NEW REFINEMENTS OF HÖLDER’S INEQUALITY

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ABSTRACT. In this paper, we define two mappings, investigate their properties, obtain some new refinements of Hölder’s inequality.

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1. Introduction

Let $a_i > 0, b_i > 0 (i = 1, 2, \cdots , n; n \geq 2)$, $p$ and $q$ be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$. If $p > 1$, then

$$\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}},$$

if $p < 1 (p \neq 0)$, then the inequality in (1.1) is reversed.

The inequality (1.1) is called the Hölder’s inequality (see [1]-[2]). For some recent results which generalize, improve and extend this classical inequality, see [1]-[4].

To go further into (1.1), we define two mappings $F_1$ and $F_2$ by

$$F_1 : \{ (n, k) | n \geq 2, k = 1, 2, \cdots , n; n \in \mathbb{N} \} \rightarrow \mathbb{R},$$

$$F_1 (n, k) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\sum_{j=1}^{k} b_{i_j}^q}{\sum_{j=1}^{k} \frac{a_{i_j} b_{i_j}^q}{\sum_{j=1}^{k} b_{i_j}^q}},$$

$$F_2 : \{ (n, k) | n \geq 2, k = 1, 2, \cdots ; n \in \mathbb{N} \} \rightarrow \mathbb{R},$$

$$F_2 (n, k) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{\sum_{j=1}^{k} b_{i_j}^q}{\sum_{j=1}^{k} \frac{a_{i_j} b_{i_j}^q}{\sum_{j=1}^{k} b_{i_j}^q}},$$

where $p$ and $q$ be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$.

The aim of this paper is to study the properties of $F_1$ and $F_2$, thus obtaining some new refinements of (1.1).

2. Main Results

**Theorem 2.1.** Let $a_i > 0, b_i > 0 (i = 1, 2, \cdots , n; n \geq 2)$, $p$ and $q$ be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$, and $F_1$ be defined as in the first section. We write $G_1 (n, k) = \left( \sum_{i=1}^{n} b_i^q \right) \left( F_1 (n, k) \right)^{\frac{1}{p}}$. We have

(1) When $p > 1$, we get the following finite refinements of (1.1)

$$\sum_{i=1}^{n} a_i b_i = G_1 (n, n) \leq G_1 (n, n-1) \leq \cdots \leq G_1 (n, 1) = \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.$$

(2) When $p < 1 (p \neq 0)$, the inequalities in (2.1) are reversed.

**Remark 1.** Theorem 2.1 is the finite refinements of Hölder’s inequality with non-repetitive sample.

**Theorem 2.2.** Let $a_i, b_i (i = 1, 2, \cdots , n; n \geq 2)$, $p$ and $q$ be defined as in the Theorem 2.1 and $F_2$ be defined as in the first section. We write $G_2 (n, k) = \left( \sum_{i=1}^{n} b_i^q \right) \left( F_2 (n, k) \right)^{\frac{1}{p}}$. We have
(1) When \( p > 1 \), we get the following infinite refinements of (1.1):

\[
\sum_{i=1}^{n} a_i b_i \leq \cdots \leq G_2(n, n) \leq G_2(n, n-1)
\]

(2.2)

\[
\leq \cdots \leq G_2(n, 1) = \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.
\]

(2) When \( p < 1 \) (\( p \neq 0 \)), the inequalities in (2.2) are reversed.

**Remark 2.** Theorem 2.2 is the infinite refinements of Hölder’s inequality with repetitive sample.

### 3. Several Lemmas

In order to prove the above theorems, we need the following two lemmas.

**Lemma 3.1.** Let \( a_i > 0, b_i > 0 \) (\( i = 1, 2, \cdots, n; n \geq 2 \)), \( p \) and \( q \) be two non-zero real numbers such that \( p^{-1} + q^{-1} = 1 \), and \( F_1 \) be defined as in the first section. We have

(1) When \( p > 1 \) or \( p < 0 \), we get

\[
\left( \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i^q} \right)^p = F_1(n, n) \leq F_1(n, n-1) \leq \cdots \leq F_1(n, 1) = \frac{\sum_{i=1}^{n} a_i^p}{\sum_{i=1}^{n} b_i^q}.
\]

(2) When \( 0 < p < 1 \), the inequalities in (3.1) are reversed.

**Proof of Lemma 3.1.** From the definition of \( F_1 \), since a simple calculation shows that

\[
F_1(n, n) = \left( \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i^q} \right)^p,
\]

\[
F_1(n, 1) = \frac{\sum_{i=1}^{n} a_i^p}{\sum_{i=1}^{n} b_i^q}.
\]

For \( k = 2, 3, \cdots, n \), using some elementary identity involved combinatorial numbers, from (3.1) and (3.6) in [6], we can get

\[
\sum_{j=1}^{k} a_i b_i = \sum_{\{r_1, \cdots, r_{k-1}\} \subset \{i_1, \cdots, i_k\}} \frac{\sum_{l=1}^{k-1} a_{r_l} b_{r_l}}{k-1}.
\]
and
\[
\sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{\{r_1, \ldots, r_{k-1}\} \subset \{i_1, \ldots, i_k\}} \left( \sum_{l=1}^{k-1} a_{r_l} b_{r_l} \right)
= (n - k + 1) \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq n} \left( \sum_{m=1}^{k-1} a_{j_m} b_{j_m} \right),
\]
respectively.

(1) When \( p > 1 \) or \( p < 0 \), \( x^p \) is convex function on \((0, +\infty)\) with \( x \). Let \( t_{r_l} = b_{r_l} q, x_{r_l} = a_{r_l} b_{r_l} (1-q) \), using Jensen’s inequality of convex function, from (3.3) in [6], we get
\[
\left( \frac{\sum_{j=1}^{k} a_{i_j} b_{i_j}}{\sum_{j=1}^{k} b_{i_j} q} \right)^p = \left( \sum_{\{r_1, \ldots, r_{k-1}\} \subset \{i_1, \ldots, i_k\}} \frac{\sum_{l=1}^{k-1} b_{r_l} q}{(k - 1) \sum_{j=1}^{k} b_{i_j} q} \left( \sum_{l=1}^{k-1} a_{r_l} b_{r_l} \right)^p \right)
\]
(3.6)
\[
\leq \sum_{\{r_1, \ldots, r_{k-1}\} \subset \{i_1, \ldots, i_k\}} \left( k - 1 \right) \sum_{j=1}^{k} b_{i_j} q \left( \sum_{l=1}^{k-1} a_{r_l} b_{r_l} \right)^p
\]
From the definition of \( F_1 \) and (3.5)-(3.6), we get
\[
F_1 (n, k) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\sum_{j=1}^{k} b_{i_j} q}{C_{n-1}^{k-1} \sum_{i=1}^{n} b_i q} \left( \sum_{j=1}^{k} a_{i_j} b_{i_j} \right)^p
\]
\[
\leq \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{\{r_1, \ldots, r_{k-1}\} \subset \{i_1, \ldots, i_k\}} \frac{\sum_{l=1}^{k-1} b_{r_l} q}{(k - 1) C_{n-1}^{k-1} \sum_{i=1}^{n} b_i q} \left( \sum_{l=1}^{k-1} a_{r_l} b_{r_l} \right)^p
\]
(3.7)
\[
= \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq n} \left( k - 1 \right) C_{n-1}^{k-1} \sum_{i=1}^{n} b_i q \left( \sum_{m=1}^{k-1} a_{j_m} b_{j_m} \right)^p
\]
\[
= \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq n} C_{n-1}^{k-2} \sum_{i=1}^{n} b_i q \left( \sum_{m=1}^{k-1} a_{j_m} b_{j_m} \right)^p = F_1 (n, k - 1).
\]
Combination of (3.2)-(3.3) and (3.7) yields (3.1).

(2) When \( 0 < p < 1 \), \( x^p \) is concave function on \((0, +\infty)\) with \( x \). Using Jensen’s inequality of concave function, we get the reverse of (3.6) and (3.7), which implies the reverse of (3.1).

This completes the proof of Lemma 3.1.
Lemma 3.2. Let \( a_i, b_i \ (i = 1, 2, \cdots, n; n \geq 2) \), \( p \) and \( q \) be defined as in the Lemma 3.1, and \( F_2 \) be defined as in the first section. We have

1. When \( p > 1 \) or \( p < 0 \), we get
   \[
   \left( \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i^q} \right)^p \leq \cdots \leq F_2 (n, n) \leq F_2 (n, n - 1) \leq \cdots \leq F_2 (n, 1) = \frac{\sum_{i=1}^{n} a_i^p}{\sum_{i=1}^{n} b_i^q}.
   \]

2. When \( 0 < p < 1 \), the inequalities in (3.8) are reversed.

Proof of Lemma 3.2. From the definition of \( F_2 \), since a simple calculation shows that
   \[
   F_2 (n, 1) = \frac{\sum_{i=1}^{n} a_i^p}{\sum_{i=1}^{n} b_i^q}.
   \]

For \( k = 2, 3, \cdots \), using some elementary identity involved combinatorial numbers, we have

\[
\sum_{i=1}^{n} a_i b_i = \frac{1}{C_{n+k-1}^{k}} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \left( \sum_{j=1}^{k} a_i b_{i_j} \right)
\]

and

\[
\sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \left( \sum_{r_1 \cdots r_{k-1}} \left( \sum_{1 \leq j \leq \cdots \leq k-1 \leq n} \left( \sum_{m=1}^{k-1} a_{r_m} b_{j_m} \right) \right) \right) = (n + k - 1) \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \left( \sum_{m=1}^{k-1} a_{j_m} b_{j_m} \right),
\]

where (3.11) can also be obtained from (2.1) in [8].

1. When \( p > 1 \) or \( p < 0 \), \( x^p \) is convex function on \((0, +\infty)\) with \( x \). Let \( t_{ij} = b_{ij}^q, x_{ij} = a_{ij} b_{ij}^{1-q} \), using Jensen’s inequality of convex function, we have
   \[
   \left( \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i^q} \right)^p \leq \left( \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{\sum_{j=1}^{k} b_{ij}^q}{\sum_{j=1}^{k} b_{ij}^q} \right)^p \leq \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{\sum_{j=1}^{k} a_{ij} b_{ij}}{\sum_{j=1}^{k} b_{ij}^q} = F_2 (n, k).
   \]

From the definition of \( F_2 \) and (3.6), (3.11), we get
\[ F_2(n, k) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{\sum_{j=1}^{k} b_{i_j} q}{n^{k-1} \sum_{i=1}^{n} b_i q} \left( \frac{\sum_{j=1}^{k} a_{i_j} b_{i_j}}{\sum_{j=1}^{k} b_{i_j} q} \right)^p \]

\[ \leq \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \sum_{(i_1, \cdots, i_k) \subset (i_1, \cdots, i_k)} \frac{\sum_{l=1}^{k-1} b_{i_l} q}{(k-1) n^{k-1} \sum_{i=1}^{n} b_i q} \left( \frac{\sum_{l=1}^{k-1} a_{i_l} b_{i_l}}{\sum_{l=1}^{k-1} b_{i_l} q} \right)^p \]

(3.13)

\[ = \sum_{1 \leq j_1 \leq \cdots \leq j_{k-1} \leq n} \sum_{(j_1, \cdots, j_{k-1}, j) \subset (j_1, \cdots, j_{k-1})} \frac{\sum_{m=1}^{k-1} b_{j_m} q}{(k-1) n^{k-1} \sum_{i=1}^{n} b_i q} \left( \frac{\sum_{m=1}^{k-1} a_{j_m} b_{j_m}}{\sum_{m=1}^{k-1} b_{j_m} q} \right)^p \]

Combination of (3.9) and (3.12)-(3.13) yields (3.8).

(2) When \(0 < p < 1\), \(x^p\) is concave function on \((0, +\infty)\) with \(x\). Using Jensen’s inequality of concave function, we get the reverse of (3.12) and (3.13), which implies the reverse of (3.8).

This completes the proof of Lemma 3.2.

4. PROOF OF THEOREMS

**Proof of Theorem 2.1.** From the definitions of \(F_1\) and \(G_1\), since a simple calculation shows that

(4.1)

\[ G_1(n, n) = \sum_{i=1}^{n} a_i b_i , \]

(4.2)

\[ G_1(n, 1) = \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} . \]

From \(a_i > 0, b_i > 0\ (i = 1, 2, \cdots, n)\), we have

\[ \sum_{i=1}^{n} a_i^p > 0, \left( \sum_{i=1}^{n} a_i b_i \right)^p > 0, \sum_{i=1}^{n} b_i^q > 0. \]

(4.3)

For \(k = 2, 3, \cdots, n\), we have

(1) When \(p > 1\), \(x^\frac{1}{p}\) is monotonically increasing on \((0, +\infty)\) with \(x\). From (3.1) and (4.3), with a simple calculation, we get

(4.4)

\[ G_1(n, k) \leq G_1(n, k - 1) . \]
Combination of (4.1)-(4.2) and (4.4) yields (2.1).

(2) When \( p < 1 \) \((p \neq 0)\).

Case 1: \( 0 < p < 1 \), \( x^{\frac{1}{p}} \) is monotonically increasing on \((0, +\infty)\) with \( x \). From the reverse of (3.1) and (4.3), since a simple calculation shows that the reverse of (4.4).

Case 2: \( p < 0 \), \( x^{\frac{1}{p}} \) is monotonically decreasing on \((0, +\infty)\) with \( x \). From (3.1) and (4.3), since a simple calculation shows that the reverse of (4.4).

From above two cases, when \( p < 1 \) \((p \neq 0)\), we have

\[
G_1(n, k) \geq G_1(n, k - 1) .
\]

Combination of (4.1)-(4.2) and (4.5) yields the reverse of (2.1).

The proof of Theorem 2.1 is completed.

Proof of Theorem 2.2: From the definitions of \( F_2 \) and \( G_2 \), we have

\[
G_2(n, 1) = \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}} .
\]

For \( k = 2, 3, \ldots \), we have

(1) When \( p > 1 \), \( x^{\frac{1}{p}} \) is monotonically increasing on \((0, +\infty)\) with \( x \). From (3.8) and (4.3), with a simple calculation, we get

\[
\sum_{i=1}^{n} a_i b_i \leq \cdots \leq G_2(n, k) \leq G_2(n, k - 1) .
\]

Combination of (4.6) and (4.7) yields (2.2).

(2) When \( p < 1 \) \((p \neq 0)\).

Case 1: \( 0 < p < 1 \), \( x^{\frac{1}{p}} \) is monotonically increasing on \((0, +\infty)\) with \( x \). From the reverse of (3.8) and (4.3), since a simple calculation shows that the reverse of (4.7).

Case 2: \( p < 0 \), \( x^{\frac{1}{p}} \) is monotonically decreasing on \((0, +\infty)\) with \( x \). From (3.8) and (4.3), since a simple calculation shows that the reverse of (4.7).

From above two cases, when \( p < 1 \) \((p \neq 0)\), we have

\[
\sum_{i=1}^{n} a_i b_i \geq \cdots \geq G_2(n, k) \geq G_2(n, k - 1) .
\]

Combination of (4.6) and (4.8) yields the reverse of (2.2).

The proof of Theorem 2.2 is completed.

REFERENCES


