THE BOUNDEDNESS OF BESSEL-RIESZ OPERATORS ON GENERALIZED MORREY SPACES

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ABSTRACT. In this paper, we prove the boundedness of Bessel-Riesz operators on generalized Morrey spaces. The proof uses the usual dyadic decomposition, a Hedberg-type inequality for the operators, and the boundedness of Hardy-Littlewood maximal operator. Our results reveal that the norm of the operators is dominated by the norm of the kernels.

Key words and phrases: Bessel-Riesz operators, Hardy-Littlewood maximal operator, generalized Morrey spaces, Boundedness, Kernels.

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1. Introduction

We begin with the definition of Bessel-Riesz operators. For \( \gamma \geq 0 \) and \( 0 < \alpha < n \), we define
\[
I_{\alpha, \gamma} f(x) := \int_{\mathbb{R}^n} K_{\alpha, \gamma}(x - y) f(y) \, dy,
\]
for every \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \), \( p \geq 1 \), where \( K_{\alpha, \gamma}(x) := \frac{|x|^{n-\alpha}}{|1 + |x||^\gamma} \). Here, \( I_{\alpha, \gamma} \) is called Bessel-Riesz operator and \( K_{\alpha, \gamma} \) is called Bessel-Riesz kernel. The name of the kernel resembles the product of Bessel kernel and Riesz kernel [13]. While the Riesz kernel captures the local behaviour, the Bessel kernel takes care the global behaviour of the function. The Bessel-Riesz kernel is used in studying the behaviour of the solution of a Schrödinger type equation [8].

For \( \gamma = 0 \), we have \( I_{0,0} = I_{\alpha} \), known as fractional integral operators or Riesz potentials [12]. Around 1930, Hardy and Littlewood [5, 6] and Sobolev [11] have proved the boundedness of \( I_{\alpha} \) on Lebesgue spaces via the inequality
\[
\|I_{\alpha} f\|_{L^p} \leq C \|f\|_{L^p},
\]
for every \( f \in L^p(\mathbb{R}^n) \), where \( 1 < p < \frac{n}{\alpha} \), and \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \). Here \( C \) denotes a constant which may depend on \( \alpha, p, q, \) and \( n \), but not on \( f \).

For \( 1 \leq p \leq q \), the (classical) Morrey space \( L^{p,q}(\mathbb{R}^n) \) is defined by
\[
L^{p,q}(\mathbb{R}^n) := \{ f \in L^{p}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,q}} < \infty \},
\]
where \( \|f\|_{L^{p,q}} := \sup_{r > 0, a \in \mathbb{R}^n} r^{n(1/q - 1/p)} \left( \int_{|x-a|<r} |f(x)|^p dx \right)^{1/p} \). For these spaces, we have an inclusion property which is presented by the following theorem.

**Theorem 1.1.** For \( 1 \leq p \leq q \), we have \( L^q(\mathbb{R}^n) = L^{q,q}(\mathbb{R}^n) \subseteq L^{p,q}(\mathbb{R}^n) \subseteq L^{1,q}(\mathbb{R}^n) \).

On Morrey spaces, Spanne [10] has shown that \( I_{\alpha} \) is bounded form \( L^{p_1,q_1}(\mathbb{R}^n) \) to \( L^{p_2,q_2}(\mathbb{R}^n) \) for \( 1 < p_1 < q_1 < \frac{n}{\alpha} \), \( \frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n} \), and \( \frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n} \). Furthermore, Adams [1] and Chiarenza and Frasca [2] reproved it and obtained a stronger result which is presented below.

**Theorem 1.2.** [Adams, Chiarenza-Frasca] If \( 0 < \alpha < n \), then we have
\[
\|I_{\alpha} f\|_{L^{p_2,q_2}} \leq C \|f\|_{L^{p_1,q_1}},
\]
for every \( f \in L^{p_1,q_1}(\mathbb{R}^n) \) where \( 1 < p_1 < q_1 < \frac{n}{\alpha} \), \( \frac{1}{p_2} = \frac{1}{p_1} \left( 1 - \frac{\alpha n}{n} \right) \), and \( \frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n} \).

For \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) and \( 1 \leq p < \infty \), we define the generalized Morrey space
\[
L^{p,\phi}(\mathbb{R}^n) := \{ f \in L^{p}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\phi}} < \infty \},
\]
where \( \|f\|_{L^{p,\phi}} := \sup_{r > 0, a \in \mathbb{R}^n} \frac{1}{\phi(r)} \left( \int_{|x-a|<r} |f(x)|^p dx \right)^{1/p} \). Here we assume that \( \phi \) is almost decreasing and \( t^{\alpha/p} \phi(t) \) is almost decreasing, so that \( \phi \) satisfies the doubling condition, that is, there exists a constant \( C \) such that \( \frac{1}{C} \leq \frac{\phi(c)}{\phi(c/2)} \leq C \) whenever \( \frac{1}{2} \leq \frac{c}{b} \leq 2 \).

In 1994, Nakai [9] obtained the boundedness of \( I_{\alpha} \) from \( L^{p_1,\phi}(\mathbb{R}^n) \) to \( L^{p_2,\psi}(\mathbb{R}^n) \) where \( 1 < p_1 < \frac{n}{\alpha} \), \( \frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha n}{n} \) and \( \int_{r}^{\infty} \frac{\psi(t)}{t} \, dt \leq C r^{\alpha} \phi(r) \leq C \psi(r) \) for every \( r > 0 \). Nakai’s result may be viewed as an extension of Spanne’s. Later on, in 2009, Gunawan and Ernadi [3] extended Adams-Chiarenza-Frasca’s result.

**Theorem 1.3.** [Gunawan-Erdani] If \( \int_{r}^{\infty} \frac{\phi(t)}{t} \, dt \leq C \phi(r) \), and \( \phi(r) \leq C r^{\beta} \) for every \( r > 0 \), \(-\frac{n}{p_1} \leq \beta < -\alpha \), \( 1 < p_1 < \frac{n}{\alpha} \), \( 0 < \alpha < n \), then we have
\[
\|I_{\alpha} f\|_{L^{p_2,\psi}} \leq C \|f\|_{L^{p_1,\phi}},
\]
for every \( f \in L^{p_1,\phi}(\mathbb{R}^n) \) where \( p_2 = \frac{\beta p_1}{\alpha + \beta} \) and \( \psi(r) = \phi(r)^{p_1/p_2} \), \( r > 0 \).
The proof of the boundedness of \( I_\alpha \) on Lebesgue spaces, Morrey spaces, or generalized Morrey spaces, usually involves \textit{Hardy-Littlewood maximal operator}, which is defined by

\[
Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy,
\]

for every \( f \in L^p_{loc}(\mathbb{R}^n) \) where \( |B| \) denotes the Lebesgue measure of the ball \( B = B(a, r) \) (centered at \( a \in \mathbb{R}^n \) with radius \( r > 0 \)). It is well known that the operator \( M \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p \leq \infty \) \cite{12, 13} and also on Morrey spaces \( L^{p,q} \) for \( 1 < p \leq q \leq \infty \) \cite{2}.

Next, we know that \( I_{\alpha,\gamma} \) is guaranteed to be bounded on generalized Morrey spaces because \( K_{\alpha,\gamma}(x) \leq K_\alpha(x) \) for every \( x \in \mathbb{R}^n \). The boundedness of \( I_{\alpha,\gamma} \) on Lebesgue spaces can also be proved by using Young’s inequality, as shown in \cite{7}.

**Theorem 1.4.** \cite{7} For \( \gamma > 0 \) and \( 0 < \alpha < n \), we have \( K_{\alpha,\gamma} \in L^t(\mathbb{R}^n) \) whenever \( \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha} \). Accordingly, we have

\[
\|I_{\alpha,\gamma}f\|_{L^t} \leq \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^p}
\]

for every \( f \in L^p(\mathbb{R}^n) \) where \( 1 \leq p < t' \), \( \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha} \), \( \frac{1}{p} + \frac{1}{t'} = 1 \).

Using the boundedness of Hardy-Littlewood maximal operator, we also know that \( I_{\alpha,\gamma} \) is bounded on Morrey spaces.

**Theorem 1.5.** \cite{7} For \( \gamma > 0 \) and \( 0 < \alpha < n \), we have

\[
\|I_{\alpha,\gamma}f\|_{L^{p_2,q_2}} \leq C \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,q_1}}
\]

for every \( f \in L^{p_1,q_1}(\mathbb{R}^n) \) where \( 1 < p_1 < q_1 < t' \), \( \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha} \), \( \frac{1}{p_1} + \frac{1}{t'} = \frac{1}{q_2} + \frac{1}{t} \).

In 1999, Kurata \textit{et al.} \cite{8} proved the boundedness of \( W \cdot I_{\alpha,\gamma} \) on generalized Morrey spaces where \( W \) is a multiplication operator. A similar result to Kurata’s can be found in \cite{3}. In the next section, we shall reprove the boundedness of \( I_{\alpha,\gamma} \) on generalized Morrey spaces using a Hedberg-type inequality and the boundedness of Hardy-Littlewood maximal operator on these spaces.

**Theorem 1.6.** (Nakai) For \( 1 < p \leq \infty \), we have

\[
\|Mf\|_{L^{p,\phi}} \leq C \|f\|_{L^{p,\phi}}
\]

for every \( f \in L^{p,\phi}(\mathbb{R}^n) \).

Our results show that the norm of Bessel-Riesz operators is dominated by the norm of their kernels on (generalized) Morrey spaces.

**2. MAIN RESULTS**

For \( \gamma > 0 \) and \( 0 < \alpha < n \), one may observe that the kernel \( K_{\alpha,\gamma} \) belongs to Lebesgue spaces \( L^t(\mathbb{R}^n) \) whenever \( \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha} \), where

\[
\sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \sim \|K_{\alpha,\gamma}\|_{L^t}
\]

(see \cite{7}). With this in mind, we obtain the boundedness of \( I_{\alpha,\gamma} \) on generalized Morrey spaces as in the following theorem.
**Theorem 2.1.** Let $\gamma > 0$ and $0 < \alpha < n$. If $\phi(r) \leq C r^\beta$ for every $r > 0$, $-\frac{n-\alpha'}{p_1} \leq \beta < -\alpha$, $1 < p_1 < t'$, and $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, then we have

$$
\|I_{\alpha,\gamma} f\|_{L^{p_2},\psi} \leq C \|K_{\alpha,\gamma}\|_{L^1} \|f\|_{L^{p_1},\psi},
$$

for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$ where $p_2 = \frac{\beta p_1}{\alpha + \beta}$, and $\psi(r) = (\phi(r))^{p_1/p_2}$.

**Proof.** Let $\gamma > 0$ and $0 < \alpha < n$. Suppose that $\phi(r) \leq C r^\beta$ for every $r > 0$, $-\frac{n-\alpha'}{p_1} \leq \beta < -\alpha$, $1 < p_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. Take $f \in L^{p_1,\phi}(\mathbb{R}^n)$ and write

$$
I_{\alpha,\gamma} f(x) := I_1(x) + I_2(x),
$$

for every $x \in \mathbb{R}^n$ where $I_1(x) := \int_{|x-y| < R} \frac{|x-y|^\alpha - n f(y)}{(1 + |x-y|)^{\alpha}} dy$ and $I_2(x) := \int_{|x-y| \geq R} \frac{|x-y|^\alpha - n f(y)}{(1 + |x-y|)^{\alpha}} dy$, $R > 0$.

Using dyadic decomposition, we have the following estimate for $I_1$:

$$
|I_1(x)| \leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|x-y|^\alpha - n |f(y)|}{(1 + |x-y|)^{\alpha}} dy.
$$

$$
\leq C_1 \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n}}{(1 + 2^k R)^{\alpha-n}} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy.
$$

$$
\leq C_2 M f(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/t}}{(1 + 2^k R)^{\gamma}}.
$$

We then use Hölder’s inequality to get

$$
|I_1(x)| \leq C_2 M f(x) \left( \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/t}}{(1 + 2^k R)^{\gamma}} \right)^{1/t} \left( \sum_{k=-\infty}^{-1} (2^k R)^{\alpha-n} \right)^{1/t'}.
$$

Because we have

$$
\left( \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/t}}{(1 + 2^k R)^{\gamma}} \right)^{1/t} \leq \left( \sum_{k \in \mathbb{Z}} \frac{(2^k R)^{\alpha-n+n/t}}{(1 + 2^k R)^{\gamma}} \right)^{1/t} \sim \|K_{\alpha,\gamma}\|_{L^1},
$$

we obtain $|I_1(x)| \leq C_3 \|K_{\alpha,\gamma}\|_{L^1} M f(x) R^{\alpha/n'}$.

To estimate $I_2$, we use Hölder’s inequality again:

$$
|I_2(x)| \leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1 + 2^k R)^{\gamma}} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy.
$$

$$
\leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1 + 2^k R)^{\gamma}} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)|^{p_1} dy \right)^{1/p_1}.
$$

It follows that

$$
|I_2(x)| \leq C_5 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n+n/t}}{(1 + 2^k R)^{\gamma}} \phi(2^k R) (2^k R)^{\alpha/n'}.
$$

$$
\leq C_6 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n+n/t}}{(1 + 2^k R)^{\gamma}} (2^k R)^{\beta+n/t'}.
$$
Another use of Hölder’s inequality gives

\[ |I_2(x)| \leq C_6 \| f \|_{L^{p_1,\phi}} \left( \sum_{k=0}^{\infty} \left( \frac{2^k R}{(1 + 2^k R)^\gamma} \right)^{(a-n)t+n} \right)^{1/t} \] 

Because \( \beta t' + n < 0 \) and \( \sum_{k=0}^{\infty} \left( \frac{2^k R}{(1 + 2^k R)^\gamma} \right)^{(a-n)t+n} \lesssim \| K_{\alpha,\gamma} \|_{L^t} \), we obtain

\[ |I_2(x)| \leq C_7 \| K_{\alpha,\gamma} \|_{L^t} \| f \|_{L^{p_1,\phi}} R^\beta R^{n/t'). \]

Summing the two estimates, we obtain

\[ |I_{\alpha,\gamma} f(x)| \leq C_8 \| K_{\alpha,\gamma} \|_{L^t} \left( Mf(x) R^{n/t'} + \| f \|_{L^{p_1,\phi}} R^{n/t' + \beta} \right), \]

for every \( x \in \mathbb{R}^n \). Now, for each \( x \in \mathbb{R}^n \), choose \( R > 0 \) such that \( R^\beta = \frac{Mf(x)}{\| f \|_{L^{p_1,\phi}}} \). Hence we get a Hedberg-type inequality for \( I_{\alpha,\gamma} f \), namely

\[ |I_{\alpha,\gamma} f(x)| \leq C_9 \| K_{\alpha,\gamma} \|_{L^t} \| f \|_{L^{p_1,\phi}} Mf(x)^{1+\alpha/\beta}. \]

Now put \( p_2 := \frac{\beta p_1}{\alpha + \beta} \). For arbitrary \( a \in \mathbb{R}^n \) and \( r > 0 \), we have

\[ \left( \int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} dx \right)^{1/p_2} \leq C_9 \| K_{\alpha,\gamma} \|_{L^t} \| f \|_{L^{p_1,\phi}}^{1-p_1/p_2} \left( \int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{1/p_2}. \]

We divide both sides by \( \phi(r)^{p_1/p_2} \) to get

\[ \frac{\left( \int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} dx \right)^{1/p_2}}{\phi(r)^{n/p_2}} \leq C_9 \| K_{\alpha,\gamma} \|_{L^t} \| f \|_{L^{p_1,\phi}}^{1-p_1/p_2} \frac{\left( \int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{1/p_2}}{\phi(r)^{p_1/p_2} \phi(r)^{n/p_2}}, \]

where \( \psi(r) := \phi(r)^{p_1/p_2} \). Taking the supremum over \( a \in \mathbb{R}^n \) and \( r > 0 \), we obtain

\[ \| I_{\alpha,\gamma} f \|_{L^{p_2,\psi}} \leq C_{10} \| K_{\alpha,\gamma} \|_{L^t} \| f \|_{L^{p_1,\phi}}^{1-p_1/p_2} \| Mf \|_{L^{p_1,\phi}}^{p_1/p_2}. \]

By the boundedness of the maximal operator on generalized Morrey spaces (Nakai’s Theorem), the desired result follows: \( \| I_{\alpha,\gamma} f \|_{L^{p_2,\psi}} \leq C \| K_{\alpha,\gamma} \|_{L^t} \| f \|_{L^{p_1,\phi}}. \]

We note that from the inclusion property of Morrey spaces, we have

\[ \| K_{\alpha,\gamma} \|_{L_s^{s,t}} \leq \| K_{\alpha,\gamma} \|_{L_{s,t}^{s,t}} = \| K_{\alpha,\gamma} \|_{L^t}. \]

whenever \( 1 \leq s \leq t, \frac{n}{\alpha + \gamma - \alpha} < t < \frac{n}{\alpha - \alpha} \). Now we wish to obtain a more general result for the boundedness of \( I_{\alpha,\gamma} \) by using the fact that the kernel \( K_{\alpha,\gamma} \) belongs to Morrey spaces.

**Theorem 2.2.** Let \( \gamma > 0 \) and \( 0 < \alpha < n \). If \( \phi(r) \leq C r^\beta \) for every \( r > 0 \), \( -\alpha t' \leq \beta < -\alpha \), \( 1 < p_1 < t' \), \( \frac{n}{\alpha + \gamma - \alpha} < t < \frac{n}{\alpha - \alpha} \), then we have

\[ \| I_{\alpha,\gamma} f \|_{L^{p_2,\psi}} \leq C \| K_{\alpha,\gamma} \|_{L_s^{s,t}} \| f \|_{L^{p_1,\phi}}, \]

for every \( f \in L^{p_1,\phi}(\mathbb{R}^n) \) where \( 1 \leq s \leq t, p_2 = \frac{\beta p_1}{\alpha + \beta}, \) and \( \psi(v) = \phi(v)^{p_1/p_2} \).
It thus follows that
\[ |I_1(x)| \leq \left| \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|x-y|^{\alpha-n} |f(y)|}{(1+|x-y|)^\gamma} dy \right| \]
\[ \leq C_1 \left| \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy \right| \]
\[ = C_2 Mf(x) \left| \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/\gamma} (2^k R)^{n/s'}}{(1+2^k R)^\gamma} \right|, \]
where \(1 \leq s \leq t\). By Hölder’s inequality,
\[ |I_1(x)| \leq C_2 Mf(x) \left( \left| \sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \right|^{1/s} \right) \left( \left| \sum_{k=-\infty}^{-1} (2^k R)^n \right|^{1/s'} \right). \]

We also have
\[ \sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \leq \int_{0<|x|<R} K_{a,\gamma}^s(x) \, dx, \]
so that
\[ |I_1(x)| \leq C_3 Mf(x) \left( \int_{0<|x|<R} K_{a,\gamma}^s(x) \, dx \right)^{1/\gamma} R^{n/s'} \leq C_3 \| K_{a,\gamma} \|_{L^{s,t}} Mf(x) R^{n/t}. \]
Next, we estimate \( I_2 \) by using Hölder’s inequality. As in the proof of Theorem 2.1, we obtain
\[ |I_2(x)| \leq C_4 \left| \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} (2^k R)^{n/p_1} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)|^{p_1} \, dy \right)^{1/p_1} \right|. \]
It thus follows that
\[ |I_2(x)| \leq C_5 \| f \|_{L^{p_1,\phi}} \left| \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha} \phi(2^k R) \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} dy \right)^{1/s}}{(2^k R)^{n/s}} \right| \]
\[ \leq C_6 \| f \|_{L^{p_1,\phi}} \phi(2^k R) (2^k R)^{n/t'} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{a,\gamma}^s(x-y) \, dy \right)^{1/s} \]
\[ \leq C_6 \| f \|_{L^{p_1,\phi}} \phi(2^k R) (2^k R)^{n/t'} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{a,\gamma}^s(x-y) \, dy \right)^{1/s} \]
\[ \leq C_7 \| K_{a,\gamma} \|_{L^{s,t}} \| f \|_{L^{p_1,\phi}} \phi(2^k R) (2^k R)^{\alpha+n/t'} \leq C_8 \| K_{a,\gamma} \|_{L^{s,t}} \| f \|_{L^{p_1,\phi}} R^{\beta R^{n/t'}}. \]

Because \( \phi(r) \leq C r^\beta \) and
\[ \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{a,\gamma}^s(x-y) \, dy \right)^{1/s} \leq \| K_{a,\gamma} \|_{L^{s,t}} \text{ for every } k = 0, 1, 2, \ldots, \]
we get
\[ |I_2(x)| \leq C_7 \| K_{a,\gamma} \|_{L^{s,t}} \| f \|_{L^{p_1,\phi}} \phi(2^k R) (2^k R)^{\alpha+n/t'} \]
\[ \leq C_8 \| K_{a,\gamma} \|_{L^{s,t}} \| f \|_{L^{p_1,\phi}} R^{\beta R^{n/t'}}. \]
From the two estimates, we obtain
\[ |I_{a,\gamma} f(x)| \leq C_9 \| K_{a,\gamma} \|_{L^{s,t}} \left( Mf(x) R^{n/t'} + \| f \|_{L^{p_1,\phi}} R^{n/t' + \beta} \right), \]
for every \( x \in \mathbb{R}^n \). Now, for each \( x \in \mathbb{R}^n \), choose \( R > 0 \) such that \( R^{\beta} = \frac{Mf(x)}{\| f \|_{L^{p_1,\phi}}} \). Hence we get
\[ |I_{a,\gamma} f(x)| \leq C_9 \| K_{a,\gamma} \|_{L^{s,t}} \| f \|_{L^{p_1,\phi}}^{\alpha/\beta} Mf(x)^{1+\alpha/\beta}. \]
Put $p_2 := \frac{\beta_1}{\alpha_3}$. For arbitrary $a \in \mathbb{R}^n$ and $r > 0$, we have
\[
\left( \int_{|x-a|<r} |I_{a,\gamma} f(x)|^{p_2} \, dx \right)^{1/p_2} \leq C_9 \| K_{a,\gamma} \|_{L^{p_1,\theta}} \| f \|_{L^{p_1,\theta}}^{1-p_1/p_2} \left( \int_{|x-a|<r} |M f(x)|^{p_1} \, dx \right)^{1/p_2}.
\]
Divide both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ and take the supremum over $a \in \mathbb{R}^n$ and $r > 0$ to get
\[
\| I_{a,\gamma} f \|_{L^{p_2,\psi}} \leq C_{10} \| K_{a,\gamma} \|_{L^{p_1,\theta}} \| f \|_{L^{p_1,\theta}}^{1-p_1/p_2} \| M f \|_{L^{p_1,\psi}}^{p_1/p_2},
\]
where $\psi(r) := \phi(r)^{p_1/p_2}$. With the boundedness of the maximal operator on generalized Morrey spaces (Nakai’s Theorem), we obtain
\[
\| I_{a,\gamma} f \|_{L^{p_2,\psi}} \leq C_{p_1,\psi} \| K_{a,\gamma} \|_{L^{p_1,\theta}} \| f \|_{L^{p_1,\theta}},
\]
as desired. \[\square\]

Note that by Theorem 2.2 and the inclusion of Morrey spaces, we recover Theorem 2.1:
\[
\| I_{a,\gamma} f \|_{L^{p_2,\psi}} \leq C \| K_{a,\gamma} \|_{L^{p_1,\theta}} \| f \|_{L^{p_1,\theta}} \leq C \| K_{a,\gamma} \|_{L^\infty} \| f \|_{L^{p_1,\theta}}.
\]

We still wish to obtain a better estimate. The following lemma presents that the Bessel-Riesz kernels belong to generalized Morrey space $L^{s,\sigma} (\mathbb{R}^n)$ for some $s \geq 1$ and a suitable function $\sigma$.

**Lemma 2.3.** Suppose that $\gamma > 0$ and $0 < \alpha < n$. If $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies
\[
\int_{0<r\leq R} r^{(\alpha-n)s+n-1} \, dr \leq C \sigma^s(R) R^n
\]
for every $R > 0$, then $K_{a,\gamma} \in L^{s,\sigma} (\mathbb{R}^n)$.

**Proof.** Suppose that the hypothesis holds. It is sufficient to evaluate the integral around 0. We observe that
\[
\int_{|x| \leq R} K_{a,\gamma}^s(x) \, dx = \int_{|x| \leq R} \frac{|x|^{(\alpha-n)s}}{(1+|x|)^{\gamma s}} \, dx \leq C \int_{0<r\leq R} r^{(\alpha-n)s+n-1} \, dr \leq C \sigma^s(R) R^n.
\]
We divide both sides of the inequality by $\sigma^s(R) R^n$ and take $s^{th}$-root to obtain
\[
\frac{\left( \int_{|x| \leq R} K_{a,\gamma}^s(x) \, dx \right)^{1/s}}{\sigma(R) R^{n/s}} \leq C^{1/s}.
\]
Now, taking the supremum over $R > 0$, we have
\[
\sup_{R>0} \frac{\left( \int_{|x| \leq R} K_{a,\gamma}^s(x) \, dx \right)^{1/s}}{\sigma(R) R^{n/s}} < \infty.
\]
Hence $K_{a,\gamma} \in L^{s,\sigma} (\mathbb{R}^n)$. \[\square\]

By the hypothesis of Lemma 2.3 we also obtain
\[
\frac{\left( \int_{|x| \leq |x|\leq k+1} K_{a,\gamma}^s(x) \, dx \right)^{1/s}}{\sigma^s(R)(2^k R)^{n/s}} \lesssim \| K_{a,\gamma} \|_{L^{s,\sigma}}
\]
for every integer $k$ and $R > 0$. Moreover,
\[
\frac{\left( \sum_{k=1}^{\infty} K_{a,\gamma}^s(2^k R)(2^k R)^n \right)^{1/s}}{\sigma^s(R)(R)^{n/s}} \lesssim \| K_{a,\gamma} \|_{L^{s,\sigma}}
\]
holds for every $R > 0$. One may observe that $1 \leq s \leq \frac{n \ln R_1}{\ln \sigma(R_1)}$ for every $R_1 > 1$. For $\sigma(R) = R^{-n/t}$, this inequality reduces to $1 \leq s \leq t$.

We shall now use the lemma to prove the following theorem.
Theorem 2.4. Suppose that $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the doubling condition and $\sigma(r) \leq C r^{\alpha}$ for every $r > 0$, so that $K_{\alpha, \gamma} \in L^{k, \sigma}(\mathbb{R}^n)$ for $1 \leq s < \frac{n}{n-\alpha}$, where $0 < \alpha < n$ and $\gamma > 0$. If $\phi(r) \leq C r^\beta$ for every $r > 0$, where $-\frac{n}{p_1} < \beta < -\alpha$, then we have

$$\|I_{\alpha, \gamma} f\|_{L^p, \psi} \leq C_{p_1, \phi} \|K_{\alpha, \gamma}\|_{L^q, \sigma} \|f\|_{L^p, \phi}$$

for every $f \in L^{p_1, \phi}(\mathbb{R}^n)$, where $1 < p_1 < \frac{n}{\alpha}$, $p_2 = \frac{\beta_1}{\beta + n - \alpha}$ and $\psi(r) = \phi(r)^{p_1/p_2}$.

Proof. Let $\gamma > 0$ and $0 < \alpha < n$. Suppose that $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the doubling condition and $\sigma(r) \leq C r^{\alpha}$ for every $r > 0$, such that $K_{\alpha, \gamma} \in L^{k, \sigma}(\mathbb{R}^n)$ for $1 \leq s < \frac{n}{n-\alpha}$. Suppose also that $\phi(r) \leq C r^\beta$ for every $r > 0$, where $-\frac{n}{p_1} < \beta < -\alpha$, $1 < p_1 < \frac{n}{\alpha}$. As in the proof of Theorem 2.1, we write $I_{\alpha, \gamma} f(x) := I_1(x) + I_2(x)$ for every $x \in \mathbb{R}^n$. As usual, we estimate $I_1$ by using dyadic decomposition:

$$|I_1(x)| \leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|x-y|^n |f(y)|}{(1+|x-y|)^\gamma} dy$$

$$\leq C_1 \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy$$

$$= C_2 M f(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/s}}{(1+2^k R)^\gamma} (2^k R)^{n/s'}$$

By using Hölder inequality, we obtain

$$|I_1(x)| \leq C_2 M f(x) \left( \sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)+n}}{(1+2^k R)^\gamma} \right)^{1/s} \left( \sum_{k=-\infty}^{-1} (2^k R)^{n} \right)^{1/s'}$$

But $\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)+n}}{(1+2^k R)} < \int_{0 < |x| < R} K_{\alpha, \gamma}^s(x) dx$, and so we get

$$|I_1(x)| \leq C_2 M f(x) \left( \int_{0 < |x| < R} K_{\alpha, \gamma}^s(x) dx \right)^{1/s} R^{n/s'}$$

$$\leq C_3 \|K_{\alpha, \gamma}\|_{L^q, \sigma} M f(x) \sigma(R) R^{n}$$

Next, we estimate $I_2$ as follows:

$$|I_2(x)| \leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| dy$$

$$\leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} (2^k R)^{n/p_1} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)|^{p_1} dy \right)^{1/p_1}$$

$$\leq C_5 \|f\|_{L^{p_1, \phi}} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha}}{(1+2^k R)^\gamma} \phi(2^k R) (2^k R)^{n} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} dy \right)^{1/s}$$

$$\leq C_6 \|f\|_{L^{p_1, \phi}} \sum_{k=0}^{\infty} \frac{(2^k R)^{n-\alpha+\beta}}{(1+2^k R)^\gamma} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha, \gamma}^s(x-y) dy \right)^{1/s}$$
Because \( \int_{|x-y|<2^{k+1}R} K_{\alpha,\gamma}^*(x-y)dy \leq \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \) for every \( k = 0, 1, 2, \ldots \), we obtain

\[
|I_2(x)| \leq C_6 \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} (2^k R)^{-\alpha+\beta}
\leq C_7 \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} R^{n-\alpha+\beta}.
\]

It follows from the above estimates for \( I_1 \) and \( I_2 \) that

\[
|I_{\alpha,\gamma} f(x)| \leq C_8 \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \left( Mf(x) R^{-\alpha} + \|f\|_{L^{p_1,\phi}} R^{n-\alpha+\beta} \right)
\]

for every \( x \in \mathbb{R}^n \). Now, for each \( x \in \mathbb{R}^n \), choose \( R > 0 \) such that \( R^\beta = \frac{Mf(x)}{\|f\|_{L^{p_1,\phi}}} \), whence

\[
|I_{\alpha,\gamma} f(x)| \leq C_9 \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}^{\frac{(\alpha-n)/\beta}{1}} Mf(x)^{1+(n-\alpha)/\beta}.
\]

Put \( p_2 := \frac{\beta n}{\beta + n - \alpha} \). For arbitrary \( a \in \mathbb{R}^n \) and \( r > 0 \), we have

\[
\left( \int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} \right)^{1/p_2} \leq C_9 \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \left( \int_{|x-a|<r} |Mf(x)|^{p_1} \right)^{1/p_2}.
\]

Divide the both sides by \( \phi(r)^{p_1/p_2} r^{-n/p_2} \) to get

\[
\left( \int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} \right)^{1/p_2} \psi(r) r^{n/p_2} = \frac{\left( \int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} \right)^{1/p_2}}{\phi(r)^{p_1/p_2} r^{-n/p_2}} \leq C_9 \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \left( \int_{|x-a|<r} |Mf(x)|^{p_1} \right)^{1/p_2},
\]

where \( \psi(r) := \phi(r)^{p_1/p_2} \). Finally, take the supremum over \( a \in \mathbb{R}^n \) and \( r > 0 \) to obtain

\[
\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C_{10} \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \|Mf\|_{L^{p_1,\phi}}.
\]

Because the maximal operator is bounded on generalized Morrey spaces (Nakai’s Theorem), we conclude that \( \|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C_{10,\phi} \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} \).

3. CONCLUDING REMARKS

The results presented in this paper, namely Theorems 2.1, 2.2, and 2.4, extend the results on the boundedness of Bessel-Riesz operators on Morrey spaces [7]. Similar to Gunawan-Eridani’s result for \( I_{\alpha} \), Theorems 2.1, 2.2, and 2.4 ensures that \( I_{\alpha,\gamma} : L^{p_1,\phi}(\mathbb{R}^n) \to L^{p_2,\psi}(\mathbb{R}^n) \). Notice that if we have \( \sigma : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for \( t \in \left( \frac{n}{n+\gamma-\alpha}, \frac{n}{n-\alpha} \right) \), then \( R^{-n/t} < \sigma(R) \) holds for every \( R > 0 \), then Theorem 2.4 gives a better estimate than Theorem 2.2. Now, if we define \( \sigma(R) := (1 + R^{n/t_1}) R^{-n/t} \) for some \( t_1 > t \), then \( \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} < \| K_{\alpha,\gamma} \|_{L^{s,t}} \). By Theorem 2.2 and the inclusion property of Morrey spaces, we obtain

\[
\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C \| K_{\alpha,\gamma} \|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} < C \| K_{\alpha,\gamma} \|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \leq C \| K_{\alpha,\gamma} \|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}.
\]

We can therefore say that Theorem 2.4 gives the best estimate among the three. Furthermore, we have also shown that, in each theorem, the norm of Bessel-Riesz operators on generalized Morrey spaces is dominated by that of Bessel-Riesz kernels.
REFERENCES


