The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 14, Issue 1, Article 10, pp. 1-17, 2017

SOME NEW INEQUALITIES OF HERMITE-HADAMARD AND FEJÉR TYPE FOR CERTAIN FUNCTIONS WITH HIGHER CONVEXITY

STEVEN G. FROM

Received 4 June, 2016; accepted 31 January, 2017; published 11 April, 2017.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA AT OMAHA, OMAHA, NEBRASKA 68182-0243.
sfrom@unomaha.edu

ABSTRACT. In this paper, we present some new inequalities of Hermite-Hadamard or Fejér type for certain functions satisfying some higher convexity conditions on one or more derivatives. An open problem is given also. Some applications to the logarithmic mean are given.

Key words and phrases: Absolutely monotonic; Completely monotonic; Convex function; Fejér inequality; Hermite-Hadamard inequality; Higher convexity; Jensen's inequality; Logarithmic mean.

2000 *Mathematics Subject Classification.* Primary 25, 26, 28, 60. Secondary 25D, 26D..

ISSN (electronic): 1449-5910

© 2017 Austral Internet Publishing. All rights reserved.

1. INTRODUCTION

To discuss the new inequalities to be given later, we shall first need some lemmas.

Lemma 1.1. *Suppose that f'' is continuous on $[a,b]$. Let $H(x)$ be a bounded, continuous nondecreasing function on $[a,b]$ with $H(a) = 0$ and $H(b) = 1$. Suppose $0 < H(x) < 1$ on (a,b). Let*

$$g(x) = \begin{cases} \frac{\int_x^b (t-x)dH(t)}{1-H(x)} & a \le x < b \\ \lim_{x\to b^-} \quad g(x) = 0, & x = b\,. \end{cases} \tag{1.1}$$

Let

$$q_1(x) = \inf\{f''(t) : x \le t \le x + g(x)\}\,, \tag{1.2}$$

and

$$q_2(x) = \sup\{f''(t) : x \le t \le x + g(x)\}\,. \tag{1.3}$$

Let

$$L_1 = \frac{1}{2}\int_a^b q_1(x)(g(x))^2 dH(x) \tag{1.4}$$

and

$$U_1 = \frac{1}{2}\int_a^b q_2(x)(g(x))^2 dH(x)\,. \tag{1.5}$$

Then

$$L_1 \le \int_a^b f(x)dH(x) - f\left(\int_a^b x dH(x)\right) \le U_1\,. \tag{1.6}$$

For a proof of Lemma 1.1, see Theorem 3.1 of From [5].

Lemma 1.2. *Suppose that the assumptions of Lemma 1.1 above hold. Then*

(a) *If g'' is continuous on $[a,b]$, then $g'(x) \ge -1$, $a \le x \le b$. Thus, $x + g(x)$ is nondecreasing on $[a,b]$.*

(b) *Suppose $h(x)$ is a probability density function absolutely continuous with respect to Lebesgue measure with $H'(x) = h(x)$ on $[a,b]$. Let $r(x) = \frac{h(x)}{1-H(x)}$ be the hazard function. If $r(x)$ is nondecreasing in x, $a \le x \le b$, then $g(x) \le \mu$, $a \le x \le b$, where $\mu = \int_a^b x \cdot dH(x)$.*

The results of Lemma 1.2 are special cases of well-known results in applied probability and reliability theory, so we omit the proofs. See, for example, Swartz [11] for part (a) and Barlow and Proschan [1] for part (b).

Lemmas 1.3 and 1.4 below will also be needed. Lemma 1.3 is a special case of a more general result given in Gupta and Gupta [7].

Lemma 1.3. *Suppose $H'(x) = h(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$, $0 \le x \le 1$, $a > 0$, $b > 0$, $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$. If $a \ge 1$ and $b \ge 1$, then $r(x) = \frac{h(x)}{1-H(x)}$ is nondecreasing in x on $[0,1]$.*

Proof. See Gupta and Gupta [7], p. 7. ∎

Remark 1.1. If $f^{(3)}(x) \ge 0$ on $[a,b]$, then from Lemma 1.1, we obtain $q_1(x) = f''(x)$ and $q_2(x) = f''(x + g(x))$. This will be needed later.

Lemma 1.4. *Suppose f'' is continuous on $[a,b]$. Then we have the identity:*

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)dt=\frac{1}{2}\int_a^b (t-a)(b-t)f''(t)dt\,.$$

Proof. See Remark 6 of Dragomir [4], p. 15. ∎

2. NEW RESULTS

In this section, we present some new inequalities of Fejér-type which complement those given in Dragomir and Gomm [2]. We assume, without loss of generality, that the interval of integration is $[0,1]$. A simple linear transformation will easily extend all results given to the interval of integration $[a,b]$.

In Dragomir and Gomm [2], the following results are given.

Theorem 2.1. (Theorem 2.1 of Dragomir and Gomm [2].) *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and such that f'' is convex on (a,b). Then*

$$\begin{aligned}\frac{1}{12}f''\left(\frac{a+b}{2}\right)\cdot(b-a)^2 &\le \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\\ &\le \frac{f''(a)+f''(b)}{24}(b-a)^2\,.\end{aligned}$$

Theorem 2.2. (Theorem 2.2 of Dragomir and Gomm [2].) *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b). If there exists a real number m such that $f''(x)\ge m$ for any $x\in(a,b)$, then*

$$\begin{aligned}&\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{240}m(b-a)^5\\ &\le \int_a^b (b-x)(x-a)f(x)dx\\ &\le \frac{f(a)+f(b)}{12}(b-a)^3-\frac{1}{60}m(b-a)^5\,.\end{aligned}$$

If there exists a real number M such that $f''(x)\le M$ for any $x\in(a,b)$, then

$$\begin{aligned}&\frac{f(a)+f(b)}{12}(b-a)^3-\frac{1}{60}M(b-a)^5\le\int_a^b (b-x)(x-a)f(x)dx\\ &\le\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{240}M(b-a)^5\,.\end{aligned}$$

We shall also obtain bounds for $\int_a^b f(x)\cdot\frac{1}{b-a}dx$ and $\int_a^b f(x)\cdot(b-x)(x-a)dx$ under assumptions on f and one or more of its derivatives which are different from those given in Theorem 2.1 or 2.2 above. For other related works, see Dragomir and Gomm [3], and Minculete and Corina-Mitrui [9].

Theorem 2.3. *Let $f^{(3)}$ be continuous. Suppose $f''(x)\ge 0$ and $f^{(3)}(x)\ge 0$ on $[0,1]$. Then*

$$\int_0^1 f(x)\cdot x(1-x)dx\le\frac{1}{6}f\left(\frac{1}{2}\right)+\frac{1}{80}\left(f'(1)-f'\left(\frac{1}{2}\right)\right)\,. \tag{2.1}$$

Proof. We apply Lemma 1.1 with $H(x) = 3x^2 - 2x^3$, $0 \le x \le 1$. Then by the 3-convexity of f,

$$\begin{aligned}\int_0^1 f(x)\cdot x(1-x)dx &= \frac{1}{6}\int_0^1 f(x)\cdot dH(x)\\ &\le \frac{1}{6}\left(f\left(\frac{1}{2}\right)+\frac{1}{2}\int_0^1 f''(x+g(x))\cdot(g(x))^2h(x)dx\right) \quad (2.2)\end{aligned}$$

where $h(x) = 6x(1-x)$, $0 \le x \le 1$. Now, $g''(x) = \frac{3}{(2x+1)^3} > 0$, $0 \le x \le 1$. So

$$\begin{aligned}&\int_0^1 f''(x+g(x))\cdot(g(x))^2h(x)dx\\ &= \int_0^1 f''(x+g(x))(1+g'(x))\cdot\left[\frac{g^2(x)h(x)}{1+g'(x)}\right]dx\,. \quad (2.3)\end{aligned}$$

By Lemma 1.2, part (a), $f''(x+g(x))\cdot(1+g'(x))$ is nondecreasing in x, since $g''(x) > 0$, $f'' \ge 0$, and $f^{(3)} \ge 0$. Also, $\frac{g^2(x)h(x)}{1+g'(x)} = \frac{(1+x)(1-x)^3}{2}$ is nonincreasing in x. By the Chebychev-Gruss inequality,

$$\begin{aligned}&\int_0^1 f''(x+g(x))\cdot(1+g'(x))\cdot\left[\frac{g^2(x)h(x)}{1+g'(x)}\right]dx\\ &\le \int_0^1 f''(x+g(x))\cdot(1+g'(x))dx\cdot\int_0^1\frac{g^2(x)h(x)}{1+g'(x)}dx\\ &= \left(f'(1)-f'\left(\frac{1}{2}\right)\right))\cdot\left(\frac{3}{20}\right)\,.\end{aligned}$$

From (2.2)–(2.3), we obtain

$$\int_0^1 f(x)\cdot x(1-x)dx \le \frac{1}{6}f\left(\frac{1}{2}\right)+\frac{1}{80}\left(f'(1)-f'\left(\frac{1}{2}\right)\right)$$

as desired, and the proof of (2.1) in Theorem 2.3 is complete. ∎

In the next theorem, we obtain a lower bound as well as an upper bound for $\int_0^1 f(x)\cdot x(1-x)dx$ without the convexity of f restriction.

Theorem 2.4. *Suppose $f^{(3)}(x)$ is continuous and $f^{(3)}(x) \ge 0$ on $[0,1]$. Let $h(x) = 6x(1-x)$, $0 \le x \le 1$,*

$$\theta = \left(152+6\sqrt{642}\right)^{1/3}, \quad \textit{and} \quad c = \frac{\theta}{12}-\frac{1}{6\theta}-\frac{1}{3} \approx 0.2022258\,. \quad (2.4)$$

Let $p(x) = (g(x))^2h(x) = \frac{3}{2}\frac{(1-x)^3(1+x)^2x}{(1+2x)^2}$, let

$$R_1 = \int_0^c p(x)dx \approx 0.016312905\,, \quad (2.5)$$

and let

$$R_2 = \int_c^1 p(x)dx \approx 0.03368710\,. \quad (2.6)$$

Then

$$\int_0^1 f(x)\cdot x(1-x)dx \geq \frac{1}{6}f\left(\frac{1}{2}\right)+\frac{1}{12}\left(\frac{R_1(f'(c)-f'(0))}{c}+R_2f''(c)\right) \tag{2.7}$$

and

$$\int_0^1 f(x)\cdot x(1-x)dx \leq \frac{1}{6}f\left(\frac{1}{2}\right)+\frac{1}{12}\left(R_1\cdot f''(c+g(c))+R_2f''(1)\right)\,, \tag{2.8}$$

where $c+g(c)\approx 0.54367748$.

Proof. By Lemma 1.1,

$$\int_0^1 f(x)\cdot x(1-x)dx = \frac{1}{6}\int_0^1 f(x)\cdot 6x(1-x)dx$$

$$\geq \frac{1}{6}f\left(\frac{1}{2}\right)+\frac{1}{2}\int_0^1 f''(x)g^2(x)h(x)dx\,. \tag{2.9}$$

Now

$$\int_0^1 f''(x)g^2(x)h(x)dx = \int_0^1 f''(x)p(x)dx\,.$$

Simple calculations show $p(x)$ is nondecreasing on $[0,c]$ and nonincreasing on $[c,1]$. Since $f^{(3)}(x)\geq 0$, f'' is nondecreasing on $[0,c]$ and

$$\int_0^1 f''(x)p(x)dx = \int_0^c f''(x)p(x)dx+\int_c^1 f''(x)p(x)dx\,,$$

applying the Chebychev-Gruss inequality on the first integral,

$$\geq \frac{1}{c}\int_0^c f''(x)dx\cdot\int_0^c p(x)dx+\int_c^1 f''(x)p(x)dx$$

$$\geq \frac{1}{c}\int_0^c f''(x)dx\cdot\int_0^c p(x)dx+f''(c)\int_c^1 p(x)dx\,, \tag{2.10}$$

using $f''(x)\geq f''(c)$ on $[c,1]$, by 3-convexity of f,

$$= \frac{R_1(f'(c)-f'(0))}{c}+f''(c)\cdot R_2\,.$$

By (2.9)-(2.10), we obtain

$$\int_0^1 f(x)\cdot x(1-x)dx \geq \frac{1}{6}f\left(\frac{1}{2}\right)+\frac{1}{12}\left(\frac{R_1(f'(c)-f'(0))}{c}+R_2f''(c)\right)\,,$$

as desired. This proves (2.7). The proof of (2.8) is very similar, except we start with, using Lemma 1.1 again,

$$\int_0^1 f(x)\cdot x(1-x)dx \leq \frac{1}{6}\left(f\left(\frac{1}{2}\right)+\frac{1}{2}\int_0^1 f''(x+g(x))\cdot g^2(x)h(x)dx\right)$$

and use the Chebychev-Gruss inequality on $\int_c^1 f''(x+g(x))\cdot g^2(x)h(x)dx$ instead. We omit the details here. ∎

Next, we obtain bounds for $\int_a^b f(t)dt$ which utilizes $\frac{f(a)+f(b)}{2}$. Heretofore, $a=0$ and $b=1$, but we shall present the result for general $[a,b]$ in this case. Another higher convexity condition is assumed.

Theorem 2.5. *Suppose* $f^{(5)}(t) \geq 0$ *on* $[a,b]$ *and is continuous on* $[a,b]$. *Then*

$$\left(\frac{f(a)+f(b)}{2}\right)\cdot(b-a)-\int_a^b f(t)dt \equiv I$$

$$\leq \frac{(b-a)^3}{12}f''\left(\frac{a+b}{2}\right)+\frac{(b-a)^5}{24}\left[f^{(4)}((b-a)\cdot(c+g(c))+a)\cdot R_1+f^{(4)}(b)\cdot R_2\right] \tag{2.11}$$

and

$$\left(\frac{f(a)+f(b)}{2}\right)\cdot(b-a)-\int_a^b f(t)dt \equiv I$$

$$\geq \frac{(b-a)^3}{12}f''\left(\frac{a+b}{2}\right)+\frac{(b-a)^4}{24}(f^{(3)}((b-a)c+a))-f^{(3)}(a))\cdot\frac{R_1}{c}$$

$$+\frac{(b-a)^5}{24}f^{(4)}((b-a)c+a)\cdot R_2\,, \tag{2.12}$$

where $c, g(c)$, R_1 *and* R_2 *were given in Theorem 2.4.*

Proof. By Lemma 1.4, we obtain

$$\left(\frac{f(a)+f(b)}{2}\right)(b-a)-\int_a^b f(t)dt=\frac{1}{2}\int_a^b(t-a)(b-t)f''(t)dt$$

$$=\frac{1}{2}\int_0^1(b-a)^3u(1-u)\cdot f''((b-a)u+a)du$$

$$=\frac{(b-a)^3}{2}\int_0^1 f''((b-a)u+a)\cdot u(1-u)du$$

$$=\frac{(b-a)^3}{2}\int_0^1 f^*(u)\cdot u(1-u)du, \text{ where } f^*(u)=f''((b-a)u+a),\ 0\leq u\leq 1\,. \tag{2.13}$$

Since $f^{(5)}(t)\geq 0$ on $[a,b]$, $(f^*)^{(3)}(u)\geq 0$ on $[0,1]$. Applying Thoerem 2.4 to the integral in (2.13), we obtain

$$\int_0^1 f^*(u)\cdot u(1-u)du$$

$$\leq \frac{1}{6}f^*\left(\frac{1}{2}\right)+\frac{1}{12}(f^*)''(c+g(c))\cdot R_1+\frac{1}{12}(f^*)''(1)\cdot R_2\,.$$

Writing the derivatives of f^* in terms of those of f, we obtain (2.11). The proof of (2.12) is very similar and is omitted. ∎

Next, we give an upper bound on $\int_0^1 f(x)\cdot h(x)dx$ for certain $h(x)$ with $f(x)=\frac{h(x)}{1-H(x)}$ nondecreasing in x on $[0,1]$. By Lemma 1.3, this includes the special cases $h(x)\equiv 1$ and $h(x)=6x(1-x)$.

Theorem 2.6. *Suppose* $f^{(3)}(x)$ *is continuous on* $[0,1]$, $f''(x)\geq 0$ *and* $f^{(3)}(x)\geq 0$ *on* $[0,1]$. *Suppose* $r(x)$ *is nondecreasing in* x *on* $[0,1)$. *Let* $\mu=\int_0^1 x\cdot h(x)dx$. *Then*

$$\int_0^1 f(x)\cdot h(x)dx\leq f(\mu)+\frac{1}{2}f''(1)\cdot\mu^2\,. \tag{2.14}$$

Proof. Apply Lemma 1.1 with $h(x) = 6x(1-x)$. Since $f'' \geq 0$ and $f^{(3)} \geq 0$ on $[0,1]$, we obtain

$$\int_0^1 f(x)h(x)dx \leq f(\mu) + \frac{1}{2}\int_0^1 f''(x+g(x)) \cdot g^2(x)h(x)dx\,.$$

By Lemma 1.2 part (b), $g(x) \leq \mu$, so

$$\leq f(\mu) + \frac{1}{2}\int_0^1 f''(1) \cdot \mu^2 h(x)dx = f(\mu) + \frac{1}{2}f''(1)\mu^2\,.$$

This completes the proof of (2.14) in Theorem 2.6. ∎

Corollary 2.7. *Under the conditions of Theorem 2.6, if $a \geq 1$ and $b \geq 1$, then*

$$\int_0^1 f(x) \cdot x^{a-1}(1-x)^{b-1}dx$$

$$\leq B(a,b) \cdot \left(f\left(\frac{a}{a+b}\right) + \frac{1}{2}f''(1) \cdot \left(\frac{a}{a+b}\right)^2 \right). \tag{2.15}$$

If $a = b = 2$, then

$$\int_0^1 f(x) \cdot x(1-x)dx \leq \frac{1}{6}f\left(\frac{1}{2}\right) + \frac{1}{48}f''(1)\,. \tag{2.16}$$

If $a = b = 1$, then

$$\int_0^1 f(x)dx \leq f\left(\frac{1}{2}\right) + \frac{1}{8}f''(1)\,. \tag{2.17}$$

Proof. The results is immediate from Theorem 2.6 and Lemma 1.3, since $\mu = \frac{a}{a+b}$. ∎

Next, we consider new inequalities of Hermite-Hadamard type for functions having certain orders of higher convexity. For these functions, one or more derivatives of $f(x)$ do not change sign on $[a,b]$, the interval of integration. These include the very important absolutely monotonic and completely montonic classes of functions as given in Widder [13].

Lemma 2.8. *Define functions $\{h_m(x)\}_{m=1}^{\infty}$, $\{g_m(x)\}_{m=1}^{\infty}$, $\{p_m(x)\}_{m=1}^{\infty}$, and $\{w_m(x)\}_{m=1}^{\infty}$ on $[0,1]$ recursively, as follows: let $h_1(x) \equiv 1$, $p_1(x) \equiv 1$, $g_1(x) = \frac{\int_x^1 (t-x)h_1(t)dt}{\int_x^1 h_1(t)dt} = \frac{1-x}{2}$, $w_1(x) = x + g_1(x) = \frac{1+x}{2}$. For $m \geq 2$, determine, in the order given:*

$$p_m(x) = (g_{m-1}(x))^2 \cdot h_{m-1}(x)\,, \tag{2.18}$$

$$h_m(x) = \frac{p_m(x)}{\int_0^1 p_m(t)dt}\,, \tag{2.19}$$

$$g_m(x) = \frac{\int_x^1 (t-x)h_m(t)dt}{\int_x^1 h_m(t)dt}, \quad 0 \leq x < 1, \quad g_m(1) = 0\,. \tag{2.20}$$

$$w_m(x) = w_{m-1}(x + g_m(x))\,. \tag{2.21}$$

Then

(a) $g_m(x) = \frac{1}{2m}(1-x), \quad m \geq 1.$
(b) $h_m(x) = (2m-1) \cdot (1-x)^{2m-2}, \quad m \geq 1.$
(c) $p_m(x) = c_m \cdot h_m(x)$, $m \geq 1$, *where* $c_m = \frac{2m-3}{4(m-1)^2(2m-1)}$, $m \geq 2$, *and* $c_1 = 1$.

(d) $w_m(x) = a_m x + b_m$, *where* $a_1 = \frac{1}{2}$, $b_1 = \frac{1}{2}$, $a_{m+1} = \left(\frac{2m+1}{2m+2}\right) a_m$, $b_{m+1} = 1 - a_{m+1}$, $m \geq 1$. *Thus,* $w_m(0) = b_m$, $w_m(1) = 1$, $m \geq 1$.

Proof. To prove (a), note that if $h^*(t) \equiv (c+1) \cdot (1-t)^c$ for some $c \geq 0$, $0 \leq t \leq 1$, then

$$g^*(x) \equiv \frac{\int_x^1 (t-x) h^*(t) dt}{\int_x^1 h^*(t) dt} = \frac{1}{c+2}(1-x), \quad 0 \leq x \leq 1 .$$

This clearly holds for $h^*(t) = h_1(t)$ with $c = 0$ and $g^*(x) = g_1(x) = \frac{1-x}{2}$. Similarly, simple calculations shows this is true for $h^*(t) = h_m(t)$ with $c = 2m-2$, $m = 2, 3, \ldots$, by letting $g^*(x)$ be replaced by $g_m(x)$, $m \geq 1$ and using a simple induction argument. Thus, $g_m(x)$ is proportional to $1-x$ for all $m \geq 1$. This proves (a), since $\frac{1}{c+2} = \frac{1}{2m}$ for $c = 2m-2$.

To prove (b), note that $p_m(x) = (g_{m-1}(x))^2 h_{m-1}(x) = \left(\frac{1}{2(m-1)}\right)^2 (1-x)^2 h_{m-1}(x)$, $m \geq 2$. For $m = 2$, $h_{m-1}(x) = 1$, so $p_m(x)$ is proportional to $(1-x)^2$. For $m = 3$, $p_m(x)$ is proportional to $(1-x)^2 \cdot (1-x)^2 = (1-x)^4$. By induction on m, $p_m(x)$ is proportional to $(1-x)^{2m-2}$. Since $h_m(x)$ is just a normalized version of $p_m(x)$, the condition $\int_0^1 h_m(x) dx = 1$ gives $h_m(x) = (2m-1)(1-x)^{2m-2}$, $m \geq 1$. This proves (b).

To prove (c), from (2.18) and part (b), we obtain

$$\begin{aligned} p_m(x) &= (g_{m-1}(x))^2 h_{m-1}(x) \\ &= \left(\frac{1}{(2m-1)}(1-x)\right)^2 \cdot ((2m-3)(1-x)^{2m-4} \\ &= \frac{(2m-3)}{4(m-1)^2} \cdot (1-x)^{2m-2} . \end{aligned}$$

Since part (b) gives $h_m(x) = (2m-1)(1-x)^{2m-2}$, we obtain

$$\frac{p_m(x)}{h_m(x)} = c_m = \frac{2m-3}{4(m-1)^2(2m-1)}, \quad m \geq 2 .$$

The result follows and the proof of part (c) is complete.

To prove (d), the result is clearly true for $m = 1$, since $w_1(x) = \frac{1+x}{2} = a_1 x + b_1$. Now

$$\begin{aligned} w_m(x) &= w_{m-1}(x + g_m(x)),\ m \geq 2 \\ &= w_{m-1}\left(\frac{(2m-1)x+1}{2m}\right) . \end{aligned} \tag{2.22}$$

Since $w_1(x) = \frac{1+x}{2}$ is linear in x, simple induction shows that $w_m(x)$ is linear in x, $m \geq 2$. Thus, there exist constants a_m, b_m such that $w_m(x) = a_m x + b_m$, $m \geq 1$. Now (2.22) requires

$$a_m x + b_m = a_{m-1}\left(\frac{(2m-1)x+1}{2m}\right) + b_{m-1}$$

for all x in $[0, 1]$. This requires

$$a_m = \left(\frac{2m-1}{2m}\right) a_{m-1}, \quad b_m = \frac{a_{m-1}}{2m} + b_{m-1} . \tag{2.23}$$

Since $a_1 + b_1 = 1$, the use of (2.23) and induction on m shows that $a_m + b_m = 1$, $m \geq 2$. Thus, $b_m = 1 - a_m$, $m \geq 1$. Replacing m by $m+1$ in (2.23) completes the proof of part (d). ∎

Now we are ready to present some results of Hermite-Hadamard type for functions $f(x)$ possessing certain types of higher order convexity. We do not require f to be convex, however.

Theorem 2.9. *Let $f(x)$ be a real-valued function on $[0,1]$. Let $f^{(j)}(x)$ denote the j^{th} derivative of $f(x)$, $j=1,2,3,\ldots$ Let $\{a_m\}_{m=1}^{\infty}$, $\{b_m\}_{m=1}^{\infty}$ and $\{c_m\}_{m=1}^{\infty}$ be the sequences given in parts (c) and (d) of Lemma 2.8. Let $d_1 = c_2$,*

$$d_m = c_2 c_3 \cdots c_{m+1} = \prod_{L=2}^{m+1} c_L, \quad m = 2,3,\ldots .$$

Let

$$\begin{aligned} A_k &= \sum_{j=1}^{k-1} \left(\frac{1}{2}\right)^j d_j f^{(2j)}(b_{j+1}), \\ B_k &= \frac{d_k}{a_k}\left(\frac{1}{2}\right)^k (f^{(2k-1)}(1) - f^{(2k-1)}(b_k)), \quad k = 1,2,3,\ldots . \end{aligned}$$

If $f^{(3)}(x) \geq 0$, $f^{(5)}(x) \geq 0, \ldots, f^{(2k+1)}(x) \geq 0$, and $f^{(2k+1)}$ is continuous on $[0,1]$, for some integer $k \geq 1$, then

(a) $\int_0^1 f(x)dx \leq f\left(\frac{1}{2}\right) + A_k + B_k$, and

(b) $\int_0^1 f(x)dx \geq f\left(\frac{1}{2}\right) + \sum_{j=1}^{k-1}\left(\frac{1}{2}\right)^j d_j f^{(2j)}(\mu_{j+1}) + \left(\frac{1}{2}\right)^k f^{(2k)}(0)\cdot d_k$, where $\mu_j = g_j(0) = \frac{1}{2j}$, $j \geq 1$, and where any impossible sum above is defined to be zero.

Before proving Theorem 2.9 above, let's write out the upper bound in part (a) and the lower bound in part (b) for the first several values of k.

For $k=1$, we obtain $d_1 = \frac{1}{12}$, and

$$\int_0^1 f(x)dx \leq f\left(\frac{1}{2}\right) + \frac{1}{12}\left(f'(1) - f'\left(\frac{1}{2}\right)\right), \tag{2.24}$$

which is just Theorem 4.2 in From [5]. Also,

$$\int_0^1 f(x)dx \geq f\left(\frac{1}{2}\right) + \frac{1}{24} f''(0). \tag{2.25}$$

For $k=2$, $c_3 = \frac{3}{80}$, $d_2 = \frac{1}{320}$ and

$$\int_0^1 f(x)dx \leq f\left(\frac{1}{2}\right) + \frac{1}{24} f''\left(\frac{5}{8}\right) + \frac{1}{480}\left(f^{(3)}(1) - f^{(3)}\left(\frac{5}{8}\right)\right) \tag{2.26}$$

and

$$\int_0^1 f(x)dx \geq f\left(\frac{1}{2}\right) + \frac{1}{24} f''\left(\frac{1}{4}\right) + \frac{1}{1280}(f^{(4)}(0)). \tag{2.27}$$

For $k=3$, $c_4 = \frac{5}{252}$, $d_3 = \frac{1}{16128}$ and

$$\begin{aligned} \int_0^1 f(x)dx \;&\leq\; f\left(\frac{1}{2}\right) + \frac{1}{24} f''\left(\frac{5}{8}\right) + \frac{1}{1280} f^{(4)}\left(\frac{11}{16}\right) \\ &\quad + \frac{1}{40320}\left(f^{(5)}(1) - f^{(5)}\left(\frac{11}{16}\right)\right). \end{aligned} \tag{2.28}$$

Also,

$$\int_0^1 f(x)dx \geq f\left(\frac{1}{2}\right) + \frac{1}{24}f''\left(\frac{1}{4}\right) + \frac{1}{1280}f^{(4)}\left(\frac{1}{6}\right) + \frac{1}{129024}f^{(6)}(0)\,. \tag{2.29}$$

In inequalities (2.24)–(2.29) above, larger values of k usually improve the bounds, but this is not always the case.

Proof. Refer to Lemma 2.8 for definitions of $h_m(x)$, $g_m(x)$, $p_m(x)$ and $w_m(x)$ used in the proof below. Then Lemma 1.1 gives, if $f^{(3)} \geq 0$,

$$\int_0^1 f(x)dx \leq f\left(\frac{1}{2}\right) + \frac{1}{2}\int_0^1 f''(w_1(x)) \cdot p_2(x)dx \tag{2.30}$$

$$= f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\int_0^1 f''(a_1x + b_1) \cdot h_2(x)dx\,. \tag{2.31}$$

Now f'' is nondecreasing on $[0,1]$ and $h_2(x) = 3(1-x)^2$ is nonincreasing on $[0,1]$. Applying the Chebychev-Gruss inequality,

$$\begin{aligned} &\leq f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\int_0^1 f''(a_1x+b_1)dx \cdot \int_0^1 h_2(x)dx \\ &= f\left(\frac{1}{2}\right) + \frac{1}{12}\left(f'(1) - f'\left(\frac{1}{2}\right)\right)\,, \end{aligned}$$

which is the result of part (a) for $k=1$, since $a_1 = 1/2$ and $c_2 = 1/12$. Note that $\int_0^1 h_m(x)dx = 1$, $m = 1,2,3,\ldots$. Now, if $k=2$, (2.31) is valid also, since $f^{(3)} \geq 0$ and $f^{(5)} \geq 0$. Thus, from Lemma 1.1, we obtain

$$\int_0^1 f(x)dx \leq f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\int_0^1 f''(w_1(x))h_2(x)dx\,. \tag{2.32}$$

A key observation that is repeatedly used in the rest of the proof is that the second integral (on the right side of (2.32)) has the same form as the integral $\int_a^b f(x)dH(x)$ in Lemma 1.1, except $f''(w_1(x))$ replaces $f(x)$ and $\int_0^x h_2(t)dt$ replaces $H(x)$ in Lemma 1.1. So we may get an upper bound on it as well. Upon doing this, we will obtain another integral bound which can be bounded by Lemma 1.1, etc. We can continue this iterative bounding procedure indefinitely. However, we shall use the Chebychev-Gruss inequality to 'terminate' this procedure for each value of k. We obtain, using various parts of Lemma 2.8

$$\leq f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\left(f''(w_1(\mu_2)) + \frac{1}{2}\int_0^1 f^{(4)}(w_2(x)) \cdot p_3(x)dx\right)\,.$$

From the Chebychev-Gruss inequality, and since $w_1(\mu_2) = b_2$,

$$\begin{aligned}
&= \leq f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\left(f''(b_2) + \frac{1}{2}\int_0^1 f^{(4)}(a_2x+b_2)\cdot c_3 \cdot \int_0^1 h_3(x)dx\right) \\
&= f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\left(f''(b_2) + \frac{1}{2}c_3 \cdot \frac{1}{a_2}\cdot (f^{(3)}(1) - f^{(3)}(b_2)\right) \\
&= f\left(\frac{1}{2}\right) + \frac{1}{2}c_2 f''(b_2) + \left(\frac{1}{2}\right)^2 (c_2c_3)\cdot \frac{1}{a_2}(f^{(3)}(1) - f^{(3)}(b_2)) \\
&= f\left(\frac{1}{2}\right) + \frac{1}{2}d_1 f''(b_2) + \left(\frac{1}{2}\right)^2 (d_2)\left(\frac{1}{a_2}\right)\cdot (f^{(3)}(1) - f^{(3)}(b_2)), \qquad (2.33)
\end{aligned}$$

which is part (a) of Theorem 2.9 for $k = 2$. Similarly, for $k = 3$, if $f^{(3)} \geq 0$, $f^{(5)} \geq 0$ and $f^{(7)} \geq 0$, we obtain

$$\begin{aligned}
\int_0^1 f(x)dx &\leq f\left(\frac{1}{2}\right) + \frac{1}{2}c_2(f''(b_2)) + \left(\frac{1}{2}c_2\right)\left(\frac{1}{2}c_3\right)f^{(4)}(b_3) \\
&\quad + \left(\frac{1}{2}c_2\right)\left(\frac{1}{2}c_3\right)\left(\frac{1}{2}c_4\right)\cdot\frac{1}{a_3}(f^{(5)}(w_3(1)) - f^{(5)}(b_3)), \\
&= f\left(\frac{1}{2}\right) + \frac{1}{2}d_1 f''(b_2) + \left(\frac{1}{2}\right)^2 d_2 \cdot f^{(4)}(b_3) \\
&\quad + \frac{d_3}{a_3}\left(\frac{1}{2}\right)^3 (f^{(5)}(1) - f^{(5)}(b_3)),
\end{aligned}$$

which is part (a) of Theorem 2.9 for $k = 3$. A simple induction argument completes the proof of part (a) of Theorem 2.9.

The proof of part (b) is very similar, so we merely indicate the parts of the proof of (b) that are different. Since we want a lower bound for $\int_0^1 f(x)dx$ instead, x replaces $w_m(x)$ in the derivative $f^{(2k}(\cdot)$ at each stage. For example, if $k = 2$, we obtain, from Lemma 1.1

$$\begin{aligned}
\int_0^1 f(x)dx &\geq f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\int_0^1 f''(x)h_2(x)dx \\
&\geq f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\left(f''(\mu_2) + \frac{c_3}{2}\int_0^1 f^{(4)}(x)h_3(x)dx\right) \\
&\geq f\left(\frac{1}{2}\right) + \frac{1}{2}c_2\left(f''(\mu_2) + \frac{c_3}{2}f^{(4)}(0)\right),
\end{aligned}$$

since $f^{(5)} \geq 0$.

$$= f\left(\frac{1}{2}\right) + \frac{1}{2}d_1(f''(\mu_2)) + \left(\frac{1}{2}\right)^2 d_2 f^{(4)}(0),$$

which is part (b) of Theorem 2.9. ∎

Corollary 2.10. *Suppose that for some integer $k \geq 1$, $f^{(3)}(t) \geq 0$, $f^{(5)}(t) \geq 0, \ldots, f^{(2k+1)}(t) \geq 0$, $a \leq t \leq b$ and $f^{(2k+1)}(t)$ is continuous in $[a,b]$. Let $\{a_m\}$, $\{b_m\}$, $\{c_m\}$ and $\{\mu_j\}$ be the* *sequences given Theorem 2.9. Let*

$$A_k = \sum_{j=1}^{k-1} \left(\frac{1}{2}\right)^j d_j f^{(2j)}(b_{j+1}) \cdot (b-a)^{2j+1} ,$$

$$B_k = \frac{d_k}{a_k} \left(\frac{1}{2}\right)^k ((b-a)^{2k}) \cdot (f^{(2k-1)}(b) - f^{(2k-1)}(a+(b-a)b_k)) .$$

Then

(a)

$$\int_a^b f(t)dt \le (b-a)f\left(\frac{a+b}{2}\right) + A_k + B_k$$

and

(b)

$$\int_a^b f(t)dt \ge (b-a)f\left(\frac{a+b}{2}\right) + \sum_{j=1}^{k-1} \left(\frac{1}{2}\right)^j d_j f^{(2j)}(\mu_{j+1}) \cdot (b-a)^{2j+1}$$

$$+ \left(\frac{1}{2}\right)^k f^{(2k)}(a) \cdot d_k \cdot (b-a)^{2k+1} .$$

Proof. Apply Theorem 2.9 to the function $f^*(x) = f(a+(b-a)x)$, $0 \le x \le 1$. Use $\int_a^b f(t)dt = (b-a)\int_0^1 f^*(x)dx$ and the fact that

$$(f^*)^{(j)}(x) = (b-a)^j \cdot f^{(j)}(a+(b-a)x), \quad 0 \le x \le 1, \quad j = 1, 2, 3, \dots$$

This result is immediate. ∎

Remark 2.1. Bounds for $\int_a^b f(x)dx$ can be obtained for functions which are either absolutely monotonic on completely monotonic on $[a, b]$ as defined by Widder [13] using theorems above since the higher convexity conditions requiring that the derivatives of odd order to be of one sign only are met by functions of the absolutely monotonic or completely monotonic classes. For completely monotonic functions, we must reverse the bounds, since Theorem 2.9 holds for $-f(x)$, not $f(x)$, in this case.

Remark 2.2. If $f^{(2j+1)}(x) \ge 0$, $0 \le x \le 1$, and for all $j \ge 1$ in Theorem 2.9, then the bounds of Theorem 2.9 hold for all k. In most cases as mentioned earlier, both the upper and lower bounds given there improve as k increases, but this is not always the case. It is an open problem to give sufficient conditions for which these bounds are guaranteed to improve as k increases.

Finally, we consider inequalities of integrals infinite intervals. Note that Lemma 1.1 holds for $b = \infty$ as well, provided all integrals given in the bottom in the lemma exist.

Remark 2.3. Alternative bounds can be given for $\int_a^b f(t)dt$ if we replace $f(t)$ by $f(a+b-t)$. Then for each k, the inequality signs will be reversed in Theorem 2.9 and Corollary 2.10, since if all odd order derivatives of order 3 or higher of $f(t)$ are nonnegative, then all odd order derivatives of order 3 or higher of $f(a+b-t)$ are nonpositive. Then, Theorem 2.9 and Corollary 2.10 would then be applicable to the function $-f(a+b-x)$. It is unclear which choice of the integrand function should be used, in advance.

Theorem 2.11. *Suppose* $f^{(2j+1)}(x) \ge 0$ *on* $[0, \infty)$, $j = 1, 2, 3, \dots$ *and all derivatives are continuous on* $[0, \infty)$. *Suppose* $I = \int_0^\infty f(x)e^{-x}dx$ *exists. Then*

(a) *If*

$$\lim_{n\to\infty}\left(\int_0^\infty |f^{(2n+2)}(x+n+1)|\cdot e^{-x}dx\right)=0\,, \tag{2.34}$$

and

$$\limsup_{n\to\infty}|f^{(2n)}(n+1)|^{1/n}<2\,, \tag{2.35}$$

then

$$\begin{aligned} I &\le f(1)+\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j\cdot f^{(2j)}(j+1)\\ &= f(1)+\frac{1}{2}f''(2)+\frac{1}{4}f^{(4)}(3)+\frac{1}{8}f^{(6)}(4)+\cdots \end{aligned}$$

If

$$\lim_{n\to\infty}\left(\int_0^\infty |f^{(2n+2)}(x)|\cdot e^{-x}dx\right)=0 \tag{2.36}$$

and

$$\limsup_{n\to\infty}|f^{(2n}(1)|^{1/n}<2\,, \tag{2.37}$$

then

(b)

$$I\ge f(1)+\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j\cdot f^{(2j)}(1)\,.$$

Thus,

$$I\ge f(1)+\sum_{j=1}^{n}\left(\frac{1}{2}\right)^j f^{(2j)}(1)$$

holds for all $n\ge 1$*, if* $f^{(2j)}(x)\ge 0$*,* $j=1,2,3,\ldots$ *hold also.*

Proof. The ideas needed are very similar to those given in the proofs of Lemma 2.8 and Theorem 2.9. Let $h_1(x)=e^{-x}$, $p_1(x)=e^{-x}$. Then $g_1(x)\equiv 1$. The function $h(x)=e^{-x}$ is a fixed point of the operator $\mathcal{L}:h\to g^2h$. Thus, by induction on m, $\mathcal{L}(h_m(x))=(g_{m-1}(x))^2h_{m-1}(x)=e^{-x}$, $m=2,3,\ldots$. Proceeding as in the proof of Theorem 2.9, we obtain

$$\begin{aligned} I &\le f(1)+\frac{1}{2}\int_0^\infty f''(x+1)\cdot e^{-x}dx\\ &\le f(1)+\frac{1}{2}\left[f''(2)+\frac{1}{2}\int_0^\infty f^{(4)}(x+2)e^{-x}dx\right]\\ &= f(1)+\frac{1}{2}f''(2)+\frac{1}{4}\int_0^\infty f^{(4)}(x+2)e^{-x}dx\\ &\le f(1)+\frac{1}{2}f''(2)+\frac{1}{4}\left(f^{(4)}(3)+\frac{1}{2}\int_0^\infty f^{(6)}(x+3)\cdot e^{-x}dx\right)\\ &= f(1)+\frac{1}{2}f''(2)+\left(\frac{1}{2}\right)^2f^{(4)}(3)+\left(\frac{1}{2}\right)^3\int_0^\infty f^{(6)}(x+3)e^{-x}dx\,. \end{aligned}$$

By induction on n, we see that

$$I \leq \left[f(1)+\sum_{j=1}^{n}\left(\frac{1}{2}\right)^{j} f^{(2j)}(j+1)\right]$$

$$+\left\{\left(\frac{1}{2}\right)^{n+1}\int_0^\infty f^{(2n+2)}(x+n+1)e^{-x}dx\right\}. \tag{2.38}$$

The expression in brackets of (2.38) converges and approaches the upper bound for I given in part (a), by the $\limsup$ condition. The expression in braces of (2.38) converges to zero by the limit condition. These follow easily by the root test of analysis and properties of convergent series. This completes the proof of part (a). ∎

The proof of part (b) is very similar, and is omitted except we mention that $x = 1$ replaces $x = j+1$ in $f^{(2j)}(x)$ at each stage.

Example 2.1. *We present a numerical example to illustrate Theorem 2.11. Let $c \in [0,1)$. Let $f(x) = e^{cx}$, $x \geq 0$. An application of Theorem 2.11 gives that the upper and lower bounds given in Theorem 2.11 parts (a) and (b) are valid, if $c^2e^c < 2$, which occurs when $c \leq 0.9012$, approximately. We give a small table of upper and lower bounds for I for various values of c, using Theorem 2.11.*

c	I	*Upper bound for I*	*Lower bound for I*
0.10	1.111111	1.111312	1.110724
0.30	1.4286	1.4376	1.4135
0.50	2.0000	2.0767	1.8843
0.70	3.3333	3.9748	2.6672
0.85	6.6667	15.1137	3.6629

The upper bound deteriorates as $c \to 0.90^-$ and more and more terms must be summed to guarantee an upper bound for I. For $c \leq 0.70$, four or five terms only need to be summed to compute rapidly converging bounds.

Remark 2.4. If f is absolutely monotonic on $[0,\infty)$, then $f^{(j)}(x) \geq 0$, $x \geq 0$, $j = 1,2,3\ldots$. In this case, f is convex, in particular, on $[0,\infty)$ and the lower bound given by

$$f(1)+\sum_{j=1}^{n}\left(\frac{1}{2}\right)^{j} f^{(2j)}(1)$$

improves on the Jensen's inequality lower bound value of $f(1)$. Also, upper and lower bounds for I can be obtained for any pattern of sign changes in $f^{(j)}(x)$ as j changes, so long as, given j, $f^{(j)}(x)$ is of one sign only for all $x \geq 0$. For absolutely monotonic sequences, the sign pattern of $f^{(j)}(x)$ is $+,+,+,+,\ldots$. For completely monotonic functions, this sign pattern of $f^{(j)}(x)$ is $+,-,+,-,+,-,\ldots$. The bounds of Theorem 2.9 are reversed in this case. Theorems for any pattern of constant signs as j varies can be obtained. The bounds for I would have the form

$$f(1)+\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{j}\cdot f^{(2j)}(\theta_j),$$

where θ_j is some real number in $[1, j+1]$, $j = 1,2,3,\ldots$. We omit the details here.

3. APPLICATIONS

The logarithmic mean of two positive real numbers a and b is given by

$$L(a,b)=\frac{a-b}{\log(a)-\log(b)}\,,\quad a<b\,. \tag{3.1}$$

See Rao and Dey [10] and the references contained therein. There are various integral representations of $L(a,b)$. Some of these are

$$L(a,b)^{-1}=\int_0^1\frac{dx}{ax+b(1-x)} \tag{3.2}$$

and

$$L(a,b)=\int_0^1 b^x a^{1-x}dx\,. \tag{3.3}$$

Letting $f(x)=\frac{1}{ax+b(1-x)}$ in (3.2) and applying Theorem 2.9, we obtain the following bounds for $L(a,b)^{-1}$. For $k=1$, (2.24)–(2.25) give, for $a<b$,

$$\frac{2}{a+b}+\frac{1}{12}\frac{(a-b)^2}{b^3}\le L(a,b)^{-1}\le\frac{2}{a+b}-\left(\frac{b-a}{12}\right)\left(\left(\frac{2}{a+b}\right)^2-\frac{1}{a^2}\right)\,. \tag{3.4}$$

For $k=2$, (2.26)–(2.27) give, for $a<b$,

$$\begin{aligned}\frac{2}{a+b}+\frac{1}{12}\frac{(b-a)^2}{\left(\frac14 a+\frac34 b\right)^3}+\frac{3}{160}\frac{(b-a)^4}{b^5}&\le L(a,b)^{-1}\\ \le\frac{2}{a+b}+\frac{(b-a)^3}{80}\left(\frac{1}{a^4}-\frac{1}{\left(\frac58 a+\frac38 b\right)^4}\right)&+\frac{(b-a)^2}{12}\cdot\frac{1}{\left(\frac58 a+\frac38 b\right)^3}\,.\end{aligned} \tag{3.5}$$

If we assume $a<b$ and use $f(x)=\frac{1}{bx+a(1-x)}$ instead, the bounds in Theorem 2.9 will reverse since $f^{(j)}(x)\le 0$. for odd $j\ge 3$ instead. This choice of $f(x)$ instead leads to improved upper bounds for $L(a,b)^{-1}$, but worse lower bounds.

What if integral representation (3.3) is used instead in Theorem 2.9? Similar inequalities can be obtained. Here, we present just a few examples.

Theorem 3.1. *Suppose $0<a<b$. Let $t=\sqrt{ab}$ be the geometric mean of a and b. Then*

(a)

$$L(a,b)\le(b-a)\cdot\left(\frac{b-t}{6\left(\sqrt{t^2+\frac{(b-t)(b-a)}{3}}-t\right)}\right)\equiv A_1 \tag{3.6}$$

and

(b)

$$L(a,b)\ge\left(t^3+\frac{(b-a)^2}{24}\right)^{1/3}\equiv A_2>t=\sqrt{ab}\,. \tag{3.7}$$

Proof. Let $f(x)=b^x a^{1-x}$ in Theorem 2.9. Then (2.24)-(2.25) give

$$L(a,b)\le\sqrt{ab}+\frac{1}{12}(\log a-\log b)\cdot(t-b)\,.$$

After some algebra, we obtain

$$\left(\frac{b-t}{12}\right)\cdot(\log b-\log a)^2+t(\log b-\log a)+(a-b)\geq 0\,.$$

We may easily solve this quadratic for $\theta=\log b-\log a$ to get

$$\log b-\log a\geq\frac{6\left(\sqrt{t^2+\frac{(b-t)(b-a)}{3}}-t\right)}{b-t}\,,$$

from which (3.6) follows upon division by $b-a$ and inversion. This completes the proof of (a).

To prove (b), we use (2.24)–(2.25). We obtain

$$L(a,b)\geq\sqrt{ab}+\frac{1}{24}(\log b-\log a)^2\,.$$

After some algebra, we get

$$L(a,b))^3-t(L(a,b))^2-\frac{(b-a)^2}{24}\geq 0\,.$$

Letting $w=L(a,b)$,

$$w^3-tw^2-\frac{(b-a)^2}{24}\equiv g(w)\geq 0\,. \tag{3.8}$$

The derivative of $g(w)$ is $g'(w)=3w^2-2wt\geq 0$, since $0<t\leq w$ is well-known. Thus, (3.8) gives, using $0\leq t\leq w$,

$$w^3-t^3-\frac{(b-a)^2}{24}\geq 0\,.$$

Solving for $w=L(a,b)$, we obtain the desired result. This completes the proof of part (b). ∎

Remark 3.1. Other bounds for $L(a,b)$ have been discussed in the literature. For example, in Jia and Cao [8], it is proven that

$$\begin{aligned} L(a,b) &< H_p(a,b)=\left(\frac{a^p+(ab)^{p/2}+b^p}{3}\right)^{1/p} \\ &< M_q(a,b)=\left(\frac{a^q+b^q}{2}\right)^{1/q}\,, \end{aligned} \tag{3.9}$$

if $p\geq 1/2$, $q\geq\frac{2p}{3}$ and that $p=1/2$ and $q=1/3$ are the best constants in (3.9). Numerical investigations have found that (3.9) is better than (3.4) and (3.5) given in this paper, but the bounds presented here use the arithmetic or geometric means only. For other bounds, see Wada [12] and Furuichi and Yanagi [6]. In Jia and Cao [8], in their Remark 4, it is given that the best lower bound for $L(a,b)$ within the family $H_p(a,b)$ occurs as $p\to 0$, namely $H_0(a,b)=t=\sqrt{ab}$. Thus, Theorem 3.1, part (b) is an improved lower bound on $L(a,b)$ which uses only the geometric mean.

The author is currently investigating inequalities for other types of means and hopefully this will be reported on in the future.

4. CONCLUDING REMARKS

Lemma 1.1 is a very useful result as far as deriving many more inequalities of either Hermite-Hadamard or Fejér type. These will be reported on in the future. It is especially useful for deriving inequalities for functions $f(x)$ having one or more derivatives of constant sign on the interval of integration $[a, b]$, since the functions $q_1(x)$ and $q_2(x)$ given in (1.2) and (1.3) have extrema at the endpoints in these higher convexity cases. In a forthcoming paper, more applications to probability theory and approximation theory will be discussed.

REFERENCES

[1] R. E. BARLOW and F. PROSCHAN, *Statistical Theory of Reliability and Life Testing*, 1981.

[2] S. S. DRAGOMIR and I. GOMM, Some applications of Fejér's inequality for convex functions (I). *Aust. J. of Math. Anal. and Appl.*, **10**(1), Article 9 (2013), pp. 1-11.

[3] S. S. DRAGOMIR and I. GOMM, Some applications of Fejér's inequality for convex functions (II). *Transylv. J. Math. Mech.*, **5** (2013), no. 1, pp. 23–33.

[4] S. S. DRAGOMIR, Inequalities of Fejér type for n-convex functions on linear spaces, 2015, *RGMIA*.

[5] S. G. FROM, Some new generalizations of Jensen's inequality with related results and applications. *Aust. J. Math. Anal. and Appl.*, **13**(1), Article 1, (2016), pp. 1–29.

[6] S. FURUICHI and K. YANAGI, Bounds of the logarithmic mean. *J. of Inequal. and Appl.*, (2013), pp. 1–11.

[7] P. L. GUPTA and R. C. GUPTA, The monotonicity of the reliability measures of the beta distribution. *Appl. Math. Lett.*, **13** (2000), pp. 5–9.

[8] G. JIA and J. CAO, A new upper bound of the logarithmic mean. *J. of Ineq. in Pure and Applied Math.*, **4**(4), Article 80, (2013), pp. 1–4.

[9] N. MINCULETE and F. CORINA-MITROI, Fejér-type inequalities. *Aust. J. of Math. Anal. and Appl.*, **9**(1), Article 12, (2012), pp. 1–8.

[10] M. RAO and A. DEY, Scope of the logarithmic mean. *Aust. J. Math. Anal. and Appl.*, **11**(1), Article 13, (2014), pp. 1–10.

[11] G. B. SWARTZ, The mean residual life function. *IEEE Trans. on Reliab.*, **22**(2) (1973), pp. 108–109.

[12] S. WADA, On some estimates for the logarithmic mean. *Aust. J. Math. Anal. and Appl.*, **11**(1), Article 13, (2014), pp. 1–5.

[13] D. WIDDER, *The Laplace Transform*, 1946.