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A SUBORDINATION THEOREM FOR ANALYTIC FUNCTIONS

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ABSTRACT. It is shown that if f is analytic in $D = \{z : |z| < 1\}$, with $f(0) = f'(0) - 1 = 0$, then for $\alpha > 0$, $\gamma > 0$, $f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} \prec \left(\frac{1+z}{1-z}\right)^{\beta(\gamma)}$ implies $\left(\frac{f(z)}{z}\right)^{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}$, where $\beta(\gamma) = \gamma + \frac{2}{\pi}\arctan\left(\frac{\gamma}{\alpha}\right)$, and that $\beta(\gamma)$ is the largest number such that this implication holds.

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1. INTRODUCTION AND DEFINITIONS

Let S be the class of analytic normalised univalent functions f, defined in $z \in D = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The Bazilevič functions with logarithmic growth $B_1(\alpha) \subset S$ defined as follows have been extensively studied (see e.g. [2, 3, 4]).

Suppose that f is analytic in D and is given by (1.1). Then for $\alpha \geq 0$, $f \in B_1(\alpha)$, if and only if,

$$Re\ f'(z)\Big(\frac{f(z)}{z}\Big)^{\alpha-1} > 0. \tag{1.2}$$

We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in D, such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$, and $f(z) = g(\omega(z))$. Marjono and Thomas [5] discussed subordination in a sector.

We will use the following well-known lemma.

2. LEMMA

The Miller-Mocanu Lemma [1]

Let F be analytic in D and G be analytic univalent in $\bar{D}$, with $F(0) = G(0)$. If $F \not\prec G$, then there is a point $z_0 \in D$ and $\zeta_0 \in \partial D$, such that $F(|z| < |z_0|) \subset G(D)$, $F(z_0) = G(\zeta_0)$ and $z_0 F'(z_0) = k\zeta_0 G'(\zeta_0)$ for $k \geq 1$.

We prove the following theorem for analytic functions, noting its relationship with the Bazilevič functions $B_1(\alpha)$ defined in (1.2).

3. THEOREM

Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$. Then for $\alpha > 0$, $\gamma > 0$ and $z \in D$,

$$f'(z)\Big(\frac{f(z)}{z}\Big)^{\alpha-1} \prec \Big(\frac{1+z}{1-z}\Big)^{\beta(\gamma)}$$

implies

$$\Big(\frac{f(z)}{z}\Big)^{\alpha} \prec \Big(\frac{1+z}{1-z}\Big)^{\gamma}, \tag{3.1}$$

where

$$\beta(\gamma) = \gamma + \frac{2}{\pi}\arctan\left(\frac{\gamma}{\alpha}\right). \tag{3.2}$$

Furthermore $\beta(\gamma)$ is the largest number such that (3.1) holds.

Proof. Write

$$p(z) = \left(\frac{f(z)}{z}\right)^{\alpha},$$

so that p is analytic in D, $p(0) = 1$ and

$$p(z) + \frac{zp'(z)}{\alpha} = \left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z).$$

Thus we need to show that

$$p(z) + \frac{zp'(z)}{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\beta}$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\gamma},$$

when $\beta = \beta(\gamma)$ is given by (3.2).

For $z \in D$, let $h(z) = [(1+z)/(1-z)]^{\beta}$ and $q(z) = [(1+z)/(1-z)]^{\gamma}$, so that $|\arg h(z)| < \dfrac{\beta\pi}{2}$ and $|\arg q(z)| < \dfrac{\gamma\pi}{2}$.

Suppose that $p \not\prec q$. Then from the Miller-Mocanu Lemma, there exists $z_0 \in D$ and $\zeta_0 \in \partial D$, such that $p(z_0) = q(\zeta_0)$, $p(|z| < |z_0|) \subset q(D)$ and $z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$, for $k \geq 1$.

Since $p(z_0) = q(\zeta_0) \neq 0$, it follows that $\zeta_0 \neq \pm 1$. Thus we can write $ri = \dfrac{1+\zeta_0}{1-\zeta_0}$ for $r \neq 0$. Hence

$$p(z_0) + \frac{z_0 p'(z_0)}{\alpha} = q(\zeta_0) + \frac{k\zeta_0 q'(\zeta_0)}{\alpha}$$

for $k \geq 1$.

Differentiating $q(z)$ we obtain

$$\begin{aligned} p(z_0) + \frac{z_0 p'(z_0)}{\alpha} &= \left(ri - \frac{k\gamma(1+r^2)}{2\alpha}\right)(ri)^{\gamma-1} \\ &= \left(ri - \frac{k\gamma(1+r^2)}{2\alpha}\right) r^{\gamma-1}\left(\cos\frac{(\gamma-1)\pi}{2} + i\sin\frac{(\gamma-1)\pi}{2}\right). \end{aligned}$$

Since $\cos\dfrac{(\gamma-1)\pi}{2} = \sin\dfrac{\gamma\pi}{2}$, and $\sin\dfrac{(\gamma-1)\pi}{2} = -\cos\dfrac{\gamma\pi}{2}$, we have

(3.3)
$$\begin{aligned} p(z_0) + \frac{z_0 p'(z_0)}{\alpha} &= \Big(ri - \frac{k\gamma(1+r^2)}{2\alpha}\Big) r^{\gamma-1}\Big(\sin\frac{\gamma\pi}{2} - i\cos\frac{\gamma\pi}{2}\Big) \\ &= \frac{r^{\gamma-1}}{2}\cos\frac{\gamma\pi}{2}\Big[2r - \frac{k\gamma}{\alpha}(1+r^2)\tan\frac{\gamma\pi}{2} + i\Big(2r\tan\frac{\gamma\pi}{2} + \frac{k\gamma}{\alpha}(1+r^2)\Big)\Big], \end{aligned}$$

and so taking arguments in (3.3), we obtain

$$\begin{aligned} \arg\Big(p(z_0) + \frac{z_0 p'(z_0)}{\alpha}\Big) &= \arctan\Big[\Big(2r\tan\frac{\gamma\pi}{2} + \frac{k\gamma}{\alpha}(1+r^2)\Big)/\Big(2r - \frac{k\gamma}{\alpha}(1+r^2)\tan\frac{\gamma\pi}{2}\Big)\Big] \\ &\geq \arctan\Big[\Big(2r\tan\frac{\gamma\pi}{2} + \frac{\gamma}{\alpha}(1+r^2)\Big)/\Big(2r - \frac{\gamma}{\alpha}(1+r^2)\tan\frac{\gamma\pi}{2}\Big)\Big] \\ &:= \Phi(r). \end{aligned}$$

Now write $\Phi(r) = \arctan\Big(\dfrac{U(r)}{V(r)}\Big)$, so that $U(r) = 2r\tan\dfrac{\gamma\pi}{2} + \dfrac{\gamma}{\alpha}(1+r^2)$ and $V(r) = 2r - \dfrac{\gamma}{\alpha}(1+r^2)\tan\dfrac{\gamma\pi}{2}$.

Next note that since $\Phi'(r)(U(r)^2+V(r)^2) = V(r)U'(r)-U(r)V'(r)$, it follows that $V(r)U'(r)-U(r)V'(r) = 0$ when $r = 1$. Thus $\Phi(r)$ attains its minimum when $r = 1$, and so

$$\begin{aligned} \Phi(r) &\geq \Phi(1) \\ &= \arctan\Big[(\alpha\tan\frac{\gamma\pi}{2} + \gamma)/(\alpha - \gamma\tan\frac{\gamma\pi}{2})\Big] \\ &= \frac{\beta(\gamma)\pi}{2}. \end{aligned}$$

Hence

$$\frac{\beta(\gamma)\pi}{2} \leq \arg\Big(p(z_0) + \frac{z_0 p'(z_0)}{\alpha}\Big) \leq \frac{\pi}{2},$$

which contradicts the fact that $|h(z)| < \dfrac{\beta(\gamma)\pi}{2}$, provided (3.1) holds. This completes the proof of the positive statement in the theorem.

To show that (3.2) is the largest number such that (3.1) holds, let

$$p(z) = \Big(\frac{1+z}{1-z}\Big)^{\gamma}.$$

Then from the minimum modulus principle for harmonic functions, it follows that

$$\inf_{|z|<1} \arg\Big(p(z) + \frac{zp'(z)}{\alpha}\Big)$$

is attained at some point $z = e^{i\theta}$, for $0 < \theta < 2\pi$.

Thus with $z = e^{i\theta}$,

$$\Big(p(z)+\frac{zp'(z)}{\alpha}\Big)\Big/\Big(\frac{\sin\theta}{1-\cos\theta}\Big)^{\gamma-1}=\frac{\cos\dfrac{\gamma\pi}{2}}{1-\cos\theta}\Big[\sin\theta-\frac{\gamma}{\alpha}\arctan\frac{\gamma\pi}{2} + i\Big(\sin\theta\tan\frac{\gamma\pi}{2}+\frac{\gamma}{\alpha}\Big)\Big].$$

Taking arguments we have

$$\arg\Big(p(z)+\frac{zp'(z)}{\alpha}\Big)=\arctan\Big[(\alpha\sin\theta\tan\frac{\gamma\pi}{2}+\gamma)/(\alpha\sin\theta-\gamma\tan\frac{\gamma\pi}{2})\Big],$$

and an elementary calculation shows that the minimum value of the right-hand side is obtained when $\sin\theta=1$.

Thus

$$\arg\Big(p(z)+\frac{zp'(z)}{\alpha}\Big)\geq\arctan\Big[(\alpha\tan\frac{\gamma\pi}{2}+\gamma)/(\alpha-\gamma\tan\frac{\gamma\pi}{2})\Big] = \frac{\beta(\gamma)\pi}{2}.$$

Hence $\beta(\gamma)$ is exact and the proof of the theorem is complete.

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