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ON SINGULAR NUMBERS OF HANKEL MATRICES OF MARKOV FUNCTIONS

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ABSTRACT. Let $E \subset (-1, 1)$ be a compact set and let μ be a positive Borel measure with support supp $\mu = E$. Let

$$D_n = \left(\frac{1}{2\pi} \int_E x^{i+j} d\mu(x) \right)_{i,j=0}^{n}.$$

In the case when $E = [a, b] \subset (-1, 1)$ and μ satisfies the condition $d\mu/dx > 0$ a.e. on E, we investigate asymptotic behavior of singular numbers $\sigma_{k_n,n}$ of the Hankel matrix D_n, where $k_n/n \to \theta \in [0, 1]$ as $n \to \infty$. Moreover, we obtain asymptotics of the Kolmogorov, Gelfand and linear k-widths, $k = k_n$, of the unit ball $\mathbf{A}_{n,2}$ of $\mathcal{P}_n \cap L_2(\Gamma)$ in the space $L_2(\mu, E)$, where $\Gamma = \{z : |z| = 1\}$ and $\mathcal{P}_n$ is the class of all polynomials of the degree at most n.

Key words and phrases: Hankel operator, Hankel matrix, Markov function, Singular number, n-width, Potential theory.

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1. INTRODUCTION

1.1. **Notation.** Let $G = \{z : |z| < 1\}$ be the open unit disk with boundary $\Gamma = \{z : |z| = 1\}$ in the complex plane $\mathbf{C}$. We assume that circumference Γ is positively oriented with respect to G. Above and thereafter $L_p(\Gamma), 1 \le p \le \infty$, stands for the Lebesgue space of functions φ measurable on Γ, with the norm

$$\|\varphi\|_p = \left(\int_\Gamma |\varphi(t)|^p |dt|\right)^{1/p} \quad \text{if} \quad 1 \le p < \infty,$$

and

$$\|\varphi\|_\infty = \operatorname*{ess\,sup}_\Gamma |\varphi(t)| \quad \text{if} \quad p = \infty.$$

Denote by $\langle \varphi, \psi \rangle_2$ the inner product in the Hilbert space $L_2(\Gamma)$:

$$\langle \varphi, \psi \rangle_2 = \int_E (\varphi\overline{\psi})(t)|dt|, \quad \varphi, \psi \in L_2(\Gamma).$$

Let $C(\Gamma)$ be the Banach space of continuous functions on Γ with the supremum norm $\|\cdot\|_\Gamma$ on Γ.

Denote by $H_p(G), 1 \le p \le \infty$, the Hardy space of analytic functions on G. Here and in what follows we consider $H_p(G)$ as a subspace of the space $L_p(\Gamma)$. We represent $L_2(\Gamma)$ as the direct sum $L_2(\Gamma) = H_2(G) \oplus H_2^\perp(G)$, where $H_2^\perp(G)$ is the orthogonal complement of $H_2(G)$ in $L_2(\Gamma)$. We remark that $a \in H_2^\perp(G)$ if and only if there exists a function $b \in H_2(G)$ such that

$$a(t) = \overline{b(t)}\frac{|dt|}{dt} \quad \text{a.e. on} \quad \Gamma.$$

1.2. **Singular numbers.** Let X and Y be Hilbert spaces and let $A : X \to Y$ be a compact linear operator, then the n-th singular number $s_n = s_n(A)$ is defined by

$$s_n(A) = \inf_K ||A - K||,$$

where the infimum is taken over all linear operators $K : X \to Y$ having rank at most n. Here $||\cdot||$ denotes the norm of the corresponding linear operator. Equivalently, s_n is given by

$$s_n(A) = \inf_{X_{-n}} ||A|_{X_{-n}}||, \tag{1.1}$$

where X_{-n} runs over all subspaces of X of codimension n (see, for example, [33]). Moreover, the sequence $\{s_n(A)\}$, $n = 0, 1, 2, \dots$, coincides with the sequence of eigenvalues of the operator $|A| = (A^*A)^{1/2}$, where $A^* : Y \to X$ is the adjoint of A.

1.3. **N-widths.** Let $\mathbf{A}$ be a convex, compact, centrally symmetric subset of a Banach space X. Here and subsequently, $d_n(\mathbf{A}, X)$ stands for Kolmogorov n-width of $\mathbf{A}$ in X:

$$d_n(\mathbf{A}, X) := \inf_{X_k} \sup_{\varphi\in\mathbf{A}} \inf_{g\in X_n} ||\varphi - g||,$$

where X_n denotes an arbitrary n dimensional subspace of X. The Gelfand n-width of $\mathbf{A}$ in X is defined as follows:

$$d^n(\mathbf{A}, X) := \inf_{X_{-n}} \sup_{x\in X_{-n}\cap\mathbf{A}} ||x||, \tag{1.2}$$

where X_{-n} varies over all subspaces of X of codimension n. The linear n-width of $\mathbf{A}$ in X is given by

$$\delta_n(\mathbf{A}, X) := \inf_K \sup_{\varphi\in\mathbf{A}} ||\varphi - K\varphi||,$$

where $K : X \to X$ runs over all linear operators of rank n (see [24] for more details about n-widths).

1.4. **N-widths and the restriction operator.** Let E be a compact subset of the open interval $(-1,1)$. We assume that E contains infinitely many points. Let μ be a positive Borel measure with support supp $\mu = E$. Denote by $L_2(\mu, E)$ the Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_{2,\mu} = \int_E (\varphi\overline{\psi})(x) d\mu(x), \quad \varphi, \psi \in L_2(\mu, E),$$

and the norm $\|\varphi\|_{2,\mu}$.

Denote by $J : H_2(G) \to L_2(\mu, E)$ the restriction operator. The operator J is given by restricting an element $\varphi \in H_2(G)$ to $E =$ supp μ: $J\varphi = \varphi_{|E}$. Let $J^* : L_2(\mu, E) \to H_2(G)$ be the adjoint of J. It is easy verified that for $\varphi \in H_2(G)$

$$(J^*J)(\varphi)(z) = \frac{1}{2\pi} \int_E \frac{\varphi(x)}{1 - zx} d\mu(x), \quad |z| < 1$$

(see, for example, [15]).

The restriction to E of the closed unit ball of $H_2(G)$ forms a compact, convex, centrally symmetric subset $\mathbf{A}_2$ of $L_2(\mu, E)$. It is possible to determine the size of the set $\mathbf{A}_2$ as measured by its n-width. For any positive integer n let $\mathcal{B}_n$ denote the class of all Blaschke products B of degree at most n:

$$B(z) = \lambda \prod_{k=1}^{l} \frac{z - \xi_1}{1 - \bar{\xi}_k z}, \quad \xi_k \in G, \quad l \leq n, |\lambda| = 1.$$

S.D. Fisher and C.A. Micchelli [12], [15] proved that

$$\begin{aligned} s_n(J) = d_n(\mathbf{A}_2, L_2(\mu, E)) &= d^n(\mathbf{A}_2, L_2(\mu, E)) = \delta_n(\mathbf{A}_2, L_2(\mu, E)) \\ &= \inf_{B \in \mathcal{B}_n} \sup_{\varphi \in \mathbf{A}_2} ||\varphi B||_{2,\mu}, \end{aligned}$$

where $d_n(\mathbf{A}_2, L_2(\mu, E)), d^n(\mathbf{A}_2, L_2(\mu, E))$, and $\delta_n(\mathbf{A}_2, L_2(\mu, E))$ are the Kolmogorov, Gelfand and linear n-widths of $\mathbf{A}_2$ in the space $L_2(\mu, E)$, and $s_n(J)$ is the singular number of the restriction operator J. It is also proved by S. D. Fisher and C.A. Micchelli that the eigenvalues for $J^*J : H_2(G) \to H_2(G)$ are simple, the corresponding eigenspaces are one dimensional and the $(n+1)$-th eigenfunction has exactly n zeros in G (see also [13], [14]).

1.5. **Connection with Hankel operators.** For any nonnegative integer n denote by $\mathcal{P}_n$ the class of all polynomials of degree at most n. Let $\mathcal{M}_{n,\infty}(G)$ be the following class of meromorphic functions in G with at most n poles:

$$\mathcal{M}_{n,\infty}(G) = \{P/Q : P \in H_\infty(G),\ Q \in \mathcal{P}_n, Q \not\equiv 0\}.$$

Let f be a continuous function on Γ. The deviation of f in $L_\infty(\Gamma)$ from the class $\mathcal{M}_{n,\infty}(G)$ is denoted by $\Delta_{n,\infty}(f; G)$:

$$\Delta_{n,\infty}(f; G) = \inf_{h \in \mathcal{M}_{n,\infty}(G)} ||f - h||_\infty.$$

The Hankel operator $A_f : H_2(G) \to H_2^\perp(G)$ is defined by

$$A_f(g) = \mathbf{P}_-(fg),$$

where $\mathbf{P}_-$ is the orthogonal projection of $L_2(\Gamma)$ onto $H_2^\perp(G)$. The Adamyan, Arov, and Kreĭn Theorem asserts [1], [2] (see also [25]) that for $f \in C(\Gamma)$, we have

$$\Delta_{n,\infty}(f; G) = s_n(A_f), \quad n = 0, 1, 2, \ldots.$$

For Markov functions f the singular numbers $s_n(A_f)$ turn out to be related to the singular numbers of the embedding operator $J : H_2(G) \to L_2(\mu, E)$ [4] (see also [5],[6] and [7]):

For the Markov function

$$f(z) = \frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z - x},$$

where μ is a positive Borel measure with support supp $\mu = E \subset (-1, 1)$ *containing infinitely many points, there holds*

$$A_f^* A_f = (J^* J)^2.$$

Moreover, for $n = 0, 1, \ldots,$

$$\Delta_{n,\infty}(f; G) = s_n(A_f) = s_n(J)^2.$$

1.6. **Motivation.** Let μ be a positive Borel measure with support supp $\mu = E \subset (-1, 1)$ containing infinitely many points. Let

$$D_n = \left(\frac{1}{2\pi} \int_E x^{i+j} d\mu(x) \right)_{i,j=0}^{n}$$

be the Hankel matrix whose elements are moments $\int_E x^k d\mu(x)/2\pi$ of $\mu/2\pi$. The theory of Hankel matrices finds applications in questions of rational approximation [3], [32], orthogonal polynomials [31], random matrices (see, for example, [9], [11]). We mention the works of G. Szegő [30], Widom and Wilf [34], and Berg and Szwarc [8], where the behavior of the smallest eigenvalue of Hankel matrices studied as the dimension of the matrices tends to infinity (a bibliography relating to this circle of problems is given in [8]; see also [20], [35], [23], [10], [36]).

In Section 2 we introduce the Hankel operator $A_{f,n}$ which acts on the class $\mathcal{P}_n$ of all polynomials of degree at most n in order to relate the AAK theory and the error of best approximation. An analogue of the AAK Theorem (the smallest singular number of $A_{f,n}$ is equal to the error of best approximation) for a continuous on the unit circle symbol f is developed, based on duality. We also establish some properties of $A_{f,n}$. We remark that in [26] we defined a Hankel-type operator acting on the class $\mathcal{P}_n$ of all polynomials of degree at most n and proved an analogue of the AAK Theorem in the case when a symbol f of the corresponding operator is an analytic function in the open disk $|z| < R, R > 1.$

The main results of the paper are presented in Sections 3 and 4. In Section 3 we investigated properties the Hankel operator $A_{f,n}$ for the special case of f being a Markov function, made a link with singular numbers of the Hankel matrix D_n, a restriction operator, k-widths, and simplified the formula for singular numbers. We also show connection singular numbers of the Hankel matrix D_n, the Kolmogorov, Gelfand and linear k-widths of the unit ball $\mathbf{A}_{n,2}$ of $\mathcal{P}_n \cap L_2(\Gamma)$ in the space $L_2(\mu, E)$, with the following extremal problem:

$$\tau_{k,n} = \inf_{u \in \mathcal{P}_k} \sup_{v \in \mathcal{P}_{n-k}} \frac{||uv||_{2,\mu}}{||uv||_2}.$$

In Section 4, in the case when $E = [a, b] \subset (-1, 1)$ and the measure μ satisfies the condition $d\mu/dx > 0$ a.e. on E, we study the n-th root behavior of singular numbers $\sigma_{k_n,n}$ of the Hankel matrix D_n, where $k_n/n \to \theta \in [0, 1]$ as $n \to \infty$. Using the works of Gonchar and Rakhmanov [18] and [19], we reduce the investigation of asymptotic behavior of singular numbers $\sigma_{k_n,n}$ of D_n to the investigation of asymptotics of the following extremal constants:

$$\chi_{k_n,n} = \inf_{u \in \mathcal{P}_{k_n}} \sup_{v \in \mathcal{P}_{n-k_n}} \frac{||uv||_E}{||uv||_\Gamma}, \quad n = 0, 1, 2, \ldots.$$

This minimax problem was investigated by Prokhorov, Saff and Yattselev [27]. The results obtained are based on extremal problems of potential theory, central among which is the problem of equilibrium in an external field.

2. THE HANKEL OPERATOR

2.1. **Definition and some properties.** Fix a nonnegative integer n. Let $\mathcal{K}_n$ be a $(n+1)$-dimensional subspace of $H_2^{\perp}(G)$ defined as follows:

$$\mathcal{K}_n = \{a : a(t) = \overline{b(t)}\frac{|dt|}{dt} \quad \text{a.e. on} \quad \Gamma, \;\; b \in \mathcal{P}_n\}.$$

Let f be a continuous function on Γ. The Hankel operator $A_{f,n} : \mathcal{P}_n \to \mathcal{K}_n$ is defined by

$$A_{f,n}(\alpha) = \mathbf{P}_n(\alpha f), \quad \alpha \in \mathcal{P}_n,$$

where $\mathbf{P}_n$ is the orthogonal projection of $L_2(\Gamma)$ onto $\mathcal{K}_n$. Denote by $H_{2,n}(G)$ a subspace of $H_2(G)$ consisting of functions $\varphi \in H_2(G)$ such that each function φ has a zero of order at least $n+1$ at $z=0$. It is not hard to see that we can represent $H_2(G)$ as the direct sum

$$H_2(G) = \mathcal{P}_n \oplus H_{2,n}(G).$$

According to the definition of the operator $A_{f,n}$ the equality

$$A_{f,n}\alpha = \alpha f - \beta - \overline{\varphi}\,\frac{|dt|}{dt} = \overline{p}\,\frac{|dt|}{dt} \text{ a.e. on } \Gamma \tag{2.1}$$

holds for any polynomial $\alpha \in \mathcal{P}_n$, where $\beta \in H_2(G)$, $\varphi \in H_{2,n}(G)$ and $p \in \mathcal{P}_n$ are uniquely determined by the relation

$$\|A_{f,n}\alpha\|_2 = \|\alpha f - \beta - \overline{\varphi}\,\frac{|dt|}{dt}\|_2 = \inf_{\beta_1 \in H_2(G), \varphi_1 \in H_{2,n}(G)} \|\alpha f - \beta_1 - \overline{\varphi}_1\,\frac{|dt|}{dt}\|_2.$$

Let us define an antilinear operator $B_{f,n} : \mathcal{P}_n \to \mathcal{P}_n$ as follows

$$B_{f,n}\alpha = p, \qquad \alpha \in \mathcal{P}_n.$$

For any $\alpha \in \mathcal{P}_n$ we have

$$A_{f,n}\alpha = \overline{B_{f,n}\alpha}\,\frac{|dt|}{dt} \quad \text{a.e. on } \Gamma. \tag{2.2}$$

For a pair of polynomials $u, v \in \mathcal{P}_n$ the bilinear symmetric form $[u,v]$ is defined by

$$[u,v] = \int_\Gamma (uvf)(t)dt.$$

On account of (2.1) and (2.2), we get

$$[u,v] = \langle v, B_{f,n}u\rangle_2 = \langle u, B_{f,n}v\rangle_2\,. \tag{2.3}$$

Consequently,

$$\langle A_{f,n}u, A_{f,n}v\rangle_2 = \langle B_{f,n}v, B_{f,n}u\rangle_2$$

and

$$\left\langle u, A^*_{f,n}A_{f,n}v\right\rangle_2 = \left\langle u, B^2_{f,n}v\right\rangle_2\,.$$

From this we obtain the following formula (compare with [28])

$$A^*_{f,n}A_{f,n} = B^2_{f,n}, \tag{2.4}$$

where $A^*_{f,n} : \mathcal{K}_n \to \mathcal{P}_n$ is the adjoint of $A_{f,n}$.

Let $\{\sigma_{k,n}\}_{k=0}^n$ be the set of all singular numbers $\sigma_{k,n} = s_k(A_{f,n})$ of $A_{f,n}$:

$$\sigma_{0,n} \geq \sigma_{1,n} \geq \cdots \geq \sigma_{n,n}.$$

Denote by $\{Q_{k,n}\}_{k=0}^{n}$ associated orthonormal polynomials (compare with [28]):

$$B_{f,n}Q_{k,n} = \sigma_{k,n}Q_{k,n} \tag{2.5}$$

and

$$A_{f,n}Q_{k,n} = \sigma_{k,n}\overline{Q}_{k,n}\,\frac{|dt|}{dt}.$$

There exist $P_{k,n} \in H_2(G), \varphi_{k,n} \in H_{2,n}(G)$ such that

$$Q_{k,n}f - P_{k,n} - \overline{\varphi}_{k,n}\,\frac{|dt|}{dt} = \sigma_{k,n}\overline{Q}_{k,n}\,\frac{|dt|}{dt} \text{ a.e. on } \Gamma. \tag{2.6}$$

By (2.6), the polynomials $Q_{k,n}$ are characterized by the double orthogonality conditions:

$$[Q_{i,n}, Q_{j,n}] = \sigma_{i,n}\delta_{ij}, \;\; \langle Q_{i,n}, Q_{j,n}\rangle_2 = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol.

Let φ be any function in the Hardy space $H_1(G)$. We can represent φ as a sum of its Taylor series:

$$\varphi(z) = \sum_{i=0}^{\infty}\varphi_i z^i, \quad z \in G.$$

Let

$$S_n(\varphi)(z) = \sum_{k=0}^{n}\varphi_k z^i$$

be the n-th partial sum of the Taylor series of φ. The equality (2.6) implies that

$$[S_n(\varphi), Q_{k,n}] = \sigma_{k,n}\langle\varphi, Q_{k,n}\rangle_2. \tag{2.7}$$

According to (1.1) and (2.3), the following formula for the n-th singular number $\sigma_{n,n}$ holds:

$$\sigma_{n,n} = \inf_{u\in\mathcal{P}_n}\frac{\|A_{f,n}u\|_2}{\|u\|_2} = \inf_{u\in\mathcal{P}_n}\sup_{v\in\mathcal{P}_n}\frac{|\int_\Gamma uvf\,dt|}{||u||_2||v||_2}. \tag{2.8}$$

2.2. **An analogue of the AAK Theorem.** We now study an analogue of the AAK Theorem for the operator $A_{f,n}$. Let n be a nonnegative integer. Denote by $H_{\infty,n}(G)$ a subspace of $H_\infty(G)$ consisting of functions $\varphi \in H_\infty(G)$ such that each function φ has a zero of order at least $n+1$ at $z = 0$. Let $\mathcal{L}_n$ be a subspace of $L_\infty(\Gamma)$ such that every function $\psi \in \mathcal{L}_n$ can be represented as a sum $\psi = \beta + \overline{\varphi}|dt|/dt$, where $\beta \in H_\infty(G)$, $\varphi \in H_{\infty,n}(G)$.

Suppose that f is a continuous function on Γ. Let us consider the following approximation problem

$$\Delta_n = \inf_{\psi\in\mathcal{L}_n,\alpha\in\mathcal{P}_n}\left\|f - \frac{\psi}{\alpha}\right\|_\infty.$$

We note that the dual of the quotient space $C(\Gamma)/H_\infty(G)$ is $H_1(G)$ (see, for example [17], [21]). Moreover, each continuous linear functional l on $C(\Gamma)/H_\infty(G)$ can be represented in the form

$$< x, l >= \int_\Gamma (gv)(t)dt,$$

where $x \in C(\Gamma)/H_\infty(G)$ is an equivalence class $g + H_\infty(G), g \in C(\Gamma)$, and $v \in H_1(G)$. From this it follows immediately that the dual of the quotient space $C(\Gamma)/\mathcal{L}_n$ is $\mathcal{P}_n$.

Fix $u \in \mathcal{P}_n$ and $u \neq 0$ on Γ. On the basis of duality relations following from the Hahn–Banach Theorem (see, for example, [22]), we get

$$\inf_{\psi\in\mathcal{L}_n}||f - \frac{\psi}{u}||_\infty = \sup_{v\in\mathcal{P}_n}\frac{|\int_\Gamma uvf\,dt|}{||uv||_1}$$

and

$$\Delta_n = \inf_{u \in \mathcal{P}_n} \sup_{v \in \mathcal{P}_n} \frac{|\int_\Gamma uvf\,dt|}{||uv||_1}, \tag{2.9}$$

where u runs over the class of all polynomials $u \in \mathcal{P}_n$ and $u \neq 0$ on Γ. Since polynomials $u \in \mathcal{P}_n$ with the condition $u \neq 0$ on Γ are dense in the uniform metric on Γ in the class of all polynomials $\mathcal{P}_n$ we can omit in (2.9) the condition $u \neq 0$ on Γ.

We now state an analogue of the AAK Theorem for the operator $A_{f,n}$. The n-th singular number of the operator $A_{f,n}$ can be characterized as an error of the best approximation of f in the space $L_\infty(\Gamma)$ by functions ψ/α, where $\psi \in \mathcal{L}_n, \alpha \in \mathcal{P}_n$.

Theorem 2.1. *We have*

$$\sigma_{n,n} = \Delta_n. \tag{2.10}$$

Proof. The inequality

$$\sigma_{n,n} \leq \Delta_n.$$

follows directly from (2.8) and (2.9), and

$$||uv||_1 \leq ||u||_2||v||_2.$$

Multiplying both sides of (2.6) by an arbitrary polynomial $v \in \mathcal{P}_n, v \not\equiv 0$, setting $k = n$ and integrating, we obtain that

$$|\int_\Gamma vQ_{n,n}f\,dt| = \sigma_{n,n}|\int_\Gamma v\overline{Q}_{n,n}|dt| \leq \sigma_{n,n}||vQ_{n,n}||_1.$$

Hence

$$\sup_{v \in \mathcal{P}_n} \frac{|\int_\Gamma vQ_{n,n}f\,dt|}{||vQ_{n,n}||_1} \leq \sigma_{n,n}.$$

From this, on account (2.9), we get

$$\Delta_n \leq \sigma_{n,n}$$

and, then, (2.10). ∎

3. MARKOV FUNCTIONS

3.1. **Hankel matrix.** Let μ be a positive Borel measure with support supp $\mu = E \subset (-1, 1)$ containing infinitely many points. Here and in what follows we assume that $f(z)$ is the Markov function

$$f(z) = \frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z - x}.$$

For any $u, v \in \mathcal{P}_n$,

$$[u, v] = \int_\Gamma (uvf)(t)dt = \int_E \frac{1}{2\pi i} \int_\Gamma \frac{(uv)(t)dt}{t - x}\, d\mu(x) = \int_E (uv)(x)d\mu(x). \tag{3.1}$$

Obviously,

$$if(z) = \frac{1}{2\pi} \int_E \frac{d\mu(x)}{z - x} = \sum_{i=0}^{\infty} \frac{f_i}{z^{i+1}}, \quad |z| > 1,$$

where

$$f_i = \frac{1}{2\pi} \int_E x^i\, d\mu(x).$$

Set

$$Q_{k,n}(z) = \sum_{i=0}^{n} a_{i,k} z^i.$$

Letting $\varphi(t) = t^l, l = 0, \dots, n$ in (2.7), and taking into account (3.1) and

$$\int_\Gamma t^i \overline{t^j}\, |dt| = 2\pi\delta_{ij},$$

we get

$$\sum_{i=0}^{n} a_{i,k} f_{i+l} = \sigma_{k,n}\overline{a}_{l,k}. \tag{3.2}$$

Let

$$D_n = \left(\frac{1}{2\pi}\int_E x^{i+j} d\mu(x)\right)_{i,j=0}^{n}.$$

By (3.2),

$$D_n \mathbf{a_k} = \sigma_{k,n}\overline{\mathbf{a_k}},$$

where

$$\mathbf{a_k} = \begin{pmatrix} a_{0,k} \\ \dots \\ a_{n,k} \end{pmatrix}.$$

So, for all $k = 0, \dots, n$, $\sigma_{k,n}$ is a singular number of the Hankel matrix D_n.

3.2. **Connection with restriction operator.** Let $J_n : \mathcal{P}_n \to L_2(\mu, E)$ be the restriction operator given by restricting an element $\alpha \in \mathcal{P}_n$ to E: $J_n\alpha = \alpha_{|E}$. It is easily seen that for $\alpha \in \mathcal{P}_n$

$$(J_n^* J_n)(\alpha)(z) = \frac{1}{2\pi}\int_E \frac{1-(zx)^{n+1}}{1-zx}\alpha(x) d\mu(x), \tag{3.3}$$

where $J_n^* : L_2(\mu, E) \to \mathcal{P}_n$ is the adjoint of J_n.

The operator $C : L_2(\mu, E) \to L_2(\mu, E)$ is defined by $C\varphi = \overline{\varphi}$, $\varphi \in L_2(\mu, E)$. The operator C satisfies the following properties: C is an antilinear operator, $C^2 = I$ is the identical operator, and

$$\langle C\varphi, \psi\rangle_{2,\mu} = \langle C\psi, \varphi\rangle_{2,\mu} \quad \text{for} \quad \varphi, \psi \in L_2(\mu, E).$$

Let $U : H_2(G) \to H_2(G)$ be the antilinear isometric conjugation on $H_2(G)$ defined by the formula

$$(U\varphi)(z) = \overline{\varphi(\overline{z})}, \quad \varphi \in H_2(G), \quad z \in G.$$

It is not hard to see that U satisfies the following properties: U is an antilinear operator, $U^2 = I$, and

$$\langle U\varphi, \psi\rangle_2 = \langle U\psi, \varphi\rangle_2 \quad \text{for} \quad \varphi, \psi \in H_2(G).$$

For the Markov function f there is a connection between the operators $A_{f,n}^* A_{f,n}$ and $(J_n^* J_n)^2$ (compare with [4], [6], [28]).

Lemma 3.1. *We have*

$$A_{f,n}^* A_{f,n} = (J_n^* C J_n)^2 = (J_n^* J_n)^2 \tag{3.4}$$

and

$$\sigma_{k,n} = s_k(J_n)^2, \qquad k = 0, \dots, n, \tag{3.5}$$

where $\sigma_{k,n}$ and $s_k(J_n)$ are the singular numbers of the operators $A_{f,n}$ and J_n, respectively.

Proof. For any pair of polynomials $u, v \in \mathcal{P}_n$,

$$\begin{aligned}[u,v] &= \int_E (uv)(x)d\mu(x) = \langle J_n u, CJ_n v\rangle_{2,\mu} \\ &= \langle u, J_n^* C J_n v\rangle_2 .\end{aligned}$$

From this, on the basis of formula (2.3), we get

$$B_{f,n} = J_n^* C J_n.$$

So, by (2.4),

$$A_{f,n}^* A_{f,n} = B_{f,n}^2 = (J_n^* C J_n)^2.$$

It follows immediately from the definitions of the operators C and U that

$$J_n = C J_n U.$$

Hence, we have

$$J_n^* J_n = J_n^* C J_n U = B_{f,n} U. \tag{3.6}$$

For any pair of polynomials $u, v \in \mathcal{P}_n$,

$$\begin{aligned}\langle J_n^* J_n u, v\rangle &= \langle J_n^* C J_n U u, v\rangle = \langle C J_n U u, J_n v\rangle_{2,\mu} \\ &= \langle C J_n v, J_n U u\rangle_{2,\mu} = \langle J_n^* C J_n v, Uu\rangle \\ &= \langle v, U J_n^* C J_n u\rangle .\end{aligned}$$

Therefore,

$$J_n^* J_n = U J_n^* C J_n = U B_{f,n}. \tag{3.7}$$

Since $U^2 = I$, from (3.6) and (3.7), we get $(J_n^* C J_n)^2 = (J_n^* J_n)^2$. ∎

By (1.1) and (3.5), the following formula holds for all integers $k = 1, \dots, n$,

$$\sigma_{k,n} = \inf_{u_1,\dots,u_k \in \mathcal{P}_n} \sup_{u \in \mathcal{P}_n} \frac{||u||_{2,\mu}^2}{||u||_2^2},$$

where the supremum is taken over all polynomials $u \in \mathcal{P}_n$, satisfying the conditions $< u, u_i >_2 = 0$, $i = 1, \dots, k$. In particularly,

$$\sigma_{n,n} = \inf_{u \in \mathcal{P}_n} \frac{||u||_{2,\mu}^2}{||u||_2^2}.$$

3.3. **Singular numbers and k-widths.** Let k and n be integers, $0 \le k \le n$. Denote by $\mathbf{A}_{n,2}$ the restriction to $L_2(\mu, E)$ of the unit ball of $\mathcal{P}_n \cap L_2(\Gamma)$. It is a convex, compact, centrally symmetric subset of $L_2(\mu, E)$. Let us define the following extremal constants:

$$\tau_{k,n} = \inf_{u \in \mathcal{P}_k} \sup_{v \in \mathcal{P}_{n-k}} \frac{||uv||_{2,\mu}}{||uv||_2},\ n = 0, 1, 2, \dots . \tag{3.8}$$

Theorem 3.2. *Let μ be a positive Borel measure with support* supp $\mu = E \subset (-1,1)$ *containing infinitely many points. For all $k = 0, 1, \dots n$,*

$$\begin{aligned}\sigma_{k,n} &= s_k(J_n)^2 = d_k(\mathbf{A}_{n,2}, L_2(\mu,E))^2 \\ &= d^k(\mathbf{A}_{n,2}, L_2(\mu,E))^2 = \delta_k(\mathbf{A}_{n,2}, L_2(\mu,E))^2 = \tau_{k,n}^2,\end{aligned}$$

where $d_k(\mathbf{A}_{n,2}, L_2(\mu,E))$, $d^k(\mathbf{A}_{n,2}, L_2(\mu,E))$, and $\delta_k(\mathbf{A}_{n,2}, L_2(\mu,E))$ are the Kolmogorov, Gelfand and linear k-widths of $\mathbf{A}_{n,2}$ in the space $L_2(\mu,E)$.

Proof. We have defined the set $\mathbf{A}_{n,2}$ as the restriction to $L_2(\mu, E)$ of the unit ball of $\mathcal{P}_n \cap L_2(\Gamma)$:

$$\mathbf{A}_{n,2} = \{J_n p : ||p||_2 \le 1, p \in \mathcal{P}_n\}.$$

It is the well-known result (see [24], Chapter IV) that the corresponding eigenvalue problem for $J_n^* J_n$ is connected with $k-$widths of $\mathbf{A}_{n,2}$, and for all $k = 0, 1, \dots n$,

$$s_k(J_n) = d_k(\mathbf{A}_{n,2}, L_2(\mu, E)) = d^k(\mathbf{A}_{n,2}, L_2(\mu, E)) = \delta_k(\mathbf{A}_{n,2}, L_2(\mu, E)).$$

Moreover, by (3.5), $\sigma_{k,n} = s_k(J_n)^2$ for all $k = 0, 1, \dots n$. To prove Theorem 3.2 it is sufficient to prove that all $k = 0, 1, \dots n$,

$$d^k(\mathbf{A}_{n,2}, L_2(\mu, E)) = \tau_{k,n}.$$

By (1.2),

$$d^k(\mathbf{A}_{n,2}, L_2(\mu, E)) = \inf_{X_{-k}} \sup_{\varphi \in X_{-k} \cap \mathbf{A}_{n,2}} \frac{||\varphi||_{2,\mu}}{||\varphi||_2}, \tag{3.9}$$

where X_{-k} denotes an arbitrary subspace of $L_2(\mu, E)$ of codimension k. From this it follows that

$$d^k(\mathbf{A}_{n,2}, L_2(\mu, E)) \le \tau_{k,n}.$$

We will show that

$$d^k(\mathbf{A}_{n,2}, L_2(\mu, E)) \ge \tau_{k,n}. \tag{3.10}$$

Without loss of generality we can assume that $1 \le k \le n$. Let S^{2k+1} be $(2k+1)$-sphere in the space $\mathbf{C^{k+1}}$:

$$S^{2k+1} = \{\mathbf{w} = (w_0, w_1, \dots, w_k) : |w_0|^2 + |w_1|^2 + \dots |w_k|^2 = 1, w_0, w_1, \dots, w_k \in \mathbf{C}\}.$$

In [12] Fisher and Micchelli constructed an odd mapping $\sigma : S^{2k+1} \to \mathcal{B}_k$ from S^{2k+1} into the set of all Blaschke products of degree at most k in $H_\infty(G)$, that is continuous when $H_\infty(G)$ is given the topology of uniform convergence on compact subsets of G. Let $\tau : \mathcal{B}_k \to \mathcal{P}_k$ be the mapping defined by

$$\tau(B) = \lambda \prod_{k=1}^{l} (z - \xi_k),$$

if

$$B(z) = \lambda \prod_{k=1}^{l} \frac{z - \xi_k}{1 - \bar{\xi}_k z}, \quad \xi_k \in G, \quad l \le k, |\lambda| = 1.$$

This mapping is odd and continuous in the topology of uniform convergence on compact subsets of G.

Fix $u \in \mathcal{P}_k, u \not\equiv 0$. Consider the following extremal problem

$$\delta = \sup_{v \in \mathcal{P}_{n-k}, v \not\equiv 0} \frac{||uv||_{2,\mu}}{||uv||_2} \tag{3.11}$$

(compare with [13]). On the basis of the normal family arguments, we can assert that there exists a polynomial $\omega \in \mathcal{P}_{n-k}, \omega \not\equiv 0$, such that

$$\delta = \frac{||u\omega||_{2,\mu}}{||u\omega||_2}.$$

The polynomial ω is zero free on G. Indeed, let $\omega(a) = 0$ for some $a \in G$. Set $\omega^*(z) = \omega(z)(1 - \bar{a}z)/(z - a)$. We have $\omega^* \in \mathcal{P}_{n-k}$, $||u\omega||_2 = ||u\omega^*||_2$ and

$$|\omega^*(z)| > |\omega(z)| \text{ for all } z \in G.$$

Hence

$$\frac{||u\omega^*||_{2,\mu}}{||u\omega^*||_2} > \frac{||u\omega||_{2,\mu}}{||u\omega||_2},$$

which contradicts the extremality of ω.

The standard variation arguments show that any extremal function ω satisfies the following equation

$$\langle uv, u\omega\rangle_{2,\mu} = \delta^2 \langle uv, u\omega\rangle_2$$

for all $v \in \mathcal{P}_n$. From this it follows that if ω and ω_1 are two extremal solutions, then $c\omega + c_1\omega_1$ is also an extremal solution. Consequently, since all extremal solutions are zero free on G, there is a unique $\omega \in \mathcal{P}_{n-k}$ such that $||u\omega||_2 = 1$, $\omega(0) > 0$ and

$$\delta = \frac{||u\omega||_{2,\mu}}{||u\omega||_2}.$$

We will denote the unique solution of (3.11) by v_u.

We now show that the mapping $u \to v_u$ is continuous on $\mathcal{P}_k$ with the topology of uniform convergence on compact subsets of $\mathbf{C}$. Let $u_l \in \mathcal{P}_k$, $u_l \to u, u \in \mathcal{P}_k, u \not\equiv 0$, uniformly on compact subsets of $\mathbf{C}$ as $l \to \infty$. We will prove that $v_{u_l} \to v_u$ uniformly on compact subsets of $\mathbf{C}$ as $l \to \infty$. Since $||u_l v_{u_l}||_2 = 1$ and $u_l \to u$ uniformly on compact subsets of $\mathbf{C}$ as $l \to \infty$, it follows that there exist a sequence $\Lambda \subset \mathbf{N}$, a polynomial $v^* \in \mathcal{P}_{n-k}$ such that $v_{u_l} \to v^*$ uniformly on compact subsets of $\mathbf{C}$ as $l \to \infty, l \in \Lambda$. Since $||u_l v_{u_l}||_2 = 1$, $v_{u_l}(0) > 0$, we get $||uv^*||_2 = 1$ and $v^*(0) > 0$. For any $v \in \mathcal{P}_{n-k}, v \not\equiv 0$,

$$\frac{||u_l v||_{2,\mu}}{||u_l v||_2} \le \frac{||u_l v_{u_l}||_{2,\mu}}{||u_l v_{u_l}||_2}.$$

Clearly,

$$||u_l v||_{2,\mu} \to ||uv||_{2,\mu} \qquad \text{and} \qquad ||u_l v||_2 \to ||uv||_2$$

as $l \to \infty,\ l \in \Lambda$. Moreover,

$$||u_l v_{u_l}||_{2,\mu} \to ||uv^*||_{2,\mu} \qquad \text{and} \qquad ||u_l v_{u_l}||_2 \to ||uv^*||_2$$

as $l \to \infty,\ l \in \Lambda$. Thus

$$\frac{||uv||_{2,\mu}}{||uv||_2} \le \frac{||uv^*||_{2,\mu}}{||uv^*||_2},$$

which implies $v^* = v_u$ and $v_{u_l} \to v_u$ uniformly on compact subsets of $\mathbf{C}$ as $l \to \infty$.

We now apply standard arguments involving the Borsuk's antipodality Theorem [24] to prove (3.10). Let $l_1, \dots l_k$ be any k continuous linear functionals on $L_2(\mu, E)$. Let us consider the mapping $\mathcal{L} : \mathcal{P}_k \to \mathbf{C}^k$ such that

$$\mathcal{L}(u) = (l_1(uv_u), \dots, l_k(uv_u)), \qquad u \in \mathcal{P}_k.$$

Thus the mapping $\mathcal{L} \circ \tau \circ \sigma$ is continuous and odd from S^{2k+1} into $\mathbf{C}^k$. It follows from the Borsuk's Theorem that this map has a zero; that there is $u_0 \in \mathcal{P}_k, u_0 \not\equiv 0$ such that

$$l_j(u_0 v_{u_0}) = 0, \quad j = 1, \dots, k.$$

So, for any subspace X_{-k} of $L_2(\mu, E)$ of codimension k

$$\sup_{\varphi \in X_{-k} \cap \mathbf{A}_{n,2}} \frac{||\varphi||_{2,\mu}}{||\varphi||_2} \ge \frac{||u_0 v_{u_0}||_{2,\mu}}{||u_0 v_{u_0}||_2} \ge \sup_{v \in \mathcal{P}_{n-k}} \frac{||u_0 v||_{2,\mu}}{||u_0 v||_2} \ge \inf_{u \in \mathcal{P}_k} \sup_{v \in \mathcal{P}_{n-k}} \frac{||uv||_{2,\mu}}{||uv||_2} = \tau_{k,n}.$$

From this we obtain (see (3.9)) the desired lower bound (3.10) for $d^k(\mathbf{A}_{n,2}, L_2(\mu, E))$ and then equality $d^k(\mathbf{A}_{n,2}, L_2(\mu, E)) = \tau_{k,n}$ which proves the theorem. ∎

3.4. **Properties of eigenfunctions.** Here and in what follows we assume that support of μ is the closed interval $E = [a, b] \subset (0, 1)$. The kernel

$$K(x, y) = \frac{1}{2\pi}(1 - (xy)^{n+1})/(1 - xy) = \frac{1}{2\pi}\sum_{i=0}^{n} x^i y^i$$

of the integral operator

$$\frac{1}{2\pi}\int_a^b \frac{1 - (xy)^{n+1}}{1 - xy}\varphi(y)d\mu(y)$$

is a continuous symmetric kernel $x, y \in E = [a, b] \subset (0, 1)$. Since a generalized Vandermonde matrix

$$||x_i^{\gamma_j}||_1^m \quad 0 < x_1 < x_2 \cdots < x_m; \quad \gamma_1 < \gamma_2 \cdots < \gamma_m$$

is totally positive, we obtain, by virtue of the Cauchy–Binet formula, that the kernel $K(x, y)$ is an oscillatory kernel $x, y \in E = [a, b] \subset (0, 1)$ (for more details about totally positive matrices, oscillatory matrices and kernels, see [16]). By the Gantmacher-Kreĭn theory [16] of integral equations with oscillatory kernels, all eigenvalues of the integral equation

$$\int_a^b K(x, y)\varphi(y)d\mu(y) = \lambda\varphi(x)$$

are positive and simple

$$\lambda_0 > \lambda_1 > \lambda_2 > \ldots,$$

the eigenfunction φ_k corresponding to the eigenvalue λ_k has k nodes on (a, b) and has no other zeros on $[a, b]$.

Lemma 3.3. *Let $E = [a, b] \subset (0, 1)$. Then all singular numbers $\sigma_{k,n}, \quad k = 0, 1, \ldots, n,$ are positive and simple, the corresponding eigenspaces are one dimensional. All eigenfunctions $Q_{k,n}$ have real coefficients and*

$$\sigma_{k,n}Q_{k,n}(x) = \frac{1}{2\pi}\int_a^b \frac{1 - (xy)^{n+1}}{1 - xy}\, Q_{k,n}(y)d\mu(y). \tag{3.12}$$

Moreover, the k-th eigenfunction $Q_{k,n}$ has exactly k simple zeros on the segment $[a, b]$.

Proof. We note that $\lambda_k = s_k(J_n)$ for $k = 0, 1, \ldots, n$. From this, on account of (3.5)

$$\sigma_{k,n} = s_k(J_n)^2,$$

we obtain that all singular numbers $\sigma_{k,n}, \quad k = 0, 1, \ldots, n,$ are positive and simple, the corresponding eigenspaces are one dimensional. Let $0 \leq k \leq n$ and let

$$Q^*_{k,n}(z) = (UQ_{k,n})(z) = \overline{Q_{k,n}(\overline{z})}.$$

Using (2.5) and (3.7), we can write

$$J_n^* J_n Q_{k,n} = UB_{f,n}Q_{k,n} = \sigma_{k,n}UQ_{k,n} = \sigma_{k,n}Q^*_{k,n}.$$

Analogously, since $U^2 = I$, on the basis of (2.5) and (3.6), we get

$$J_n^* J_n Q^*_{k,n} = J_n^* J_n UQ_{k,n} = B_{f,n}Q_{k,n} = \sigma_{k,n}Q_{k,n}.$$

Therefore, by (3.4), $Q^*_{k,n}$ is an eigenfunction of $A^*_{f,n}A_{f,n}$. Since the corresponding eigenspace is one dimensional, we can conclude that $Q_{k,n} = c\varphi_k, c \in \mathbf{R}$, $Q_{k,n}$ has real coefficients and, by 3.3 satisfies the integral equation (3.12).

By Theorem 3.2,

$$\sigma_{k,n} = \frac{||Q_{k,n}||^2_{2,\mu}}{||Q_{k,n}||^2_2} = \inf_{u\in\mathcal{P}_k}\sup_{v\in\mathcal{P}_{n-k}}\frac{||uv||^2_{2,\mu}}{||uv||^2_2}. \tag{3.13}$$

By properties of extremal solutions of (3.13), $Q_{k,n}$ has no more than k zeros on $E = [a, b]$. Since the eigenfunction $Q_{k,n}$ has exactly k nodes on (a, b) and has no other zeros on $[a, b]$, we obtain that $Q_{k,n}$ has k simple zeros $x_{1,n}, x_{2,n}, \dots, x_{k,n}$ on the segment $[a, b]$, and

$$\sigma_{k,n} = \frac{||Q_{k,n}||^2_{2,\mu}}{||Q_{k,n}||^2_2} = \frac{||w_{k,n}\beta_{k,n}||^2_{2,\mu}}{||w_{k,n}\beta_{k,n}||^2_2} = \sup_{v\in\mathcal{P}_{n-k}} \frac{||w_{k,n}v||^2_{2,\mu}}{||w_{k,n}v||^2_2},$$

where $w_{k,n}(x) = \prod_{i=1}^{k}(x - x_{i,n})$, $Q_{k,n} = w_{k,n}\beta_{k,n}$, and a polynomial $\beta_{k,n}$ is constructed from the zeros of $Q_{k,n}$ in $C \setminus G$. ∎

3.5. **Orthogonality.** We have the following lemma (compare with [7]).

Lemma 3.4. *Let $E = [a, b] \subset (0, 1)$. The following orthogonality relations are valid:*

$$\int_a^b S_n(\frac{x^i w^*_{k,n} - x^{k-i}w_{k,n}}{w^*_{k,n}}\beta_{k,n})(x)Q_{k,n}(x)d\mu(x) = 0, \qquad i = 0, \dots, k, \tag{3.14}$$

and

$$\int_a^b S_n(\frac{x^i}{w^*_{k,n}(x)})(x)Q_{k,n}(x)d\mu(x) = 0, \qquad i = 0, \dots, k-1, \tag{3.15}$$

where $w^*_{k,n}(x) = \prod_{i=1}^{k}(1 - x_{i,n}x)$.

Proof. Let φ be any function in the Hardy space $H_1(G)$. By (2.7) and (3.1),

$$[S_n(\varphi), Q_{k,n}] = \int_a^b S_n(\varphi)Q_{k,n}(x)d\mu(x) = \sigma_{k,n}\langle\varphi, Q_{k,n}\rangle_2,$$

where $S_n(\varphi)$ is the n-th partial sum of the Taylor series of φ. Letting $\varphi = x^i\beta_{k,n}, \quad i = 0, \dots, k$, we get

$$I = [x^i\beta_{k,n}, Q_{k,n}] = \sigma_{k,n}\langle t^i\beta_{k,n}, Q_{k,n}\rangle_2. \tag{3.16}$$

Since I is a real number,

$$I = \sigma_{k,n}\overline{\langle t^i\beta_{k,n}, Q_{k,n}\rangle_2} = \sigma_{k,n}\langle\frac{t^{k-i}Q_{k,n}(t)}{w^*_{k,n}(t)}, Q_{k,n}\rangle_2 = [S_n(\frac{x^{k-i}Q_{k,n}}{w^*_{k,n}}), Q_{k,n}].$$

From this, on account of (3.16), we get

$$[S_n(x^i\beta_{k,n} - \frac{x^{k-i}Q_{k,n}}{w^*_{k,n}}), Q_{k,n}] = 0, \qquad i = 0, \dots, k,$$

and, then, (3.14).

By (3.12),

$$\int_a^b S_n(\frac{1}{1 - x_{i,n}x})(x)Q_{k,n}(x)d\mu(x) = 0 \quad \text{for} \quad i = 1, \dots, k.$$

Thus for any polynomial ω, $\deg\omega \le k - 1$,

$$\int_a^b S_n(\frac{\omega}{w^*_{k,n}})(x)Q_{k,n}(x)d\mu(x) = 0$$

Letting $\omega(x) = x^i, i = 0, \dots, k-1$, we get (3.15). This finishes the proof. ∎

4. ASYMPTOTICS OF THE EXTREMAL CONSTANTS

4.1. **The main Theorem.** Let $E = [a, b] \subset (-1, 1)$. Let us consider the following extremal constants:

$$\chi_{k,n} = \inf_{u \in \mathcal{P}_k} \sup_{v \in \mathcal{P}_{n-k}} \frac{||uv||_E}{||uv||_\Gamma}, \quad n = 0, 1, 2, \ldots . \tag{4.1}$$

In [27] asymptotics of $\chi_{k_n,n}$, $k_n/n \to \theta \in [0, 1]$ as $n \to \infty$, investigated in the terms of the solutions of a minimal energy problem associated with the sets E and Γ.

Let K be a compact set in the complex plane $\mathbf{C}$. Denote by $\Lambda_\delta(K)$, $\delta > 0$, the set of all positive Borel measures λ of mass $\delta = \int d\lambda$ compactly supported on K. The sequence of measures $\{\tau_n\}_{n=1}^\infty$, supp $\tau_n \subseteq K$, weak-star converges to a measure τ, supp $\tau \subseteq K$:

$$\tau_n \xrightarrow{*} \tau \quad \text{as} \quad n \to \infty,$$

if for any continuous function φ on K,

$$\int \varphi d\tau_n \to \int \varphi d\tau \quad \text{as} \quad n \to \infty.$$

Let τ be a positive Borel measure with compact support supp τ, $|\tau| = \int d\tau$. The logarithmic potential

$$U^\tau(z) = \int \log \frac{1}{|z - t|} d\tau(t)$$

of the measure τ is superharmonic in $\mathbf{C}$ and harmonic in $\mathbf{C} \setminus$ supp τ. Denote by

$$I(\tau) = \int \int \log \frac{1}{|z - t|} d\tau(t) d\tau(z)$$

the logarithmic energy of the measure τ. Let $\mathrm{cap}(E, \Gamma)$ be the condenser (Green) capacity of E with respect to G.

We now consider the minimal energy problem of the logarithmic potential theory that we will use in asymptotical analysis of the extremal constants. The following can be proved by the well-known methods (see [29], [27]).
For each $0 \leq \theta \leq 1$ *there exist unique measures* $\lambda_\theta \in \Lambda_{1-\theta}(\Gamma)$ *and* $\mu_\theta \in \Lambda_\theta(E)$ *such that*

$$I(\lambda_\theta + \mu_\theta) = \min_{\lambda, \mu} I(\lambda + \mu).$$

where $\lambda \in \Lambda_{1-\theta}(\Gamma)$ *and* $\mu \in \Lambda_\theta(E)$*. Moreover, there exist constants* m_θ *and* $\widehat{m_\theta}$ *such that*

$$U^{\lambda_\theta + \mu_\theta} = \widehat{m_\theta} \quad \textit{on} \quad \text{supp}\ \mu_\theta = E \tag{4.2}$$

and

$$U^{\lambda_\theta + \mu_\theta} = \min_E U^{\lambda_\theta + \mu_\theta} = m_\theta + \widehat{m_\theta} \quad \textit{on} \quad \text{supp}\ \lambda_\theta \subseteq \Gamma. \tag{4.3}$$

In general, (4.2) and (4.3) hold only q.e.(quazi-everywhere) on supp λ_θ and supp μ_θ, respectively. The regularity of E and Γ is sufficient for this property to hold at every point of supp λ_θ and supp μ_θ. We also remark that the measures λ_θ and μ_θ are uniquely deteremined by conditions (4.2) and (4.3) (see [29], [27]).

In the case when $E = [a, b] \subset (0, 1)$, we represent $Q_{k_n,n}$ in the form $Q_{k_n,n} = w_{k_n,n}\beta_{k_n,n}$, where $w_{k_n,n}$ is constructed from zeros $x_{1,n}, x_{2,n}, \ldots, x_{k,n}$ of $Q_{k_n,n}$ on E. Denote by $\nu(w_{k_n,n})$ the zero counting measure associated with polynomial $w_{k_n,n}$:

$$\nu(w_{k_n,n}) = \frac{1}{n} \sum_{i=0}^{k} \delta_{x_{i,n}}$$

where, δ_z is the unit Dirac measure with support at $z \in \mathbf{C}$.

We now formulate a theorem characterizing asymptotics of the singular numbers $\sigma_{k_n,n}$, where $k_n/n \to \theta \in [0,1]$ as $n \to \infty$.

Theorem 4.1. *We assume that the measure μ has support $E = [a,b] \subset (-1,1)$ and satisfies the following condition*

$$\frac{d\mu}{dx} > 0 \quad \textit{a.e. on} \quad E. \tag{4.4}$$

We have

$$\lim_{n\to\infty} \frac{\log \sigma_{k_n,n}}{n} = 2m_\theta.$$

In particular, when $\theta = 0$, $k_n \to \infty$ and $k_n = o(n)$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{\log \sigma_{k_n,n}}{k_n} = -\frac{2}{\operatorname{cap}(E,\Gamma)}.$$

Moreover, in the case when $E = [a,b] \subset (0,1)$,

$$\nu(w_{k_n,n}) \xrightarrow{*} \mu_\theta \quad \textit{as} \quad n \to \infty.$$

By the works of Gonchar and Rakhmanov (see [18] and [19]), for any sequence of polynomials $p_n \in \mathcal{P}_n, p_n \not\equiv 0,$ we have

$$\lim_{n\to\infty} \left(\frac{||p_n||_2}{||p_n||_\Gamma} \right)^{1/n} = 1,$$

and

$$\lim_{n\to\infty} \left(\frac{||p_n||_{2,\mu}}{||p_n||_E} \right)^{1/n} = 1,$$

where measure μ satisfies condition (4.4). Then Theorem 4.1 directly follows from results obtained in [27], where asymptotics of the extremal constants $\chi_{k_n,n}$ and asociated extremal polynomials is investigated when $k_n/n \to \theta \in [0,1]$ as $n \to \infty$.

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