N is $k-$quasi class Q^* operator. ∎

Corollary 2.2. *Let S be a $k-$quasi class Q^* operator and M any positive operator such that $M^{-1} = M^*$. Then $T = M^{-1}SM$ is $k-$quasi class Q^* operator.*

Proof. Let S be a $k-$quasi class Q^* operator. Then

$$S^{*k}(S^{*2}S^2 - 2SS^* + I)S^k \geq O.$$

Consider,

$$\begin{aligned}&T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k\\&= (M^{-1}SM)^{*k}((M^{-1}SM)^{*2}(M^{-1}SM)^2 - 2(M^{-1}SM)(M^{-1}SM)^* + I)(M^{-1}SM)^k\\&=M^*S^*M^{-1*}M^*S^*M^{-1*}...M^*S^*M^{-1*}(M^*S^*M^{-1*}M^*S^*M^{-1*}M^{-1}SMM^{-1}SM\\&\quad - 2M^*S^*M^{-1*}M^{-1}SM + I)M^{-1}SMM^{-1}SM...M^{-1}SM\\&=M^*S^{*k}(S^{*2}S^2 - 2SS^* + I)S^kM \geq O.\end{aligned}$$

hence, $T = M^{-1}SM$ is $k-$quasi class Q^* operator. ∎

Theorem 2.3. *Let be $T \in L(H)$. Then $\tilde{T}$ is $k-$quasi class Q^* operator if and only if $\tilde{T}^{(*)}$ is $k-$quasi class Q^* operator.*

Proof. Assume that $\tilde{T}$ is $k-$quasi class Q^* then

$$\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^k \geq 0.$$

We need to prove that $\tilde{T}^{(*)}$ is $k-$quasi -class Q^* operator.

$$
\begin{aligned}
&\tilde{T}^{(*)*k}(\tilde{T}^{(*)*2}\tilde{T}^{(*)2} - 2\tilde{T}^{(*)}\tilde{T}^{(*)*} + I)\tilde{T}^{(*)k} \\
=&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*k} \\
&((|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*2}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^2 - 2(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^* + I) \\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^k \\
=&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^* \cdots (|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^* \\
&[(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \\
&-2(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}) + I] \\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \cdots (|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \\
=&(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}) \cdots (|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}) \\
&[(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \\
&-2(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}) + I] \\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \cdots (|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \\
=&UU^*(|T^*|^{\frac{1}{2}}U^*|T^*|U^* \cdots U^*|T^*|^{\frac{1}{2}})UU^* \\
&[|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*U|T^*|U|T^*|^{\frac{1}{2}} - 2|T^*|^{\frac{1}{2}}U|T^*|U^*|T^*|^{\frac{1}{2}} + I] \\
&UU^*(|T^*|^{\frac{1}{2}}U|T^*|U \cdots U|T^*|^{\frac{1}{2}})UU^* \\
=&U(U^*|T^*|^{\frac{1}{2}}U^*|T^*|U^* \cdots U^*|T^*|^{\frac{1}{2}}U)[U^*|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*U|T^*|U|T^*|^{\frac{1}{2}}U \\
&-2U^*|T^*|^{\frac{1}{2}}U|T^*|U^*|T^*|^{\frac{1}{2}}U + I](U^*|T^*|^{\frac{1}{2}}U|T^*|U \cdots U|T^*|^{\frac{1}{2}}U)U^* \\
=&U(|T|^{\frac{1}{2}}U^*|T|U^* \cdots U^*|T|^{\frac{1}{2}}U)[|T|^{\frac{1}{2}}U^*|T|U^*|T|U|T|U|T|^{\frac{1}{2}} \\
&-2|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}} + I](|T|^{\frac{1}{2}}U|T|U \cdots U|T|^{\frac{1}{2}})U^* \\
=&U(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}) \cdots (|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}) \\
&[(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&-2(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}) + I] \\
&(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \cdots (|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})U^* \\
=&U\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^kU^* \geq 0.
\end{aligned}
$$

Therefore

$$\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^k \geq 0.$$

Hence $\tilde{T}^{(*)}$ is $k-$quasi class Q^* operator.
Conversely, assume that $\tilde{T}^{(*)}$ is $k-$quasi class Q^* operator, then

$$\tilde{T}^{(*)*k}(\tilde{T}^{(*)*2}\tilde{T}^{(*)2} - 2\tilde{T}^{(*)}\tilde{T}^{(*)*} + I)\tilde{T}^{(*)k} \geq 0.$$

We need to prove that $\tilde{T}$ is $k-$quasi class Q^*.
Consider

$$
\begin{aligned}
&\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^k \\
=&U^*U(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})\cdots(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})U^*U \\
&[(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) - 2(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}) + I] \\
&U^*U(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})\cdots(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})U^*U \\
=&U^*(U|T|^{\frac{1}{2}}U^*|T|U^*\cdots U^*|T|^{\frac{1}{2}}U^*) \\
&[U|T|^{\frac{1}{2}}U^*|T|U^*|T|U|T|U|T|^{\frac{1}{2}}U^* - 2U|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}U^* + I] \\
&(U|T|^{\frac{1}{2}}U|T|U\cdots U|T|^{\frac{1}{2}}U^*)U \\
=&U^*(|T^*|^{\frac{1}{2}}U^*|T^*|U^*\cdots U^*|T^*|^{\frac{1}{2}}) \\
&[|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*|U|T^*|U|T^*|^{\frac{1}{2}} - 2|T^*|^{\frac{1}{2}}U|T^*|U^*|T^*|^{\frac{1}{2}} + I] \\
&(|T^*|^{\frac{1}{2}}U|T^*|U\cdots U|T^*|^{\frac{1}{2}})U \\
=&U^*(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})\cdots(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}) \\
&[(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) \\
&-2(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}) + I] \\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\cdots(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})U \\
=&U^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*k} \\
&[(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*2}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^2 \\
&-2(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}) + I](|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^kU \\
=&U^*\tilde{T}^{(*)*k}(\tilde{T}^{(*)*2}\tilde{T}^{(*)2} - 2\tilde{T}^{(*)}\tilde{T}^{(*)*} + I)\tilde{T}^{(*)k}U
\end{aligned}
$$

Therefore

$$\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^k) \geq 0.$$

Hence $\tilde{T}$ is $k-$quasi class Q^* operator. ∎

Proposition 2.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be the operator defined as*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

If A is operator of class Q^* and $BB^* = 0$, then T is an operator of $k-$quasi class Q^*.

Proof. A simple calculation shows that:

$$T^* = \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix},$$
$$T^{*k} = \begin{pmatrix} A^{*k} & 0 \\ B^*A^{*(k-1)} & 0 \end{pmatrix},$$
$$T^k = \begin{pmatrix} A^k & A^{(k-1)}B \\ 0 & 0 \end{pmatrix},$$
$$T^{*k}TT^*T^k = \begin{pmatrix} A^{*k}(AA^*+BB^*)A^k & A^{*k}(AA^*+BB^*)A^{(k-1)}B \\ B^*A^{*(k-1)}(AA^*+BB^*)A^k & B^*A^{*(k-1)}(AA^*+BB^*)A^{(k-1)}B \end{pmatrix}$$
$$= \begin{pmatrix} A^{*k}AA^*A^k & A^{*k}AA^*A^{(k-1)}B \\ B^*A^{*(k-1)}AA^*A^k & B^*A^{*(k-1)}AA^*A^{(k-1)}B \end{pmatrix},$$
$$T^{*(k+2)} = \begin{pmatrix} A^{*(k+2)} & 0 \\ B^*A^{*(k+1)} & 0 \end{pmatrix},$$
$$T^{(k+2)} = \begin{pmatrix} A^{(k+2)} & A^{(k+1)}B \\ 0 & 0 \end{pmatrix},$$
$$T^{*(k+2)}T^{(k+2)} = \begin{pmatrix} A^{*(k+2)}A^{(k+2)} & A^{*(k+2)}A^{(k+1)}B \\ B^*A^{*(k+1)}A^{(k+2)} & B^*A^{*(k+1)}A^{(k+1)}B \end{pmatrix}.$$
$$T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k = T^{*(k+2)}T^{(k+2)} - 2T^{*k}TT^*T^k + T^{*k}T^k =$$
$$\begin{pmatrix} A^{*k}(A^{*2}A^2 - 2AA^* + I)A^k & A^{*k}(A^{*2}A^2 - 2AA^* + I)A^{(k-1)}B \\ B^*A^{*(k-1)}(A^{*2}A^2 - 2AA^* + I)A^k & B^*A^{*(k-1)}(A^{*2}A^2 - 2AA^* + I)A^{(k-1)}B \end{pmatrix}$$

Let $u = x \oplus y \in \mathcal{H} \oplus \mathcal{H}$. Then,

$$\begin{aligned}
&\langle (T^{*(k+2)}T^{(k+2)} - 2T^{*k}TT^*T^k + T^{*k}T^k)u, u\rangle \\
&= \langle A^{*k}(A^{*2}A^2 - 2AA^* + I)A^k x, x\rangle + \langle A^{*k}(A^{*2}A^2 - 2AA^* + I)A^{(k-1)}By, x\rangle \\
&+ \langle B^*A^{*(k-1)}(A^{*2}A^2 - 2AA^* + I)A^k x, y\rangle \\
&+ \langle B^*A^{*(k-1)}(A^{*2}A^2 - 2AA^* + I)A^{(k-1)}By, y\rangle \\
&= \langle (A^{*2}A^2 - 2AA^* + I)A^k x, A^k x\rangle \\
&+ \langle (A^{*2}A^2 - 2AA^* + I)A^{(k-1)}By, A^k x\rangle \\
&+ \langle (A^{*2}A^2 - 2AA^* + I)A^k x, A^{(k-1)}By\rangle \\
&+ \langle (A^{*2}A^2 - 2AA^* + I)A^{(k-1)}By, A^{(k-1)}By\rangle \\
&= \langle (A^{*2}A^2 - 2AA^* + I)(A^k x + A^{(k-1)}By), (A^k x + A^{(k-1)}By)\rangle \geq 0
\end{aligned}$$

because A is operator of class Q^* then, $A^{*2}A^2 - 2AA^* + I \geq O$, so this prove the result. ∎

Proposition 2.5. *Every quasi hyponormal operator is operator of quasi class Q^*.*

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ be a quasi hypormal operator, then

$$\|T^*Tx\| \leq \|T^2x\|.$$

Since every quasi hyponormal operator is paranormal [3, Corollary 3.15] then we have

$$\begin{aligned}\|T^*Tx\| &\leq \|T^2x\|^2 = \|T\left(\frac{Tx}{\|Tx\|}\right)\|^2 \cdot \|Tx\|^2 \\ &\leq \|T^2\left(\frac{Tx}{\|Tx\|}\right)\| \cdot \|Tx\|^2 = \|T^3x\| \cdot \|Tx\| \\ &\leq \frac{1}{2}\left(\|T^3x\|^2 + \|Tx\|^2\right),\end{aligned}$$

So, T is operator of quasi class Q^*. ∎

In following example we will prove that exist an operator $k-$quasi class Q^* which is quasi nilpotent but it is not quasi hyponormal.

Example 2.1. *Consider the operator $T : l^2 \to l^2$ defined by*

$$T(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots)$$

where $\alpha_1 = \dfrac{1}{2^n}$ for $n \geq 1$. Operator T is of $k-$quasi class Q^* and quasi nilpotent but it is not quasi hyponormal.
Given $T(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots)$. Then $T^*(x) = (\alpha_1 x_1, \alpha_2 x_2, \ldots)$,

$$\begin{aligned}
&T^2(x) = (0, 0, \alpha_1\alpha_2 x_1, \alpha_2\alpha_3 x_2, \ldots),\\
&T^k(x) = (\overbrace{0,0,0,\ldots,0}^{k-time}, \alpha_1\alpha_2\ldots\alpha_k x_1, \alpha_2\alpha_3\ldots\alpha_{k+1}x_2, \ldots),\\
&T^*T^k(x) = (\overbrace{0,0,0,\ldots,0}^{(k-1)-time}, \alpha_1\alpha_2\ldots\alpha_k^2 x_1, \alpha_2\alpha_3\ldots\alpha_{k+1}^2 x_2, \ldots),\\
&T^{*k}T^k(x) = (\alpha_1^2\alpha_2^2\ldots\alpha_k^2 x_1, \alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2 x_2, \ldots),\\
&T^{*k}T^{k+1}(x) = (0, \alpha_1^2\alpha_2^2\ldots\alpha_k^2\alpha_{k+1}x_1, \alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2\alpha_{k+2}x_2, \ldots),\\
&T^{*(k+1)}T^{k+1}(x) = (\alpha_1^2\alpha_2^2\ldots\alpha_k^2\alpha_{k+1}^2 x_1, \alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2\alpha_{k+2}^2 x_2, \ldots),\\
&T^{*(k+1)}T^{k+2}(x) = (0, \alpha_1^2\alpha_2^2\ldots\alpha_{k+1}^2\alpha_{k+2}x_1, \alpha_2^2\alpha_3^2\ldots\alpha_{k+2}^2\alpha_{k+3}x_2, \ldots),\\
&T^{*(k+2)}T^{k+2}(x) = (\alpha_1^2\alpha_2^2\ldots\alpha_{k+1}^2\alpha_{k+2}^2 x_1, \alpha_2^2\alpha_3^2\ldots\alpha_{k+2}^2\alpha_{k+3}^2 x_2, \ldots),\\
&TT^*T^k(x) = (\overbrace{0,0,0,\ldots,0}^{(k)-time}, \alpha_1\alpha_2\ldots\alpha_k^3 x_1, \alpha_2\alpha_3\ldots\alpha_{k+1}^3 x_2, \ldots),\\
&T^*TT^*T^k(x) = (\overbrace{0,0,0,\ldots,0}^{(k-1)-time}, \alpha_1\alpha_2\ldots\alpha_k^4 x_1, \alpha_2\alpha_3\ldots\alpha_{k+1}^4 x_2, \ldots),\\
&T^{*k}TT^*T^k(x) = (\alpha_1^2\alpha_2^2\ldots\alpha_{k-1}^2\alpha_k^4 x_1, \alpha_2^2\alpha_3^2\ldots\alpha_k^2\alpha_{k+1}^4 x_2, \ldots).
\end{aligned}$$

Now consider

$$\begin{aligned}
&\langle T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k x, x\rangle\\
&\quad=\langle (T^{*(k+2)}T^{k+2} - 2T^{*k}TT^*T^k + T^{*(k)}T^k)x, x\rangle\\
&\quad=\langle (\alpha_1^2\alpha_2^2\ldots\alpha_{k+1}^2\alpha_{k+2}^2 - 2\alpha_1^2\alpha_2^2\ldots\alpha_{k-1}^2\alpha_k^4 + \alpha_1^2\alpha_2^2\ldots\alpha_k^2)x_1, x_1\rangle\\
&\quad+\langle (\alpha_2^2\alpha_3^2\ldots\alpha_{k+2}^2\alpha_{k+3}^2 - 2\alpha_2^2\alpha_3^2\ldots\alpha_k^2\alpha_{k+1}^4 + \alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2)x_2, x_2\rangle + \ldots\\
&\quad=\alpha_1^2\alpha_2^2\ldots\alpha_k^2(\alpha_{k+1}^2\alpha_{k+2}^2 - 2\alpha_k^2 + 1)\|x_1\|^2\\
&\quad+\alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2(\alpha_{k+2}^2\alpha_{k+3}^2 - 2\alpha_{k+1}^2 + 1)\|x_2\|^2 + \ldots \geq 0.
\end{aligned}$$

Because

$$\begin{aligned}&\alpha_{n+k}^2\alpha_{n+k+1}^2-2\alpha_{n+k-1}^2+1\\ =&\left(\frac{1}{2^{n+k}}\right)^2\cdot\left(\frac{1}{2^{n+k+1}}\right)^2-2\left(\frac{1}{2^{n+k-1}}\right)^2+1\geq 0, k\geq 1, n\geq 1.\end{aligned}$$

From $T(e_k)=\dfrac{1}{2^k}e_{k+1}$ *we have*

$$\begin{aligned}T^2(e_k)&=\frac{1}{2^k}\cdot\frac{1}{2^{k+1}}e_{k+2}\\ &.\\ &.\\ &.\\ T^n(e_k)&=\frac{1}{2^k}\cdot\frac{1}{2^{k+1}}\cdot\ldots\cdot\frac{1}{2^{k+n-1}}e_{k+n}.\end{aligned}$$

Since $\|T^n\|=\sup\limits_k\|\dfrac{1}{2^k}\cdot\dfrac{1}{2^{k+1}}\cdots\dfrac{1}{2^{k+n-1}}\|=\dfrac{1}{2}\cdot\dfrac{1}{2^2}\cdots\dfrac{1}{2^n}$,

$$r(T)=\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}=\lim_{n\to\infty}\left(\frac{1}{2^{\frac{n\cdot(n+1)}{2}}}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{2^{\frac{n+1}{2}}}=0.$$

Hence, operator T is quasi nilpotent.

But operator T is not quasi hypornormal.

From [3, Proposition 3.4] *we have that the operator T is quasi hyponormal if and only if $|\alpha_n|\leq|\alpha_{n+1}|$. In this case we have that $|\alpha_n|\nleq|\alpha_{n+1}|$, so it is not quasi hyponormal.*

In following example we will prove that the $k-$quasi class Q^* and $k-$quasi $-*-$ paranormal operator are two different classes.

Example 2.2. *Let T_x be the weighted shift operator with nonzero weights where*

$$\alpha_0=x,\alpha_1=\sqrt{\frac{2}{3}},\alpha_2=\sqrt{\frac{3}{4}},\cdots,\alpha_n=\sqrt{\frac{n+1}{n+2}},\cdots,\alpha_{n+k}=\sqrt{\frac{n+k+1}{n+k+2}},n\geq 1,k\geq 1.$$

Then we have the following results:

1) From [5, Corrolary 2.2] *operator T_x is of an operator of $k-$quasi class Q^* if and only if*

$$\alpha_{n+k}^2\alpha_{n+k+1}^2-2\alpha_{n+k-1}^2+1\geq 0.$$

So after some calculation we see that this is true only if

$$0<x\leq\frac{\sqrt{3}}{2}.$$

2) From [8, Example 1.2] *operator T_x is of an operator of $k-$quasi $-*-$ paranormal if and only if*

$$\alpha_{n+k-1}^2\leq\alpha_{n+k}\alpha_{n+k+1}.$$

So after some calculation we see that this is true only if

$$0<x\leq\frac{1}{\sqrt[4]{2}}.$$

3) So, if

$$\frac{1}{\sqrt[4]{2}} < x \leq \frac{\sqrt{3}}{2},$$

operator T_x is an operator of $k-$quasi class Q^* but not $k-$quasi $- * -$ paranormal.

REFERENCES

[1] A. ALUTHGE, On $p-$hyponormal operators for $0 < p < 1$. *Integral Equations Operator Theory*, **13** (1990), 307–315.

[2] S. C. ARORA and J. K. THUKRAL, On a class of operators, *Glasnik Math.* **21**(1986), no. 41, pp. 381–386.

[3] N. L. BRAHA, M.LOHAJ, F.H. MAREVCI, Sh.LOHAJ, Some properties of paranormal and hyponormal operators, *Bulletin of Mathematical analysis and Applications*, **1**,(2009), no. 2, pp 23–35.

[4] T. FURUTA, On The Class of Paranormal Operators, *Proc. Jap. Acad.* **43**, (1967), pp. 594–598.

[5] V.R. HAMITI,Sh. LOHAJ and Q. GJONBALAJ, On $k-$quasi class Q^* operators, *Turkish Journal of Analysis and Number Theory*, **4**, (2016), no.4, pp. 87–91.

[6] D. SENTHILKUMAR and T. PRASAD, Composition Operators of Class Q^*, *Int. Journal of Math. Analysis*, **4** (2010), no. 21, pp.1035 – 1040.

[7] T. YAMAZAKI, Parallelisms between Aluthge transformation and powers of operators, *Acta Sci. Math. (Szeged)*, **67**, (2001), pp. 809–820.

[8] F. ZUO and H. ZUO, Structural and Spectral Properties of $k-$quasi$- * -$paranormal operators, *Korean J. Math.*, **23**, (2005), no. 2, pp. 249–257.