SOME PROPERTIES OF $k$–QUASI CLASS $Q^*$ OPERATORS
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Received 12 February, 2017; accepted 20 April, 2017; published 6 July 2017.

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ABSTRACT. In this paper, we give some results of $k$–quasi class $Q^*$ operators. We proved that if $T$ is an invertible operator and $N$ be an operator such that $N$ commutes with $T^*T$, then $N$ is $k$–quasi class $Q^*$ if and only if $TNT^{-1}$ is of $k$–quasi class $Q^*$. With example we proved that exist an operator $k$–quasi class $Q^*$ which is quasi nilpotent but it is not quasi hyponormal.

Key words and phrases: $k$–quasi class $Q^*$ operator; quasi nilpotent; quasi hyponormal.

2000 Mathematics Subject Classification Primary 47B47, 47B20.

ISSN (electronic): 1449-5910
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1. Introduction

Let \( \mathcal{L}(\mathcal{H}) \) denote the \( C^* \) algebra of all bounded operators on \( \mathcal{H} \) and let \( \mathcal{H} \) be a complex Hilbert space with inner product \( \langle \cdot , \cdot \rangle \). For \( T \in \mathcal{L}(\mathcal{H}) \), we denote by \( \sigma(T) \) the spectrum of \( T \) and by \( r(T) \) the spectral radius of operator \( T \) which is defined by \( r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} \). The null operator and the identity on \( \mathcal{H} \) will be denoted by \( 0 \) and \( I \), respectively. If \( T \) is an operator, then \( T^* \) is its adjoint, and \( \|T\| = \|T^*\| \).

An operator \( T \in \mathcal{L}(\mathcal{H}) \) is a positive operator, \( T \geq 0 \), if \( \langle Tx, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \). If two operator \( A \in \mathcal{L}(\mathcal{H}) \) and \( B \in \mathcal{L}(\mathcal{H}) \) are positive operators and if \( AB = BA \) then the product \( AB \) is also positive operator. The \( \|T\| = (T^*T)^{\frac{1}{2}} \) is a positive operator and we have that \( |T|^2 = T^*T \) and \( |T^*|^2 = TT^* \).

The operator \( T \) is called unitary operator if \( T^*T = TT^* = I \). The operator \( T \) is normaloid if \( r(T) = \|T\| \) and it is quasi nilpotent if \( r(T) = 0 \).

An operator \( T \in \mathcal{L}(\mathcal{H}) \), is said to be paranormal \([4]\), if \( \|Tx\|^2 \leq \|T^2x\| \) for any unit vector \( x \) in \( \mathcal{H} \). Further, \( T \) is said to be quasi hyponormal \([3]\), if \( \|T^*Tx\| \leq \|T^2x\| \) for any unit vector \( x \) in \( \mathcal{H} \).

An operator \( T \) is called \( k \)-quasi-\(*\)-paranormal if \( \|T^{*}Tkx\|^2 \leq \|T^{k+2}x\|\|Tkx\| \), for all \( x \in \mathcal{H} \), where \( k \) is a natural number \([8]\).

An operator \( T \in \mathcal{L}(\mathcal{H}) \) belongs to class \( Q^* \) if \( T^{*}T^2 - 2TT^* + I \geq 0 \) or equivalent if \( \|T^*x\|^2 \leq \frac{1}{2} (\|T^2x\|^2 + \|x\|^2) \), for all \( x \in \mathcal{H} \) \([6]\).

An operator \( T \in \mathcal{L}(\mathcal{H}) \) belongs to \( k \)-quasi class \( Q^* \) if
\[
\|T^{*}Tkx\|^2 \leq \frac{1}{2} (\|Tk^2x\|^2 + \|Tkx\|^2),
\]
for all \( x \in \mathcal{H} \), where \( k \) is a natural number. Equivalently, operator \( T \in \mathcal{L}(\mathcal{H}) \) belongs to \( k \)-quasi class \( Q^* \) if \( T^{*k}(T^{2}T^2 - 2TT^* + I)Tk \geq 0 \), where \( k \) is a natural number \([5]\).

Aluthge in \([1]\) define a transformation \( \tilde{T} \) of operator \( T \) by \( \tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2} \), where \( T = U|T| \) is the polar decomposition of operator \( T \). \( \tilde{T} \) is called Aluthge transformation.

Yamazaki in \([7]\) define the \(*\)-Aluthge transformation of operator \( T \). The \(*\)-Aluthge transformation is defined by \( \tilde{T}^{(*)} \overset{\text{def}}{=} (\tilde{T}^*)^* = |T|^\frac{1}{2}U|T^*|^\frac{1}{2} \).

It is proved that \( U^*|T^*|^\frac{1}{2} = |T|^\frac{1}{2}U^* \), \( U^*|T|^\frac{1}{2} = |T|U^* \), \( U|T|^\frac{1}{2} = |T^*|^\frac{1}{2}U \), \( U|T| = |T^*|U \).

2. Main results

In this section we prove some properties of \( k \)-quasi class \( Q^* \) operators.

**Theorem 2.1.** Let be \( T \) an invertible operator and \( N \) be an operator such that \( N \) commutes with \( T^*T \). Then \( N \) is \( k \)-quasi class \( Q^* \) if and only if \( TTN^{-1} \) is of \( k \)-quasi class \( Q^* \).

**Proof.** Let \( N \) be a \( k \)-quasi class \( Q^* \) operator.
\[
N^k(N^{*2}N^2 - 2NN^* + I)N^k \geq 0.
\]

From this we have that:
\[
TN^k[N^{*2}N^2 - 2NN^* + I]N^kT^* \geq 0.
\]

Consider,
\[
TN^k[N^{*2}N^2 - 2NN^* + I]N^kT^*[TT^*]
\]
\[
= TN^k[N^{*2}N^2 - 2NN^* + I]N^k[T^*T]T^*
\]
\[
= T[T^*T]N^k[N^{*2}N^2 - 2NN^* + I]N^kT^*
\]
\[
= [TT^*]TN^k[N^{*2}N^2 - 2NN^* + I]N^kT^*.
\]
So, we see that operator $TT^*$ commutes with operator 

$$TN^k[N^s2N^2 - 2NN^* + I]N^kT^*.$$ 

Then operator $[TT^*]^{-1}$ also commutes with operator 

$$TN^k[N^s2N^2 - 2NN^* + I]N^kT^*.$$ 

Since the operators $[TT^*]^{-1}$ and $TN^k[N^s2N^2 - 2NN^* + I]N^kT^*$ are positive and since they commute with each other we have that their product is also positive operator: 

$$T^k[N^s2N^2 - 2NN^* + I]N^k[TT^*]^{-1} \geq 0.$$ 

Since operator $N$ commutes with operator $T^*T$, we get, 

$$(TNT^{-1})^k = (TNT^{-1})^s(TNT^{-1})^s \cdots (TNT^{-1})^s$$

(2.1) 

$$= T^{s^{-1}}N^sT^*T^{s^{-1}}N^sT^* \cdots T^{s^{-1}}N^sT^* = T^{s^{-1}}N^kT^*$$

(2.2) 

$$(TNT^{-1})^k = TNT^{-1}TNT^{-1} \cdots TNT^{-1} = TN^kT^{-1}$$

(2.3) 

$$(TNT^{-1})^s(TNT^{-1})^2 = TN^kN^2T^{-1}$$

(2.4) 

$$(TNT^{-1})(TNT^{-1})^s = TNT^{-1}T^{s^{-1}}N^sT^* = TNN^sT^{-1}$$

To prove that $TNT^{-1}$ is $k$–quasi class $Q^*$ operator, the equation (2.1), (2.2), (2.3) and (2.4) we substitute in above expression:

$$(TNT^{-1})^k[(TNT^{-1})^s(TNT^{-1})^2 - 2(TNT^{-1})(TNT^{-1})^s + I](TNT^{-1})^k$$

and we have

$$(TNT^{-1})^k[(TNT^{-1})^s(TNT^{-1})^2 - 2(TNT^{-1})(TNT^{-1})^s + I](TNT^{-1})$$

$$= T^{s^{-1}}N^kT^*[TN^s2N^2T^{-1} - 2TN^sT^{-1} + I]TN^kT^{-1}$$

$$= T^{s^{-1}}N^kT^*[N^s2N^2 - 2NN^* + I]T^{-1}TN^kT^{-1}$$

$$= T^{s^{-1}}N^kT^*[N^s2N^2 - 2NN^* + I]N^kT^{-1}$$

$$= TN^k[N^s2N^2 - 2NN^* + I]N^kT^{-1}$$

Now we have to prove that that the last expression is positive. From the fact that we prove that 

$$TN^k[N^s2N^2 - 2NN^* + I]N^kT^*[TT^*]^{-1} \geq 0$$

we have that:

$$\Rightarrow TN^k[N^s2N^2 - 2NN^* + I]N^kT^*T^{-1}T^{-1} \geq 0$$

$$\Rightarrow TN^k[N^s2N^2 - 2NN^* + I]N^kT^{-1} \geq 0$$

Hence, $TNT^{-1}$ is $k$–quasi class $Q^*$ operator.

Conversely, let $TNT^{-1}$ be a $k$–quasi class $Q^*$ operator. 

$$(TNT^{-1})^k[(TNT^{-1})^s(TNT^{-1})^2 - 2(TNT^{-1})(TNT^{-1})^s + I](TNT^{-1})^k \geq 0.$$
Then similar as before, after substituting the equation (2.1), (2.2), (2.3) and (2.4) we have:

\[ T N^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^{-1} \geq 0 \]
\[ T^*TN^{*k}[N^{*2}N^2 - 2NN^* + I]N^kT^{-1}T \geq 0 \]
\[ [T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k \geq 0. \]

Since operator \([T^*T]\) commutes with operator \(N\) and hence commute with operator

\[ [T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k. \]

The also the operator \([T^*T]^{-1}\) commutes with operator

\[ [T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k. \]

Since the operators \([T^*T]^{-1}\) and \([T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k\) are positive and since they commute with each other we have:

\[ [T^*T]^{-1}[T^*T]N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k \geq 0. \]

Therefore,

\[ N^{*k}[N^{*2}N^2 - 2NN^* + I]N^k \geq 0. \]

Hence, \(N\) is \(k\)-quasi class \(Q^*\) operator. \(\blacksquare\)

**Corollary 2.2.** Let \(S\) be a \(k\)-quasi class \(Q^*\) operator and \(M\) any positive operator such that \(M^{-1} = M^*\). Then \(T = M^{-1}SM\) is \(k\)-quasi class \(Q^*\) operator.

**Proof.** Let \(S\) be a \(k\)-quasi class \(Q^*\) operator. Then

\[ S^{*k}(S^{*2}S^2 - 2SS^* + I)S^k \geq 0. \]

Consider,

\[ T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k = (M^{-1}SM)^{*k}((M^{-1}SM)^{*2}(M^{-1}SM)^2 - 2(M^{-1}SM)(M^{-1}SM)^* + I)(M^{-1}SM)^k = M^*S^*M^{-1*}M^*S^*M^{-1*}...M^*S^*M^{-1*}(M^*S^*M^{-1*}M^*S^*M^{-1*}M^{-1}SMM^{-1}SM
- 2M^*S^*M^{-1*}M^{-1}SM + I)M^{-1}SMM^{-1}SM...M^{-1}SM
= M^*S^{*k}(S^{*2}S^2 - 2SS^* + I)S^kM \geq O. \]

hence, \(T = M^{-1}SM\) is \(k\)-quasi class \(Q^*\) operator. \(\blacksquare\)

**Theorem 2.3.** Let be \(T \in L(H)\). Then \(\tilde{T}\) is \(k\)-quasi class \(Q^*\) operator if and only if \(\tilde{T}^{(e)}\) is \(k\)-quasi class \(Q^*\) operator.

**Proof.** Assume that \(\tilde{T}\) is \(k\)-quasi class \(Q^*\) then

\[ \tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^k \geq 0. \]
We need to prove that $\tilde{T}(s)$ is $k$–quasi class $Q^*$ operator.

\[
\tilde{T}(s)^{k}(\tilde{T}(s)^{2}\tilde{T}(s)^{2} - 2\tilde{T}(s)\tilde{T}(s) + I)\tilde{T}(s)^{k} = (|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2})^{k}
\]

\[
= ((|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2})^{2} - 2(|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2})((|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2}) + I)
\]

\[
= ((|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2})^{2} ... ((|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2})^{k})
\]

\[
= (|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2})(|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2}) ... (|T^{s}|\frac{1}{2}U|T^{s}|\frac{1}{2})
\]

\[
= UU^{*}(|T^{s}|\frac{1}{2}U|T^{s}|U^{*} ... U^{*}|T^{s}|\frac{1}{2})UU^{*}
\]

\[
= U(U^{*}|T^{s}|\frac{1}{2}U^{*}|T^{s}|U^{*} ... U^{*}|T^{s}|\frac{1}{2})U^{*}|T^{s}|\frac{1}{2}U^{*} ... U^{*}|T^{s}|\frac{1}{2}U^{*}
\]

\[
= U(U^{*}|T^{s}|\frac{1}{2}U^{*}|T^{s}|U^{*} ... U^{*}|T^{s}|\frac{1}{2})U^{*}|T^{s}|\frac{1}{2}U^{*} ... U^{*}|T^{s}|\frac{1}{2}U^{*}
\]

\[
= U(U^{*}|T^{s}|\frac{1}{2}U^{*}|T^{s}|U^{*} ... U^{*}|T^{s}|\frac{1}{2})U^{*}|T^{s}|\frac{1}{2}U^{*} ... U^{*}|T^{s}|\frac{1}{2}U^{*}
\]

\[
= U(U^{*}|T^{s}|\frac{1}{2}U^{*}|T^{s}|U^{*} ... U^{*}|T^{s}|\frac{1}{2})U^{*}|T^{s}|\frac{1}{2}U^{*} ... U^{*}|T^{s}|\frac{1}{2}U^{*}
\]

Therefore

\[
\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^{*2} - 2\tilde{T}^{*}\tilde{T}^{*} + I)\tilde{T}^{*k} \geq 0.
\]

Hence $\tilde{T}(s)$ is $k$–quasi class $Q^*$ operator.

Conversely, assume that $\tilde{T}(s)$ is $k$–quasi class $Q^*$ operator, then

\[
\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^{*2} - 2\tilde{T}^{*}\tilde{T}^{*} + I)\tilde{T}^{*k} \geq 0.
\]
We need to prove that $\tilde{T}$ is $k$–quasi class $Q^*$.
Consider

$$
\tilde{T}^*k(\tilde{T}^*2\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^k
$$

$$
= U^*U(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})\ldots (|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})U^*U
$$

$$
[(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}) - 2(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})]
$$

$$
(U|T|^{\frac{1}{2}}U^*|T|U|T|U|T|^{\frac{1}{2}}U^* - 2U|T|^{\frac{1}{2}}U^*|T|U|T|U|T|^{\frac{1}{2}}U^* + I)
$$

$$
(U|T|^{\frac{1}{2}}U^*|T|U\ldots U|T|^{\frac{1}{2}}U^*)U
$$

$$
= U^*((|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}})U
$$

$$
[(|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}})((|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}}) - 2(|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}})]
$$

$$
(U|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}}U^* - 2U|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}}U^* + I)
$$

$$
(U|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}}U^*)U
$$

$$
= U^*((|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}})^k
$$

$$
[(|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}})^k((|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}})^k - 2(|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}})^kU
$$

$$
(U|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}}^kU^* - 2U|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}}^kU^* + I)((|T^*|^{\frac{1}{2}}U^*|T^*|U\ldots U|T^*|^{\frac{1}{2}})^kU
$$

$$
= U^*\tilde{T}^*k(\tilde{T}^*2\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^k
$$

Therefore

$$
\tilde{T}^*k(\tilde{T}^*2\tilde{T}^2 - 2\tilde{T}\tilde{T}^* + I)\tilde{T}^k \geq 0.
$$

Hence $\tilde{T}$ is $k$–quasi class $Q^*$ operator. $\blacksquare$

**Proposition 2.4.** Let $T \in \mathcal{L}(\mathcal{H})$ be the operator defined as

$$
T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.
$$

If $A$ is operator of class $Q^*$ and $BB^* = 0$, then $T$ is an operator of $k$–quasi class $Q^*$.  

Proof. A simple calculation shows that:

\[ T^* = \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix}, \]

\[ T^{*k} = \begin{pmatrix} A^{*k} & 0 \\ B^* A^{*(k-1)} & 0 \end{pmatrix}, \]

\[ T^k = \begin{pmatrix} A^k & A^{(k-1)}B \\ 0 & 0 \end{pmatrix}, \]

\[ T^{*k} T^k = \begin{pmatrix} A^{*k} (A^* + BB^*) A^k & A^{*k} (A^* + BB^*) A^{(k-1)}B \\ B^* A^{*(k-1)} (A^* + BB^*) A^k & B^* A^{*(k-1)} (A^* + BB^*) A^{(k-1)}B \end{pmatrix}, \]

\[ T^{*k} T^k = \begin{pmatrix} A^{*k} A A^k & A^{*k} A A^k A^{(k-1)}B \\ B^* A^{*(k-1)} A A^k & B^* A^{*(k-1)} A A^k A^{(k-1)}B \end{pmatrix}, \]

\[ T^{*k} T^{*(k+2)} - 2 T^{*k} T^k = T^{*k} T^{*(k+2)} - 2 T^{*k} T^k = \begin{pmatrix} A^{*k} (A^2 A^2 - 2 A A^* + I) A^k & A^{*k} (A^2 A^2 - 2 A A^* + I) A^{(k-1)}B \\ B^* A^{*(k-1)} (A^2 A^2 - 2 A A^* + I) A^k & B^* A^{*(k-1)} (A^2 A^2 - 2 A A^* + I) A^{(k-1)}B \end{pmatrix}. \]

Let \( u = x \oplus y \in \mathcal{H} \oplus \mathcal{H} \). Then,

\[ \langle (T^{*(k+2)} T^{(k+2)} - 2 T^{*k} T^k) u, u \rangle \]

\[ = \langle A^{*k} (A^2 A^2 - 2 A A^* + I) A^k, x, x \rangle + \langle A^{*k} (A^2 A^2 - 2 A A^* + I) A^{(k-1)}B, y, x \rangle 
+ \langle B^* A^{*(k-1)} (A^2 A^2 - 2 A A^* + I) A^k, x, y \rangle 
+ \langle B^* A^{*(k-1)} (A^2 A^2 - 2 A A^* + I) A^{(k-1)}B, y, y \rangle 
= \langle (A^2 A^2 - 2 A A^* + I) A^k, x, x \rangle 
+ \langle (A^2 A^2 - 2 A A^* + I) A^{(k-1)}B, y, x \rangle 
+ \langle (A^2 A^2 - 2 A A^* + I) A^k, x, (k-1)B y \rangle 
+ \langle (A^2 A^2 - 2 A A^* + I) A^{(k-1)}B, y, (k-1)B y \rangle 
= \langle (A^2 A^2 - 2 A A^* + I) (A^k x + A^{(k-1)}B y), (A^k x + A^{(k-1)}B y) \rangle \geq 0 \]

because \( A \) is operator of class \( Q^* \) then, \( A^2 A^2 - 2 A A^* + I \geq O \), so this prove the result. \( \blacksquare \)

**Proposition 2.5.** Every quasi hyponormal operator is operator of quasi class \( Q^* \).

**Proof.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be a quasi hyponormal operator, then

\[ \| T^* T x \| \leq \| T^2 x \|. \]
Since every quasi hyponormal operator is paranormal \cite[Corollary 3.15]{3} then we have
\[
\|T^*Tx\| \leq \|T^2x\|^2 = \|T\left(\frac{Tx}{\|Tx\|}\right)\|^2 \cdot \|Tx\|^2 \\
\leq \|T^2\left(\frac{Tx}{\|Tx\|}\right)\| \cdot \|Tx\|^2 = \|T^3x\| \cdot \|Tx\| \\
\leq \frac{1}{2} \left(\|T^3x\|^2 + \|Tx\|^2\right),
\]

So, \(T\) is operator of quasi class \(Q^*\). 

In the following example we will prove that exist an operator \(k\)–quasi class \(Q^*\) which is quasi nilpotent but it is not quasi hyponormal.

**Example 2.1.** Consider the operator \(T : l^2 \to l^2\) defined by
\[
T(x) = (0, \alpha_1x_1, \alpha_2x_2, \ldots)
\]
where \(\alpha_1 = \frac{1}{2^n}\) for \(n \geq 1\). Operator \(T\) is of \(k\)–quasi class \(Q^*\) and quasi nilpotent but it is not quasi hyponormal.

Given \(T(x) = (0, \alpha_1x_1, \alpha_2x_2, \ldots)\). Then \(T^*(x) = (\alpha_1x_1, \alpha_2x_2, \ldots)\),
\[
T^2(x) = (0, 0, \alpha_1\alpha_2x_1, \alpha_2\alpha_3x_2, \ldots),
\]

\(T^k(x) = (0, 0, \ldots, 0, \alpha_1\alpha_2 \ldots \alpha_kx_1, \alpha_2\alpha_3 \ldots \alpha_kx_2, \ldots)\),
\[
T^{(k-1)}T^k(x) = (0, 0, \ldots, 0, \alpha_1\alpha_2 \ldots \alpha_k^2x_1, \alpha_2\alpha_3 \ldots \alpha_k^2x_2, \ldots),
\]
\[
T^{(k-1)}T^k(x) = (\alpha_1^2\alpha_2^2 \ldots \alpha_k^2x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_k^2x_2, \ldots),
\]
\[
T^{(k+1)}T^{(k+1)}(x) = (0, \alpha_1^2\alpha_2^2 \ldots \alpha_k^2\alpha_{k+1}x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_k^2\alpha_{k+1}x_2, \ldots),
\]
\[
T^{(k+1)}T^{(k+1)}(x) = (\alpha_1^2\alpha_2^2 \ldots \alpha_k^2\alpha_{k+1}^2x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_k^2\alpha_{k+1}^2x_2, \ldots),
\]
\[
TT^*TT^k(x) = (0, 0, \ldots, 0, \alpha_1\alpha_2 \ldots \alpha_k^3x_1, \alpha_2\alpha_3 \ldots \alpha_k^3x_2, \ldots),
\]
\[
T^{(k-1)}TT^k(x) = (\alpha_1^2\alpha_2^2 \ldots \alpha_k^2\alpha_{k-1}^3x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_k^2\alpha_{k-1}^3x_2, \ldots),
\]

Now consider
\[
\langle T^*(T^{(k+2)}T^2 - 2TT^* + I)T^k, x \rangle
\]
\[
= \langle (T^{(k+2)}T^2 - 2TT^* + I)T^k, x \rangle
\]
\[
= \langle (\alpha_1^2\alpha_2^2 \ldots \alpha_{k+2}^2x_1 - 2\alpha_1^2\alpha_2^2 \ldots \alpha_{k-1}^2x_1, x_1) \rangle
\]
\[
+ \langle (\alpha_1^2\alpha_2^2 \ldots \alpha_{k+2}^2x_1 - 2\alpha_1^2\alpha_2^2 \ldots \alpha_{k-1}^2x_1, x_2) \rangle + \ldots
\]
\[
= \alpha_1^2\alpha_2^2 \ldots \alpha_k^2\alpha_{k+2}^2x_1^2 - 2\alpha_1^2\alpha_2^2 \ldots \alpha_{k-1}^2x_1^2 + \alpha_2^2\alpha_3^2 \ldots \alpha_k^2\alpha_{k+2}^2x_1^2 + \alpha_2^2\alpha_3^2 \ldots \alpha_k^2\alpha_{k+2}^2x_2^2 + \ldots \geq 0.
\]
Because

\[ \alpha_{n+k}^2 \alpha_{n+k+1}^2 - 2\alpha_{n+k-1}^2 + 1 = \left( \frac{1}{2n+k} \right)^2 \cdot \left( \frac{1}{2n+k+1} \right)^2 - 2\left( \frac{1}{2n+k-1} \right)^2 + 1 \geq 0, \quad k \geq 1, \quad n \geq 1. \]

From \( T(e_k) = \frac{1}{2k} e_{k+1} \) we have

\[ T^2(e_k) = \frac{1}{2k} \cdot \frac{1}{2k+1} e_{k+2} \]

\[ \ldots \]

\[ T^n(e_k) = \frac{1}{2k} \cdot \frac{1}{2k+1} \cdot \ldots \cdot \frac{1}{2k+n-1} e_{k+n}. \]

Since \( \| T^n \| = \sup_k \frac{1}{2k} \cdot \frac{1}{2k+1} \ldots \frac{1}{2k+n-1} = \frac{1}{2} \cdot \frac{1}{2} \ldots \frac{1}{2n} \),

\[ r(T) = \lim_{n \to \infty} \frac{1}{2n} = \lim_{n \to \infty} \left( \frac{1}{n \cdot (n+1)} \right) = \frac{1}{2} \cdot \frac{1}{2} = 0. \]

Hence, operator \( T \) is quasi nilpotent.

But operator \( T \) is not quasi hyponormal.

From [3, Proposition 3.4] we have that the operator \( T \) is quasi hyponormal if and only if

\[ |\alpha_n| \leq |\alpha_{n+1}|. \]

In this case we have that \( |\alpha_n| \nless |\alpha_{n+1}| \), so it is not quasi hyponormal.

In following example we will prove that the \( k \)-quasi class \( Q^* \) and \( k \)-quasi \(-*\) paranormal operator are two different classes.

**Example 2.2.** Let \( T_x \) be the weighted shift operator with nonzero weights where

\[ \alpha_0 = x, \quad \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_2 = \sqrt{\frac{3}{4}}, \ldots, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}}, \ldots, \quad \alpha_{n+k} = \sqrt{\frac{n+k+1}{n+k+2}}, \quad n \geq 1, \quad k \geq 1. \]

Then we have the following results:

1) From [5, Corrolary 2.2] operator \( T_x \) is of an operator of \( k \)-quasi class \( Q^* \) if and only if

\[ \alpha_{n+k}^2 \alpha_{n+k+1}^2 - 2\alpha_{n+k-1}^2 + 1 \geq 0. \]

So after some calculation we see that this is true only if

\[ 0 < x \leq \frac{\sqrt{3}}{2}. \]

2) From [8, Example 1.2] operator \( T_x \) is of an operator of \( k \)-quasi \(-*\) paranormal if and only if

\[ \alpha_{n+k}^2 \leq \alpha_{n+k} \alpha_{n+k+1}. \]

So after some calculation we see that this is true only if

\[ 0 < x \leq \frac{1}{\sqrt{2}}. \]
3) So, if 
\[ \frac{1}{\sqrt{2}} \leq x \leq \frac{\sqrt{3}}{2}, \]
operator \( T_x \) is an operator of \( k- \)quasi class \( Q^* \) but not \( k- \)quasi \( - \) paranormal.

REFERENCES


