A SUBORDINATION THEOREM FOR ANALYTIC FUNCTIONS
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Received 4 November, 2016; accepted 25 January, 2017; published 20 February, 2017.

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ABSTRACT. It is shown that if $f$ is analytic in $D = \{z : |z| < 1\}$, with $f(0) = f'(0) - 1 = 0$, then for $\alpha > 0$, $\gamma > 0$, $f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} \prec \left(\frac{1 + z}{1 - z}\right)^{\beta(\gamma)}$ implies $\left(\frac{f(z)}{z}\right)^{\alpha} \prec \left(\frac{1 + z}{1 - z}\right)^{\gamma}$, where $\beta(\gamma) = \gamma + \frac{2}{\pi} \arctan \left(\frac{\gamma}{\alpha}\right)$, and that $\beta(\gamma)$ is the largest number such that this implication holds.

Key words and phrases: Analytic function, Univalent function, Bazilević function, Subordination.

2010 Mathematics Subject Classification. Primary 30C45. Secondary 30C50.

ISSN (electronic): 1449-5910
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1. Introduction and Definitions

Let \( S \) be the class of analytic normalised univalent functions \( f \), defined in \( z \in D = \{ z : |z| < 1 \} \) and given by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

The Bazilevič functions with logarithmic growth \( B_1(\alpha) \subset S \) defined as follows have been extensively studied (see e.g. [2, 3, 4]).

Suppose that \( f \) is analytic in \( D \) and is given by (1.1). Then for \( \alpha \geq 0 \), \( f \in B_1(\alpha) \), if and only if,

\[
\text{Re} \ f'(z) \left( \frac{f(z)}{z} \right)^{\alpha-1} > 0.
\]

We say that an analytic function \( f \) is subordinate to an analytic function \( g \), and write \( f(z) \prec g(z) \), if and only if there exists a function \( \omega \), analytic in \( D \), such that \( \omega(0) = 0 \), \( |\omega(z)| < 1 \) for \( z \in D \), and \( f(z) = g(\omega(z)) \). Marjono and Thomas [5] discussed subordination in a sector.

We will use the following well-known lemma.

2. Lemma

The Miller-Mocanu Lemma [1]

Let \( F \) be analytic in \( D \) and \( G \) be analytic univalent in \( D \), with \( F(0) = G(0) \). If \( F \not\prec G \), then there is a point \( z_0 \in D \) and \( \zeta_0 \in \partial D \), such that \( F(|z| < |z_0|) \subset G(D) \), \( F(z_0) = G(\zeta_0) \) and \( z_0 F'(z_0) = k \zeta_0 G'(\zeta_0) \) for \( k \geq 1 \).

We prove the following theorem for analytic functions, noting its relationship with the Bazilevič functions \( B_1(\alpha) \) defined in (1.2).

3. Theorem

Let \( f \) be analytic in \( D \), with \( f(0) = f'(0) - 1 = 0 \). Then for \( \alpha > 0 \), \( \gamma > 0 \) and \( z \in D \),

\[
f'(z) \left( \frac{f(z)}{z} \right)^{\alpha-1} \prec \left( \frac{1 + z}{1 - z} \right)^{\beta(\gamma)}
\]

implies

\[
\left( \frac{f(z)}{z} \right)^{\alpha} \prec \left( \frac{1 + z}{1 - z} \right)^{\gamma},
\]

where
A Subordination Theorem for Analytic Functions

\[(3.2) \quad \beta(\gamma) = \gamma + \frac{2}{\pi} \arctan \left( \frac{\gamma}{\alpha} \right).\]

Furthermore, \(\beta(\gamma)\) is the largest number such that \((3.1)\) holds.

**Proof.** Write

\[p(z) = \left( \frac{f(z)}{z} \right)^{\alpha},\]

so that \(p\) is analytic in \(D\), \(p(0) = 1\) and

\[p(z) + \frac{zp'(z)}{\alpha} = \left( \frac{f(z)}{z} \right)^{\alpha-1} f'(z).\]

Thus we need to show that

\[p(z) + \frac{zp'(z)}{\alpha} \prec \left( \frac{1 + z}{1 - z} \right)^{\beta}\]

implies

\[p(z) \prec \left( \frac{1 + z}{1 - z} \right)^{\gamma},\]

when \(\beta = \beta(\gamma)\) is given by \((3.2)\).

For \(z \in D\), let \(h(z) = [(1+z)/(1-z)]^{\beta}\) and \(q(z) = [(1+z)/(1-z)]^{\gamma}\), so that \(|\arg h(z)| < \frac{\beta \pi}{2}\) and \(|\arg q(z)| < \frac{\gamma \pi}{2}\).

Suppose that \(p \not\prec q\). Then from the Miller-Mocanu Lemma, there exists \(z_0 \in D\) and \(\zeta_0 \in \partial D\), such that \(p(z_0) = q(\zeta_0)\), \(p(|z| < |z_0|) \subset q(D)\) and \(z_0 p'(z_0) = k \zeta_0 q'(\zeta_0)\), for \(k \geq 1\).

Since \(p(z_0) = q(\zeta_0) \neq 0\), it follows that \(\zeta_0 \neq \pm 1\). Thus we can write \(r i = \frac{1 + \zeta_0}{1 - \zeta_0}\) for \(r \neq 0\). Hence

\[p(z_0) + \frac{z_0 p'(z_0)}{\alpha} = q(\zeta_0) + \frac{k \zeta_0 q'(\zeta_0)}{\alpha}\]

for \(k \geq 1\).

Differentiating \(q(z)\) we obtain

\[p(z_0) + \frac{z_0 p'(z_0)}{\alpha} = \left( r i - \frac{k \gamma (1 + r^2)}{2 \alpha} \right) (r i)^{\gamma - 1}\]

\[= \left( r i - \frac{k \gamma (1 + r^2)}{2 \alpha} \right) r^{\gamma - 1} \left( \cos \left( \frac{(\gamma - 1) \pi}{2} \right) + i \sin \left( \frac{(\gamma - 1) \pi}{2} \right) \right).\]
Since $\cos \frac{(\gamma - 1)\pi}{2} = \sin \frac{\gamma \pi}{2}$, and $\sin \frac{(\gamma - 1)\pi}{2} = -\cos \frac{\gamma \pi}{2}$, we have

(3.3)

\[
p(z_0) + \frac{z_0 p'(z_0)}{\alpha} = \left( r + \frac{k\gamma(1 + r^2)}{2\alpha} \right) r^{\gamma - 1} \left( \sin \frac{\gamma \pi}{2} - i \cos \frac{\gamma \pi}{2} \right)
\]

\[
= \frac{r^{\gamma - 1}}{2} \cos \frac{\gamma \pi}{2} \left[ 2r - \frac{k\gamma}{\alpha}(1 + r^2) \tan \frac{\gamma \pi}{2} + i \left( 2r \tan \frac{\gamma \pi}{2} + \frac{k\gamma}{\alpha}(1 + r^2) \right) \right],
\]

and so taking arguments in (3.3), we obtain

\[
\arg \left( p(z_0) + \frac{z_0 p'(z_0)}{\alpha} \right) = \arctan \left[ \left( 2r \tan \frac{\gamma \pi}{2} + \frac{\gamma}{\alpha}(1 + r^2) \tan \frac{\gamma \pi}{2} \right) \right]
\]

\[
\geq \arctan \left[ \left( 2r \tan \frac{\gamma \pi}{2} + \frac{\gamma}{\alpha}(1 + r^2) \tan \frac{\gamma \pi}{2} \right) \right] := \Phi(r).
\]

Now write $\Phi(r) = \arctan \left( \frac{U(r)}{V(r)} \right)$, so that $U(r) = 2r \tan \frac{\gamma \pi}{2} + \frac{\gamma}{\alpha}(1 + r^2)$ and $V(r) = 2r - \frac{\gamma}{\alpha}(1 + r^2) \tan \frac{\gamma \pi}{2}$.

Next note that since $\Phi'(r) U(r)^2 + V(r)^2 = V(r) U'(r) - U(r) V'(r)$, it follows that $V(r) U'(r) - U(r) V'(r) = 0$ when $r = 1$. Thus $\Phi(r)$ attains its minimum when $r = 1$, and so

\[
\Phi(r) \geq \Phi(1)
\]

\[
= \arctan \left( \frac{\gamma}{\alpha}(1 + r^2) \tan \frac{\gamma \pi}{2} \right)
\]

\[
= \frac{\beta(\gamma) \pi}{2}.
\]

Hence

\[
\frac{\beta(\gamma) \pi}{2} \leq \arg \left( p(z_0) + \frac{z_0 p'(z_0)}{\alpha} \right) \leq \frac{\pi}{2},
\]

which contradicts the fact that $|h(z)| < \frac{\beta(\gamma) \pi}{2}$, provided (3.1) holds. This completes the proof of the positive statement in the theorem.

To show that (3.2) is the largest number such that (3.1) holds, let

\[
p(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma.
\]

Then from the minimum modulus principle for harmonic functions, it follows that

\[
\inf_{|z| < 1} \arg \left( p(z) + \frac{z p'(z)}{\alpha} \right)
\]

is attained at some point $z = e^{i\theta}$, for $0 < \theta < 2\pi$.

Thus with $z = e^{i\theta}$.
\[
\left( p(z) + \frac{zp'(z)}{\alpha} \right) / \left( \frac{\sin \theta}{1 - \cos \theta} \right)^{\gamma - 1} = \frac{\cos \frac{\gamma \pi}{2}}{1 - \cos \theta} \left[ \sin \theta - \gamma \arctan \frac{\gamma \pi}{2} \right] + i \left( \sin \theta \tan \frac{\gamma \pi}{2} + \frac{\gamma}{\alpha} \right).
\]

Taking arguments we have
\[
\arg \left( p(z) + \frac{zp'(z)}{\alpha} \right) = \arctan \left( (\alpha \sin \theta \tan \frac{\gamma \pi}{2} + \gamma) / (\alpha \sin \theta - \gamma \tan \frac{\gamma \pi}{2}) \right),
\]
and an elementary calculation shows that the minimum value of the right-hand side is obtained when \( \sin \theta = 1 \).

Thus
\[
\arg \left( p(z) + \frac{zp'(z)}{\alpha} \right) \geq \arctan \left( (\alpha \tan \frac{\gamma \pi}{2} + \gamma) / (\alpha - \gamma \tan \frac{\gamma \pi}{2}) \right)
\]
\[
= \frac{\beta(\gamma) \pi}{2}.
\]

Hence \( \beta(\gamma) \) is exact and the proof of the theorem is complete.

\section*{References}


