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HYPONORMAL AND k-QUASI-HYPONORMAL OPERATORS ON SEMI-HILBERTIAN SPACES

OULD AHMED MAHMOUD SID AHMED AND ABDELKADER BENALI

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MATHEMATICS DEPARTMENT, COLLEGE OF SCIENCE, ALJOUF UNIVERSITY, ALJOUF 2014, SAUDI ARABIA,
sididahmed@ju.edu.sa

MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, HASSIBA BENBOUALI UNIVERSITY OF CHLEF, B.P. 151 HAY ESSALEM, CHLEF 02000, ALGERIA.,
benali4848@gmail.com

ABSTRACT. Let $\mathcal{H}$ be a Hilbert space and let A be a positive bounded operator on $\mathcal{H}$. The semi-inner product $\langle u \mid v\rangle_A := \langle Au \mid v\rangle,\ \ u,v \in \mathcal{H}$ induces a semi-norm $\|.\|_A$ on $\mathcal{H}$. This makes $\mathcal{H}$ into a semi-Hilbertian space. In this paper we introduce the notions of hyponormalities and k-quasi-hyponormalities for operators on semi Hilbertian space $\big(\mathcal{H},\ \|\,.\,\|_A\big)$, based on the works that studied normal, isometry, unitary and partial isometries operators in these spaces. Also, we generalize some results which are already known for hyponormal and quasi-hyponormal operators. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be (A,k)-quasi-hyponormal if

$$(T^\sharp)^k\Big(T^\sharp T - TT^\sharp\Big)T^k \geq_A 0 \ \text{ or equivalently } \ A(T^\sharp)^k\Big(T^\sharp T - TT^\sharp\Big)T^k \geq 0.$$

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1. INTRODUCTION AND PRELIMINARIES RESULTS

Throughout this manuscript, $\mathcal{H}$ denotes a complex Hilbert space with the inner product $\langle . \mid . \rangle$, $\mathcal{B}(\mathcal{H})$ is the Banach algebra of all bounded linear operators defined on $\mathcal{H}$ and $\mathcal{B}(\mathcal{H}^+$ is the cone of all positive (semidefinite) operators of $\mathcal{B}(\mathcal{H})$,i.e.;

$$\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}) : \ \langle Au \mid u\rangle \geq 0 \ \ \forall\, u \in \mathcal{H} \ \}.$$

For every $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{N}(T), \mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ stand for, respectively, the null space, the range and the closure of the range of T and its adjoint operator by T^*. In addition, if $T, S \in \mathcal{B}(\mathcal{H})$ then $T \geq S$ means that $T - S \geq 0$. Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$.

Given $A \in \mathcal{B}(\mathcal{H})^+$, we consider the semi-inner product $\langle \, . \mid . \, \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ defined by

$$\langle \ u \mid v \ \rangle_A = \langle \ Au \mid v \ \rangle \ \ \ \forall\, u, v \in \mathcal{H}.$$

Naturally, this semi-inner product induces a semi-norm,$\| \, . \, \|_A$, defined by

$$\|u\|_A = \big(\langle Au \mid u\rangle \big)^{\frac{1}{2}} = \left\| A^{\frac{1}{2}} u \right\|.$$

It is easily seen that $\| \, . \, \|_A$ is a norm on $\mathcal{H}$ if and only if A is injective operator,and the semi-normed space $\big(\mathcal{B}(\mathcal{H}), \ \| \, . \, \|_A\big)$ is complete if and only if $\mathcal{R}(A)$ is closed.

The above seminorm induces a seminorm on the subspace $\mathcal{B}^A(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$

$$\mathcal{B}^A(\mathcal{H}) := \{ \ T \in \mathcal{B}(\mathcal{H}) / \ \exists\, c > 0 \ : \ \|Tu\|_A \leq c\|u\|_A \ \ \forall\, u \in \mathcal{H} \ \}.$$

Indeed, if $T \in \mathcal{B}^A(\mathcal{H})$ then

$$\|T\|_A := \sup \Big\{ \ \frac{\|Tu\|_A}{\|u\|_A}, \quad u \in \overline{\mathcal{R}(A)}, \ \ u \neq 0 \ \Big\} < \infty.$$

Moreover

$$\|T\|_A = \sup\{ |\langle Tu \mid v\rangle_A|; \ u, v \in \mathcal{H}, \ : \|u\| \leq 1, \|v\| \leq 1 \ \}.$$

Operators in $\mathcal{B}^A(\mathcal{H})$ are called A-bounded operators.

For $u, v \in \mathcal{H}$, we say that u and v are A-orthogonal if $\langle u \mid v\rangle_A = 0$.

Definition 1.1. [3] For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if for every $u, v \in \mathcal{H}$

$$\langle Tu \mid v\rangle_A = \langle u \mid Sv\rangle_A,$$

i.e., $AS = T^*A$.

If T is an A-adjoint of itself, then T is called an A-selfadjoint operator $\big(AT = T^*A\big)$.

Remark 1.1. It is possible that an operator T does not have an A-adjoint, and if S is an A-adjoint of T we may find many A-adjoints; in fact if $AR = 0$ for some $R \in \mathcal{B}(\mathcal{H})$, then $S + R$ is an A-adjoint of T.

The set of all A-bounded operators which admit an A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas Theorem (see [9, 11]) we have that

$$\mathcal{B}_A(\mathcal{H}) = \big\{ \ T \in \mathcal{B}(\mathcal{H}) \ / \ \mathcal{R}(T^*A) \subset \mathcal{R}(A) \ \big\}.$$

If $T \in \mathcal{B}_A(\mathcal{H})$, then there exists a distinguished A-adjoint operator of T, namely,the reduced solution of equation $AX = T^*A$, i.e., $A^\dagger T^*A$. We denote this operator by $T^\sharp$. Therefore, $T^\sharp = A^\dagger T^*A$ and

$$AT^\sharp = T^*A, \ \mathcal{R}(T^\sharp) \subset \overline{\mathcal{R}(A)} \ \text{ and } \ \mathcal{N}(T^\sharp) = \mathcal{N}(T^*A).$$

Note that in which $A^\dagger$ is the Moore-Penrose inverse of A. For more details see [3, 4] and [5].

In the next proposition we collect some properties of $T^\sharp$ and its relationship with the seminorm $\| \,.\, \|_A$. For the proof see [3] and [4].

Proposition 1.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold.*

(1) $T^\sharp \in \mathcal{B}_A(\mathcal{H}), (T^\sharp)^\sharp = P_{\overline{R(A)}} T P_{\overline{R(A)}}$ and $((T^\sharp)^\sharp)^\sharp = T^\sharp$.

(2) If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^\sharp = S^\sharp T^\sharp$.

(3) $T^\sharp T$ and $TT^\sharp$ are A-selfadjoint.

(4) $\|T\|_A = \|T^\sharp\|_A = \|T^\sharp T\|_A^{\frac{1}{2}} = \|TT^\sharp\|_A^{\frac{1}{2}}$.

(5) $\|S\|_A = \|T^\sharp\|_A$ for every $S \in \mathcal{B}(\mathcal{H})$ which is an A-adjoint of T.

(6) If $S \in \mathcal{B}_A(\mathcal{H})$ then $\|TS\|_A = \|ST\|_A$.

The classes of normal, quasinormal, isometries, partial isometries, quasi-isometry and m-isometries on Hilbert spaces have been generalized to semi-Hilbertian spaces by many authors in [2, 3, 4, 5, 12, 16, 17, 19, 20, 24] and other papers.

Definition 1.2. Any operator $T \in \mathcal{B}_A(\mathcal{H})$ is called

(1) A-normal if $TT^\sharp = T^\sharp T$ (see [20]).

(2) A-isometry if $T^\sharp T = P_{\overline{\mathcal{R}(A)}}$ (see [3]).

(3) A-unitary if $T^\sharp T = TT^\sharp = P_{\overline{\mathcal{R}(A)}}$ (see [3]).

(4) A-quasinormal if $TT^\sharp T = T^\sharp T^2$ (see [18]).

(5) (A, m)-isometry if

$$\sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \left\|T^k u\right\|_A^2 = 0 \Longleftrightarrow \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} T^{\sharp k} T^k = 0 \qquad \text{(see [17]).}$$

Definition 1.3. ([20]) Let $T \in \mathcal{B}(\mathcal{H})$. The A-spectral radius and the A-numerical radius of T are denoted respectively $r_A(T)$ and $w_A(T)$ and defined by

$$r_A(T) = \limsup_{n \longrightarrow \infty} \|T^n\|_A^{\frac{1}{n}}$$

and

$$w_A(T) = \sup \left\{ \; |\langle Tu \mid u \rangle_A| \,, \;\; u \in \mathcal{H} \,, \|u\|_A = 1 \; \right\}.$$

Remark 1.2. If $T \in \mathcal{B}_A(\mathcal{H})$ is A-selfadjoint,then $\|T\|_A = w_A(T)$ (see [20]).

Theorem 1.2. ([20] *Theorem 3.1*)

A necessary and sufficient condition for an operator $T \in \mathcal{B}_A(\mathcal{H})$ to be A-normal is that $\mathcal{R}(TT^\sharp) \subset \overline{\mathcal{R}(A)}$ and $\|T^\sharp T u\|_A = \|TT^\sharp u\|_A$ for all $u \in \mathcal{H}$.

We recapitulate very briefly the following definitions. For more details, the interested reader is referred to [14] and the references therein.

Definition 1.4. ([14]) An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be

(1) p-hyponormal if $(T^*T)^p - (TT^*)^p \ge 0$ for $0 < p \le 1$.

(2) p-quasi-hyponormal if $T^*\bigg((T^*T)^p - (TT^*)^p\bigg)A \geq 0, \ \ 0 < p \leq 1$

(3) k-quasihyponormal operator if $T^{*k}(T^*T - TT^*)T^k \geq 0$ for positive integer k.

(4) (p, k)-quasihyponormal if $T^{*k}\bigg((T^*T)^p - (TT^*)^p\bigg)T^k \geq 0, \ 0 < p \leq 1$ and k is positive integer.

A (p, k)-quasi-hyponormal is an extension of p-hyponormal, p-quasi-hyponormal and k-quasi-hyponormal.

The contents of the paper are the following. In Section 1 we set up certain terminology that is used throughout the this paper and to list some properties which are important for the discussion of our result. In Section 2 we study the concepts of A-hyponormal and k-quasihyponormal operatros and we investigate various structural properties of this classes of operators. Finally, in Section 3 we consider the tensor product of some classes of A-operators.

2. $(A,\ k)$-QUASI-HYPONORMAL OPERATORS IN SEMI-HILBERTIAN SPACES $\big(\mathcal{H}, \langle . \mid . \rangle_A\big)$

Hyponormal and k-quasi-hyponormal operators on Hilbert spaces have received considerable attention in the current literature [6, 8, 21] and [23]. From which our inspraton cames.

In this section, we introduce the concept of A-hyponormal and (A, k)-quasi-hyponormal operators on semi-Hilbertian spaces.

2.1. A-HYPONORMAL OPERATORS.

Definition 2.1. We say that $T \in \mathcal{B}(\mathcal{H})$ is an A-positive if $AT \in \mathcal{B}(\mathcal{H})^+$ or equivalently

$$\langle Tu \mid u\rangle_A \geq 0 \ \ \forall u \ \in \mathcal{H}.$$

We note $T \geq_A 0$.

Example 2.1. *If $T \in \mathcal{B}_A(\mathcal{H})$, then $T^\sharp T$ and $TT^\sharp$ are A-positive i.e.,*

$$TT^\sharp \geq_A 0 \ \text{ and } \ T^\sharp T \geq_A 0.$$

Remark 2.1. An operator T is A-positive if and only if $A^{\frac{1}{2}}T$ is $A^{\frac{1}{2}}$-positive.

Remark 2.2. We can define a order relation by

$$T \geq_A S \Longleftrightarrow T - S \geq_A 0.$$

Remark 2.3. Inequality de Cauchy-Schwarz for A-positive operator.

It $T \in \mathcal{B}(\mathcal{H})$ is A-positive, then

$$\big|\langle Tu \mid v\rangle_A\big|^2 \leq \langle Tu \mid u\rangle_A \langle Tv \mid v\rangle_A \ \text{ for all } \ u, v \in \mathcal{H}.$$

The following lemma is useful for our study.

Lemma 2.1. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $T \geq_A S$ and let $R \in \mathcal{B}_A(\mathcal{H})$. Then following properties hold*

(1) $R^\sharp TR \geq_A R^\sharp SR$.

(2) $RTR^\sharp \geq_A RSR^\sharp$.

(3) If R is A-selfadjoint then $RTR \geq_A RSR$.

Proof. (1) Let $T \geq_A S$. Then we get the following relation for all $u \in \mathcal{H}$:

$$\begin{aligned} \langle (R^\sharp TR - R^\sharp SR)u \mid u\rangle_A &= \langle R^\sharp (T-S)Ru \mid u\rangle_A \\ &= \langle AR^\sharp (T-S)Ru \mid u\rangle \\ &= \langle R^* A(T-S)Ru \mid u\rangle \\ &= \langle A(T-S)Ru \mid Ru\rangle \\ &\geq 0. \end{aligned}$$

(2) Let $T \geq_A S$. Then we get the following relation for all $u \in \mathcal{H}$:

$$\begin{aligned} \langle (RTR^\sharp - RSR^\sharp)u \mid u\rangle_A &= \langle R(T-S)R^\sharp u \mid u\rangle_A \\ &= \langle AR(T-S)R^\sharp u \mid u\rangle \\ &= \langle (AR^\sharp)^* (T-S)R^\sharp u \mid u\rangle \\ &= \langle A(T-S)R^\sharp u \mid R^\sharp u\rangle \\ &\geq 0. \end{aligned}$$

(3) Let $T \geq_A S$. Then we get the following relation for all $u \in \mathcal{H}$:

$$\begin{aligned} \langle (RTR - RSR)u \mid u\rangle_A &= \langle R(T-S)Ru \mid u\rangle_A \\ &= \langle AR(T-S)Ru \mid u\rangle \\ &= \langle R^* A(T-S)Ru \mid u\rangle \quad (\text{since } R \text{ is } A-\text{selfadjoint}) \\ &= \langle A(T-S)Ru \mid Ru\rangle \\ &\geq 0. \end{aligned}$$

∎

Definition 2.2. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be A-hyponormal if $T^\sharp T - TT^\sharp$ is A-positive i.e., $T^\sharp T - TT^\sharp \geq_A 0$ or equivalently

$$\langle (T^\sharp T - TT^\sharp)u \mid u\rangle_A \geq 0 \text{ for all } u \in \mathcal{H}.$$

We start by our first result which provides a characterization of A-hyponormal operators.

Proposition 2.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then T is A-hyponormal if and only if*

$$\|Tu\|_A \geq \|T^\sharp u\|_A \text{ for all } u \in \mathcal{H}.$$

Proof. Assume that T is A-hyponormal, it follows that for all $u \in \mathcal{H}$

$$\begin{aligned} \langle (T^\sharp T - TT^\sharp)u \mid u\rangle_A \geq 0 &\implies \langle AT^\sharp Tu \mid u\rangle \geq \langle ATT^\sharp u \mid u\rangle \\ &\implies \langle T^* ATu \mid u\rangle \geq \langle (AT^\sharp)^* T^\sharp u \mid u\rangle \\ &\implies \langle ATu \mid Tu\rangle \geq \langle T^\sharp u \mid AT^\sharp u\rangle \\ &\implies \|Tu\|_A^2 \geq \|T^\sharp u\|_A^2. \end{aligned}$$

Conversely assume that $\|Tu\|_A^2 \geq \|T^\sharp u\|_A^2$ for all $u \in \mathcal{H}$, then we get

$$\begin{aligned} \|Tu\|_A^2 \geq \|T^\sharp u\|_A^2 &\implies \langle ATu \mid Tu\rangle \geq \langle T^\sharp u \mid AT^\sharp u\rangle \\ &\implies \langle T^* ATu \mid u\rangle \geq \langle (AT^\sharp)^* T^\sharp u \mid u\rangle \\ &\implies \langle AT^\sharp Tu \mid u\rangle \geq \langle ATT^\sharp u \mid u\rangle \\ &\implies \langle (T^\sharp T - TT^\sharp)u \mid u\rangle_A \geq 0. \end{aligned}$$

∎

Proposition 2.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$, then T is A-hyponormal if and only if*

$$T^\sharp T + 2\lambda TT^\sharp + \lambda^2 T^\sharp T \geq_A 0, \quad \textit{for all} \;\; \lambda \in \mathbb{R}.$$

Proof. Let $u \in \mathcal{H}$ and $\lambda \in \mathbb{R}$, we have that T is A-hyponormal if and only if

$$\begin{aligned} \|Tu\|_A \geq \left\|T^\sharp u\right\|_A &\iff 4\left\|T^\sharp u\right\|_A^4 - 4\|Tu\|_A^4 \leq 0 \\ &\iff \|Tu\|_A^2 + 2\lambda\left\|T^\sharp u\right\|_A^2 + \lambda^2\|Tu\|_A^2 \geq 0 \\ &\iff \left\langle AT^\sharp Tu \mid u\right\rangle + 2\lambda\left\langle ATT^\sharp u \mid u\right\rangle + 2\lambda^2\left\langle AT^\sharp Tu \mid u\right\rangle \geq 0 \\ &\iff \left\langle \left(T^\sharp T + 2\lambda TT^\sharp + 2\lambda^2 T^\sharp T\right)u \mid u\right\rangle_A \geq 0 \\ &\iff T^\sharp T + 2\lambda TT^\sharp + \lambda^2 T^\sharp T \geq_A 0. \end{aligned}$$

∎

Proposition 2.4. *Let $T \in \mathcal{B}_A(\mathcal{H})$ is A-hyponormal, then $\|Tu\|_A = \left\|T^\sharp u\right\|_A$ if and only if $\left(T^\sharp T - TT^\sharp\right)u \in \mathcal{N}(A)$.*

Proof. Assume that $\|Tu\|_A = \left\|T^\sharp u\right\|_A$. A simple computation shows that

$$\|Tu\|_A = \left\|T^\sharp u\right\|_A \Longrightarrow \left\langle\left(T^\sharp T - TT^\sharp\right)u \mid u\right\rangle_A = 0.$$

Since $T^\sharp T - TT^\sharp$ is A-positive we have for all $v \in \mathcal{H}$ by the A-Cauchy Schwarz inequality (Remark 2.3);

$$\left|\left\langle\left(T^\sharp T - TT^\sharp\right)u \mid v\right\rangle_A\right|^2 \leq \left\langle\left(T^\sharp T - TT^\sharp\right)u \mid u\right\rangle_A \left\langle\left(T^\sharp T - TT^\sharp\right)v \mid v\right\rangle_A = 0.$$

Hence, $\left\langle\left(T^\sharp T - TT^\sharp\right)u \mid v\right\rangle_A = 0 \quad \forall\, v \in \mathcal{H}$ and this implies that $A\left(T^\sharp T - TT^\sharp\right)u = 0$.

Conversely if $A\left(T^\sharp T - TT^\sharp\right)u = 0$ it is clear that $\left\langle\left(T^\sharp T - TT^\sharp\right)u \mid u\right\rangle_A = 0$ and hence

$$\|Tu\|_A = \left\|T^\sharp u\right\|_A.$$

∎

Theorem 2.5. *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(A)$ in invariant subspace for T. The following statement hold:*

(1) T and $T^\sharp$ are A-hyponormal if and only if $\|Tu\|_A = \left\|T^\sharp u\right\|_A, \;\; \forall\, u \in \mathcal{H}$.

(2) If A is injective then T an $T^\sharp$ are A-hyponormal if and only if T is A-normal.

Proof. (1) Assume that T and $T^\sharp$ are A-hyponormal.We have

$$\left(T^\sharp T - TT^\sharp\right) \geq_A 0 \;\text{ and }\; \left((T^\sharp)^\sharp T^\sharp - T^\sharp (T^\sharp)^\sharp\right) \geq_A 0.$$

Since $\mathcal{N}(A)$ in invariant subspace for T we have $TP_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}T$ and $P_{\overline{\mathcal{R}(A)}}A = AP_{\overline{\mathcal{R}(A)}} = A$. Therefore in view of the fact that $(T^\sharp)^\sharp = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ we have for all $u \in \mathcal{H}$

$$\begin{aligned} \left\langle\left((T^\sharp)^\sharp T^\sharp - T^\sharp (T^\sharp)^\sharp\right)u \mid u\right\rangle \geq_A 0 &\iff \left\langle\left(P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}T^\sharp - T^\sharp P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}\right)u \mid u\right\rangle_A \geq 0 \\ &\iff \left\langle\left(AP^2_{\overline{\mathcal{R}(A)}}TT^\sharp - AT^\sharp TP^2_{\overline{\mathcal{R}(A)}}\right)u \mid u\right\rangle \geq 0 \\ &\iff \left\langle\left(ATT^\sharp - \left(T^\sharp T\right)^* A\right)u \mid u\right\rangle \geq 0 \\ &\iff TT^\sharp \geq_A T^\sharp T. \end{aligned}$$

It follows that

$$T^\sharp T \geq_A TT^\sharp \geq_A T^\sharp T$$

and hence

$$\|Tu\|_A \geq \left\|T^\sharp u\right\|_A \geq \|Tu\|_A .$$

Conversely, assume that $\|Tu\|_A = \left\|T^\sharp u\right\|_A \quad \forall\ u \in \mathcal{H}$. From which it is clear that T is A-hyponormal and $\left\langle \left(T^\sharp T - TT^\sharp\right)u \mid u\right\rangle_A = 0\ \ \forall\, u \in \mathcal{H}$.

Now, we have

$$\begin{aligned} 0 &= \left\langle \left(T^\sharp T - TT^\sharp\right)u \mid u\right\rangle_A = \left\langle A\left(T^\sharp T - TT^\sharp\right)u \mid u\right\rangle \\ &= \left\langle \left(T^\sharp P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}} - P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}} T^\sharp\right)u \mid u\right\rangle_A \\ &= \left\langle \left(T^\sharp \left(T\sharp\right)^\sharp - \left(T^\sharp\right)^\sharp T^\sharp\right)u \mid u\right\rangle_A \\ &= \left\|T^\sharp u\right\|_A^2 - \left\|(T^\sharp)^\sharp u\right\|_A^2 . \end{aligned}$$

Thus, $T^\sharp$ is A-hyponormal.

(2) If we assume that T and $T^\sharp$ are A-hyponormal, it follows that $\|Tu\|_A = \left\|T^\sharp u\right\|_A \ \ \forall\ u \in \mathcal{H}$. Applying Proposition 2.4 and taking into account A is injective we see that $T^\sharp T - TT^\sharp = 0$. We clearly have T is A-normal.

Conversely, if T is A-normal, we have $T^\sharp$ is A-normal ([20], Corollary 3.2) and hence T and $T^\sharp$ are A-hyponormal. The proof is complete. ∎

Theorem 2.6. *Let $T \in \mathcal{B}_A(\mathcal{H})$ is A-hyponormal, then $r_A(T) = \|T\|_A$.*

Proof. Let $u \in \overline{\mathcal{R}(A)},\ u \neq 0$. Since $T \in \mathcal{B}_A(\mathcal{H})$ belongs to the A-hyponormality class, then we have

$$\begin{aligned} \|Tu\|_A^2 &= \langle Tu \mid Tu\rangle_A = \langle |AT^\sharp Tu \mid u\rangle = \langle A^{\frac{1}{2}} T^\sharp Tu \mid A^{\frac{1}{2}} u\rangle \\ &\leq |A^{\frac{1}{2}} T^\sharp Tu\| \|A^{\frac{1}{2}} u\| \\ &= \|T^\sharp Tu\|_A \|u\|_A \\ &\leq \|T^2 u\|_A \|u\|_A \end{aligned}$$

divided by $\|u\|_A^2$ we obtain

$$\frac{\|Tu\|_A^2}{\|u\|_A^2} \leq \frac{\|T^2 u\|_A \|u\|_A}{\|u\|_A^2} = \frac{\|T^2 u\|_A}{\|u\|_A}.$$

This,is tern, implies

$$\|T\|_A^2 \leq \|T^2\|_A.$$

Since we know that for $T \in \mathcal{B}_A(\mathcal{H})$,

$$\left\|T^2\right\|_A = \sup\left\{\ \left|\left\langle T^2 u \mid v\right\rangle_A\right| \ \ , \|u\|_A \leq 1,\ \|v\|_A \leq 1\right\}$$

and therefore, we have

$$\left|\langle T^2 u \mid v\rangle_A\right| = \left|\langle Tu \mid AT^\sharp v\rangle_A\right| \leq \|Tu\|_A \left\|T^\sharp v\right\|_A \leq \|T\|_A \left\|T^\sharp\right\|_A \|u\|_A \|v\|_A .$$

This shows that $\quad \|T^2\|_A \leq \|T\|_A^2 .$

So we have

$$\left\|T^2\right\|_A = \|T\|_A^2 .$$

Take any integer $n \geq 1$. Observe that

$$\begin{aligned} \|T^n u\|_A^2 &= \langle T^n u \mid T^n u\rangle_A = \left\langle T^* A T^n u \mid T^{n-1}u\right\rangle = \left\langle T^\sharp T^n u \mid T^{n-1}u\right\rangle_A \\ &\leq \left\|T^\sharp T^n u\right\|_A \left\|T^{n-1}u\right\|_A \\ &\leq \left\|T^{n+1}u\right\|_A \left\|T^{n-1}u\right\|_A, \ \ \forall \ \ u \in \mathcal{H}, \end{aligned}$$

which implies

$$\|T^n\|_A^2 \leq \left\|T^{n+1}\right\|_A \left\|T^{n-1}\right\|_A,$$

and hence

$$\frac{\|T^n\|_A}{\|T^{n-1}\|_A} \leq \frac{\|T^{n+1}\|_A}{\|T^n\|_A}.$$

Combining this with the equality above, a simple induction argument yields

$$\|T^n\|_A = \|T\|_A^n \ \ \text{for} \ \ n = 1, 2, ...$$

Consequently

$$r_A(T) = \limsup_{n \longrightarrow \infty} \|T^n\|_A^{\frac{1}{n}} = \|T\|_A.$$

∎

Theorem 2.7. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be an A-hyponormal operator such that $\mathcal{N}(A)$ is a invariant subspace for T. Then the following properties hold*

(1) $T - \lambda$ is A-hyponormal for all $\lambda \in \mathbb{C}$.

(2) If $(T - \lambda)u_0 = 0$ for $u_0 \in \mathcal{H}$, then $T^\sharp u_0 = \overline{\lambda} P_{\overline{\mathcal{R}(A)}} u_0$.

(3) If $Tu = \lambda u$ and $Tv = \mu v$ with $\lambda \neq \mu$ then $\langle u \mid v\rangle_A = 0$.

Proof. Since $T\big(\mathcal{N}(A)\big) \subset \mathcal{N}(A)$, we have that

$$P_{\overline{\mathcal{R}(A)}} T = T P_{\overline{\mathcal{R}(A)}} \ \ \text{and} \ \ A P_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}} A = A$$

(1) It suffice to prove that $\big(T - \lambda I\big)\big(T - \lambda I\big)^\sharp \geq_A \big(T - \lambda I\big)^\sharp \big(T - \lambda I\big)$. In fact

$$\begin{aligned} \big(T - \lambda I\big)\big(T - \lambda I\big)^\sharp &= \big(T - \lambda I\big)\big(T^\sharp - \overline{\lambda} P_{\overline{\mathcal{R}(A)}}\big) \\ &= T T^\sharp - \overline{\lambda} T P_{\overline{\mathcal{R}(A)}} - \lambda T^\sharp + |\lambda|^2 P_{\overline{\mathcal{R}(A)}} \\ &\geq_A T^\sharp T + \overline{\lambda} P_{\overline{\mathcal{R}(A)}} T - \lambda T^\sharp + |\lambda|^2 P_{\overline{\mathcal{R}(A)}} \\ &\geq_A \big(T - \lambda I\big)^\sharp \big(T - \lambda I\big). \end{aligned}$$

(2) From (1) we have $\|\big(T - \lambda I\big)u\|_A \geq \|\big(T - \lambda I\big)^\sharp\|_A$ for all $u \in \mathcal{H}$.
If $\big(T - \lambda I\big)u_0 = 0$ then

$$\|\big(T^\sharp - \overline{\lambda} P_{\overline{\mathcal{R}(A)}}\big)u_0\|_A = 0.$$

This implies that $\big(T^\sharp - \overline{\lambda} P_{\overline{\mathcal{R}(A)}}\big)u_0 \in \mathcal{N}(A)$. On the other hand

$$\mathcal{R}(T^\sharp - \overline{\lambda} P_{\overline{\mathcal{R}(A)}}) \subset \overline{\mathcal{R}(A)} = \mathcal{N}(A)^\perp.$$

Hence,

$$(T^\sharp - \overline{\lambda} P_{\overline{\mathcal{R}(A)}}\big)u_0 = 0.$$

(3)

$$\begin{aligned}\langle Tu \mid v\rangle_A = \langle u \mid T^\sharp v\rangle_A &\Longrightarrow \lambda\langle u \mid v\rangle_A = \langle u \mid \overline{\mu}P_{\overline{\mathcal{R}(A)}}v\rangle_A \\ &\Longrightarrow \lambda\langle u \mid v\rangle_A = \langle u \mid \overline{\mu}AP_{\overline{\mathcal{R}(A)}}v\rangle \\ &\Longrightarrow \lambda\langle u \mid v\rangle_A = \mu\langle u \mid v\rangle_A \\ &\Longrightarrow (\lambda-\mu)\langle u \mid v\rangle_A = 0\end{aligned}$$

As $\lambda \neq \mu$, it follows that $\langle u \mid v\rangle_A = 0$. ∎

Corollary 2.8. *Let $T \in \mathcal{B}_A(\mathcal{H})$ is A-hyponormal such that $\mathcal{N}(A)$ is a invariant subspace for T and let $u \in \mathcal{H}:\ \|u\|_A = 1$. Then*

$$\left\|(T-\lambda)^n u\right\|_A \geq \left\|(T-\lambda)u\right\|_A^n \ \text{ for all } \ \lambda \in \mathbb{C}.$$

Proof. Since the A-hyponormal is translation invariant, we may assume that $\lambda = 0$.

It is obvious that

$$\begin{aligned}\|Tu\|_A^2 &= \langle Tu \mid; Tu\rangle_A = \langle AT^\sharp Tu \mid u\rangle = \langle A^{\frac{1}{2}}T^\sharp Tu \mid A^{\frac{1}{2}}u\rangle \\ &\leq |A^{\frac{1}{2}}T^\sharp Tu\|\|A^{\frac{1}{2}}u\| \\ &= \|T^\sharp Tu\|_A\|u\|_A \\ &\leq \|T^2u\|_A.\end{aligned}$$

Analogously, we obtain for any positive integer n

$$\|T^n u\|_A^2 \leq \|T^{n+1}u\|_A\|T^{n-1}u\|_A.$$

To prove the corollary,we will use the induction on n. For $n = 1$, trivial. For $n = 2$, again it holds. Now assume that the result is true for any positive integer $n \geq 1$. We show that it holds for $n+1$.

From the inequality $\|T^n u\|_A^2 \leq \|T^{n+1}u\|_A\|T^{n-1}u\|_A$ and the inductions hypothesis,we have

$$\frac{\|Tu\|_A^{2n}}{\|Tu\|_A^{n-1}} \leq \frac{\|T^{n+1}u\|_A}{\|Tu\|_A^{n-1}},$$

it follows that

$$\|Tu\|_A^{n+1} \leq \left\|T^{n+1}u\right\|_A.$$

Hence by indication the result follows. ∎

Corollary 2.9. *Let $T \in \mathcal{B}_A(\mathcal{H})$ is A-hyponormal such that $\mathcal{N}(A)$ is a invariant subspace for T and let $\lambda, \mu \in \mathbb{C},\ \lambda \neq \mu$ for which there exist $(u_n)_n$ and $(v_n)_n\ \in \mathcal{H}:\ \|u_n\|_A = \|v_n\|_A = 1$.If*

$$\left\|(T-\lambda)u_n\right\|_A \longrightarrow 0 \text{ and } \left\|(T-\mu)v_n\right\|_A \longrightarrow 0 \ \text{ as } n \longrightarrow \infty$$

then

$$\langle u_n \mid v_n\rangle_A \longrightarrow 0 \ \text{ as } n \longrightarrow \infty.$$

Proof. Since the A-hyponormal is translation invariant it is obvious that

$$\left\|(T-\lambda)^\sharp u_n\right\|_A \longrightarrow 0 \text{ and } \left\|(T-\mu)^\sharp v_n\right\|_A \longrightarrow 0 \ \text{ as } n \longrightarrow \infty.$$

It is an immediate consequence of the following equalities

$$\begin{aligned}(\lambda-\mu)\langle u_n \mid v_n\rangle_A &= (\lambda-\mu)\left\langle u_n \mid \overline{\mu}P_{\overline{\mathcal{R}(A)}}v_n\right\rangle_A \\ &= \langle (T-\lambda)u_n \mid v_n\rangle_A + \left\langle u_n \mid \left(T^\sharp - \overline{\mu}P_{\overline{\mathcal{R}(A)}}\right)v_n\right\rangle_A \\ &= -\left\langle (T-\lambda)u_n \mid v_n\right\rangle_A + \left\langle u_n \mid (T-\mu)^\sharp v_n\right\rangle_A\end{aligned}$$

that

$$\langle u_n \mid v_n \rangle_A \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

∎

Consider two normal (resp.hyponormal) operators T and S on a Hilbert space. It is know that in general,TS is not normal (resp. not hyponormal). In the following theorems, we give some conditions for which the product of operators will be normal and hyponormal on semi-Hilbertian spaces.

Theorem 2.10. *Let T and $S \in \mathcal{B}_A(\mathcal{H})$ are A-normal operator, then TS and ST are A-normal if the the following statements hold*

(1) $STT^\sharp = TT^\sharp S,$

(2) $TSS^\sharp = SS^\sharp T.$

Proof. Suppose that the conditions (1) and (2) hold. It is known that for any $T, S \in \mathcal{B}_A(\mathcal{H})$, $TS \in \mathcal{B}_A(\mathcal{H})$ and $\left(TS\right)^\sharp = S^\sharp T^\sharp$. Since T and S are A-normal, we have for all $u \in \mathcal{H}$

$$\begin{aligned}
\left\|(TS)^\sharp u\right\|_A^2 &= \left\langle (TS)^\sharp u \mid A(TS)^\sharp u\right\rangle \\
&= \left\langle S^\sharp T^\sharp u \mid AS^\sharp T^\sharp u\right\rangle \\
&= \left\langle S^\sharp T^\sharp u, S^* AT^\sharp u\right\rangle \\
&= \left\langle SS^\sharp T^\sharp u, AT^\sharp u\right\rangle \\
&= \left\langle SS^\sharp T^\sharp u, T^* Au\right\rangle \\
&= \left\langle TSS^\sharp T^\sharp u, Au\right\rangle \\
&= \left\langle SS^\sharp TT^\sharp u, Au\right\rangle \\
&= \left\langle STT^\sharp u, \left(S^\sharp\right)^* Au\right\rangle \\
&= \left\langle STT^\sharp u, ASu\right\rangle \\
&= \left\langle T^\sharp TSu, ASu\right\rangle \\
&= \left\langle TSu, \left(T^\sharp\right)^* ASu\right\rangle \\
&= \langle TSu, ATSu\rangle = \|(TS)u\|_A^2 .
\end{aligned}$$

On the other hand

$$\mathcal{R}(TS(TS)^\sharp) = \mathcal{R}(SS^\sharp TT^\sharp) \subset \mathcal{R}(SS^\sharp) \subset \overline{\mathcal{R}(A)}, \quad (\text{since } S \text{ is } A-\text{normal}).$$

Hence TS is A-normal operator. A similar argument shows that ST is also A-normal operator.Therefore the proof is complete.

∎

Theorem 2.11. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ are A-hyponormal operators, The following statements hold:*

(1) If $ST^\sharp = T^\sharp S$ then TS is A-hyponormal.

(2) If $TS^\sharp = S^\sharp T$ then ST is A-hyponormal.

Proof. (1) It is known that for any $T, S \in \mathcal{B}_A(\mathcal{H})$, $TS \in \mathcal{B}_A(\mathcal{H})$ and $\big(TS\big)^\sharp = S^\sharp T^\sharp$. Suppose that $ST^\sharp = T^\sharp S$. Since T and S are A-hyponormal, we have for all $u \in \mathcal{H}$

$$\begin{aligned}\left\|\big(TS\big)^\sharp u\right\|_A^2 &= \left\langle \big(TS\big)^\sharp u \mid \big(TS\big)^\sharp u\right\rangle = \left\langle S^\sharp T^\sharp u \mid S^\sharp T^\sharp u\right\rangle_A \\ &= \left\langle AS^\sharp T^\sharp u \mid S^\sharp T^\sharp u\right\rangle = \left\langle T^\sharp u \mid SS^\sharp T^\sharp u\right\rangle_A \\ &\leq \left\langle T^\sharp u \mid S^\sharp S T^\sharp u\right\rangle_A = \left\langle T^\sharp S u \mid AT^\sharp S u\right\rangle \\ &\leq \left\|T^\sharp S u\right\|_A^2 \\ &\leq \left\|TSu\right\|_A^2.\end{aligned}$$

Hence, TS is A-hyponormal operator as required.

Finally, we have to prove statement (2). It is verified by the same way as in statement (1). ∎

2.2. (A, k)-QUASI-HYPONORMAL OPERATORS.

The operator $T \in \mathcal{B}(\mathcal{H})$ is k-quasi-hyponormal for some positive integer k if $\left\|T^* T^k u\right\| \leq \left\|T^{k+1} u\right\|$, for every $u \in \mathcal{H}$. It can be written as $T^{*k}\big(T^*T - TT^*\big)T^k \geq 0$.

By analogy with this, we could define the (A, k)-quasi-hyponormal operators in indefinite inner product spaces.

Definition 2.3. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be $(A,\ k)$-quasi-hyponormal if

$$(T^\sharp)^k\big(T^\sharp T - TT^\sharp)T^k \geq_A 0,$$

where k is a positive integer.

It is convenient to write it as

$$A(T^\sharp)^k\big(T^\sharp T - TT^\sharp)T^k \geq 0.$$

Remark 2.4. (1) $(A, 1)$-quasi-hyponrmal is A-quasi-hyponormal.

(2) From Lemma 2.1, it is clear that every A-hyponormal operator is (A, k)-quasi-hyponormal.

(3) Every (A, k)-quasi-hyponormal operator is $(A, k+1)$-quasi-hyponormal operator.

Example 2.2. *The following example shows that T is (A, k)-quasi-hyponormal normal operator that is neither$A$-normal nor A-hyponormal.*

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. *It easy to check that*

$$A \geq 0, \mathcal{R}(T^*A) \subset \mathcal{R}(A) \textit{and } \ T^\sharp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}, T^\sharp T \neq TT^\sharp \textit{and } \ \|Tu\|_A \ngeq \|T^\sharp u\|_A.$$

is neither$A$-normal nor A-hyponormal.
Moreover

$$T^{\sharp k}\big(T^\sharp T - TT^\sharp)T^k = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq_A 0.$$

So T is (A, k)-quasi-hyponormal normal operator of all positive integer k.

In the following theorem we give characterization of (A,k)-quasi-hyponormal operators.

Theorem 2.12. *Let $T\in\mathcal{B}_A(\mathcal{H})$. Then T is (A,k)-quasi-hyponormal if and only if*

$$\|T^{\sharp}T^ku\|_A\leq\|T^{k+1}u\|_A.$$

Proof.

$$\begin{aligned}(T^{\sharp})^k\big(T^{\sharp}T-TT^{\sharp})T^k\geq_A 0 &\iff \langle A\Big((T^{\sharp})^k\big(T^{\sharp}T-TT^{\sharp})T^k\Big)u\mid u\rangle\geq 0\\ &\iff \langle A(T^{\sharp})^{k+1}T^{k+1}u\mid u\rangle\geq\langle A(T^{\sharp})^kTT^{\sharp}T^ku\mid u\rangle\\ &\iff \langle T^{*k+1}AT^{k+1u}\mid u\rangle\geq\langle T^{*k}ATT^{\sharp}T^ku\mid u\rangle\\ &\iff \langle AT^{k+1}u\mid T^{k+1}u\rangle\geq\langle T^{\sharp}T^ku\mid AT^{\sharp}T^ku\rangle\\ &\iff \|T^{k+1}u\|_A^2\geq\|T^{\sharp}T^ku\|_A^2.\end{aligned}$$

∎

Lemma 2.13. *Let $T\in\mathcal{B}_A(\mathcal{H})$ be an $(A,\ k)$-quasi-hyponormal,then*

$$\|T^ku\|_A^2\leq\|T^{k+1}u\|_A\|T^{k-1}u\|_A\ \text{ for all }\ u\in\mathcal{H}.$$

Proof.

$$\begin{aligned}\|T^ku\|_A^2 &= \langle T^ku\mid T^ku\rangle_A\\ &= \langle AT^ku\mid T^ku\rangle\\ &= \langle T^*AT^ku\mid T^{k-1}u\rangle\\ &= \langle AT^{\sharp}T^ku\mid T^{k-1}u\rangle\\ &= \langle A^{\frac12}T^{\sharp}T^ku\mid A^{\frac12}T^{k-1}\rangle\\ &\leq \big\|A^{\frac12}T^{\sharp}T^ku\big\|\big\|A^{\frac12}T^{k-1}u\big\|\\ &= \|T^{\sharp}T^ku\|_A\|T^{k-1}u\|_A\\ &\leq \|T^{k+1}u\|_A\|T^{k-1}u\|_A.\end{aligned}$$

∎

In the following propositions, we give some conditions for which the product of operators will be (A,k)-quasi-hyponormal on semi-Hilbertian spaces.

Proposition 2.14. *Let $T,S\in\mathcal{B}_A(\mathcal{H})$ are A-quasi-hyponormal operators. Then the following properties hold*

(1) If $S(T^{\sharp}T)=(T^{\sharp}T)S$ and $(ST)^2=S^2T^2$, then TS is A-quasi-hyponormal.

(2) If $T(S^{\sharp}S)=(S^{\sharp}S)T$ and $(TS)^2=T^2S^2$, then ST is A-quasi-hyponormal.

Proof. (1) For all $u\in\mathcal{H}$, we have that

$$\begin{aligned}\big\|(TS)^{\sharp}TSu\big\|_A &= \big\|S^{\sharp}T^{\sharp}TSu\big\|_A=\big\|S^{\sharp}ST^{\sharp}Tu\big\|_A\\ &\leq \big\|S^2T^{\sharp}Tu\big\|_A\\ &\leq \big\|T^{\sharp}TS^2u\big\|_A\\ &\leq \big\|T^2S^2u\big\|_A\\ &= \big\|(TS)^2u\big\|_A\end{aligned}$$

(2) Same prove. ∎

Proposition 2.15. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ are $(A;k)$-quasi-hyponormal operators for some positive integer $k \geq 2$. The following statements hold*

(1) If $T^\sharp T^k S = ST^\sharp T^k$ and $T^j S^j = (TS)^j$ for $j \in \{k, k+1\}$,then TS is $(A\ ;k)$- quasi-hyponormal.

(2) If $S^\sharp S^k T = TS^\sharp S^k$ and $S^j T^j = (ST)^j$ for $j \in \{k, k+1\}$,then ST is $(A;\ k)$- quasi-hyponormal.

Proof. (1) For all $u \in \mathcal{H}$ we have

$$\begin{aligned}
\left\|(TS)^\sharp (TS)^k u\right\|_A &= \left\|S^\sharp T^\sharp T^k S^k u\right\|_A = \left\|S^\sharp S^k T^\sharp T^k u\right\|_A \\
&\leq \|S^{k+1} T^\sharp T^k u\|_A \quad \text{(since } S \text{ is } (A;k) - \text{quasi-hyponormal} \\
&\leq \left\|T^\sharp T^k S^{k+1}\right\|_A \\
&\leq \left\|T^{k+1} S^{k+1} u\right\|_A \quad \text{(since } T \text{ is } (A;k) - \text{ quasi-hyponormal)} \\
&= \left\|(TS)^{k+1} u\right\|_A .
\end{aligned}$$

Thus, TS- is $(A;k)$-quasi-hyponormal.

(2) Using the same argument as in (1) we get the desired result. ∎

3. TENSOR PRODUCTS OF k-QUASI-HYPONORMAL OPERATORS IN SEMI-HILBERTIAN SPACES

Let $\mathcal{H}\overline{\otimes}\mathcal{H}$ denote the completion, endowed with a reasonable uniform crose-norm, of the algebraic tensor product $\mathcal{H}\otimes\mathcal{H}$ of $\mathcal{H}$ with $\mathcal{H}$. Given non-zero $T, S \in \mathcal{B}(\mathcal{H})$, let $T \otimes S \in \mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})$ denote the tensor product on the Hilbert space $\mathcal{H}\overline{\otimes}\mathcal{H}$, when $T \otimes S$ is defined as follows

$$\langle T \otimes S(\xi_1 \otimes \eta_1)|\ (\xi_2 \otimes \eta_2)\rangle = \langle T\xi_1|\ \xi_2\rangle\langle S\eta_1|\ \eta_2\rangle.$$

The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in \mathcal{B}(\mathcal{H})$, but by no means all of them. Thus, whereas $T \otimes S$ is normal if and only if T and S are normal [13], there exist paranormal operators T and S such that $T \otimes S$ is not paranormal [1]. In [10], Duggal showed that if for non-zero $T, S \in \mathcal{B}(\mathcal{H}), T \otimes S$ is p-hyponormal if and only if T and S are p-hyponormal. Thus result was extended to p-quasi-hyponormal operators in [14] .

In the following study we will prove a necessary and sufficient condition for $T \otimes S$ to be A-normal,A-hyponormal and A-quasi-hyponormal, where T and S are both non-zero operators.

Recall that $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ and so, by the uniqueness of positive square roots,$|T \otimes S|^r = |T|^r \otimes |S|^r$ for any positive rational number r.From the density of of the rationales in the reals, we obtain $|T \otimes S|^p = |T|^p \otimes |S|^p$ for every positive real number p. Observe also that

$$A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I).$$

The following elementary results on tensor products of operators will be used often (and without further reference) in the sequel: $T_1 \otimes S_1 = T_2 \otimes S_2$ if and only if there exists a scalar $d \neq 0$ such that $T_1 = dT_2$ and $S_1 = d^{-1}S_2$. If T_k and S_k $(k = 1, 2)$ are positive operators, then $T_1 \otimes S_1 = T_2 \otimes S_2$ if and only if there exists a scalar $d > 0$ such that $T_1 = dT_2$ and $S_1 = d^{-1}S_2$. The proofs to these results are to be found in the papers by Hou [13] and Stochel [22].

Lemma 3.1. *Let $T_k, S_k \in \mathcal{B}(\mathcal{H}),\ k = 1, 2$ and Let $A, B \in \mathcal{B}(\mathcal{H})^+$, such that $T_1 \geq_A T_2 \geq_A 0$ and $S_1 \geq_B S_2 \geq_B 0$, then*

$$\left(T_1 \otimes S_1\right) \geq_{A\otimes B} \left(T_2 \otimes S_2\right) \geq_{A\otimes B} 0.$$

Proof. By Assumptions we have

$$\langle AT_1u|\ u\rangle \geq \langle AT_2u \mid u\rangle \geq \ \text{ and } \langle BS_1v \mid v\rangle \geq \langle S_2v|\ v\rangle \geq 0\ \ \forall\, u,\ v \in \mathcal{H}.$$

It follows that

$$\langle AT_1u \mid u\rangle\langle BS_1v|\ v\rangle\ \geq \langle AT_2u \mid u\rangle\rangle\langle BS_2v \mid v\rangle,$$

and hence

$$\langle AT_1 \otimes BS_1(u \otimes v) \mid u \otimes v\rangle \geq \langle AT_2 \otimes BS_2(u \otimes v) \mid u \otimes v\rangle.$$

Or equivalently

$$\Big\langle \big(A \otimes B\big)\big(T_1 \otimes S_1\big)(u \otimes v) \mid u \otimes v\Big\rangle \geq \Big\langle \big(A \otimes B\big)\big(T_2 \otimes S_2\big)(u \otimes v) \mid u \otimes v\Big\rangle\,.$$

Since $A \otimes B$ is positive,we deduce that

$$\Big\langle \big(T_1 \otimes S_1\big)(u \otimes v) \mid u \otimes v\Big\rangle \geq_{\big(A\otimes B\big)} \Big\langle \big(T_2 \otimes S_2\big)(u \otimes v) \mid u \otimes v\Big\rangle\,.$$

∎

The following proposition is an extension of Proposition 2.2 in [22] to the concept of A-positivity.

Proposition 3.2. *Let $T_1, T_2, S_1, S_2 \in \mathcal{B}(\mathcal{H})$ and let $A, B \in \mathcal{B}(\mathcal{H})^+$ such that T_k is A- positive and S_k is B-positive for $k = 1, 2$. If $T_1 \neq 0$ and $S_1 \neq 0$,then the following conditions are equivalents*

(1) $T_2 \otimes S_2 \geq_{A\otimes B} T_1 \otimes S_1$

(2) there exists $d > 0$ such that $dT_2 \geq_A T_1$ and $d^{-1}S_2 \geq_B S_1$.

Proof. $1 \Longrightarrow 2$. Since;

$$T_2 \otimes S_2 \geq_{A\otimes B} T_1 \otimes S_1 \Longleftrightarrow AT_2 \otimes BS_2 \geq AT_1 \otimes BS_1.$$

As AT_k and BS_k are positive operators, we deduce form [22], Proposition 2.2 $)$ that there exists a constant $d > 0$ such that

$$dAT_2 \geq AT_1 \ \text{ and } \ d^{-1}BS_2 \geq BS_1.$$

On the other hand

$$dAT_2 \geq AT_1 \Longleftrightarrow A(dT_2 - T_1) \geq 0 \Longleftrightarrow dT_2 \geq_A T_1$$

and

$$d^{-1}BS_2 \geq BS_1 \Longleftrightarrow B\big(d^{-1}S_2 - S_1\big) \geq 0 \Longleftrightarrow d^{-1}S_2 \geq_B S_1.$$

$2 \Longrightarrow 1$. This implication follows form Lemma 3.1. ∎

The following theorem gives a necessary and sufficient condition for $T \otimes S$ to be a $(A \otimes B, k)$ quasi-hyponormal operator when T and S are both nonzero operators.

Theorem 3.3. *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. If $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_B(\mathcal{H})$ are nonzero operators, then the following properties hold.*

(1) $T \otimes S$ is $(A \otimes B)$-normal if and only if T is A-normal and S is B-normal.

(2) $T \otimes S$ is $(A \otimes B)$-hyponormal if and only if T is A-hyponormal and S is B-hyponormal.

(3) $T \otimes S$ is $(A \otimes B)$-quasi-hyponormal if and only if T is A-quasi-hyponormal and S is B-quasi-hyponormal.

Proof. (1) Assume that T is A-normal and S is B-normal operators. Then

$$\begin{aligned}\big(T\otimes S\big)\big(T\otimes S\big)^{\sharp} &= \big(T\otimes S\big)\big(T^{\sharp}\otimes S^{\sharp}\big)\\ &= TT^{\sharp}\otimes SS^{\sharp}\\ &= T^{\sharp}T\otimes S^{\sharp}S\\ &= \big(T\otimes S\big)^{\sharp}\big(T\otimes S\big).\end{aligned}$$

Which implies that $T\otimes S$ is $A\otimes B$-normal operator.

Conversely, assume that $T\otimes S$ is $A\otimes B$-normal operator.We aim to show that T is A-normal and S is B-normal. Since $T\otimes S$ is a $A\otimes A$-normal operator, we have

$$\begin{aligned}(T\otimes S)\text{ is }\ (A\otimes B)-\text{normal} &\iff (T\otimes S)^{\sharp}(T\otimes S)=(T^{\sharp}\otimes S^{\sharp})(T\otimes S)\\ &\iff T^{\sharp}T\otimes S^{\sharp}S=TT^{\sharp}\otimes SS^{\sharp}\\ &\iff \exists\ d>0: T^{\sharp}T=dTT^{\sharp}\text{ and }\ S^{\sharp}S=d^{-1}SS^{\sharp}\end{aligned}$$

Passing to $\|.\|_A$ we have that

$$\|T^{\sharp}T\|_A=d\|TT^{\sharp}\|_A\quad\text{and}\quad\|S^{\sharp}S\|_B=d^{-1}\|SS^{\sharp}\|_B.$$

Since $\|T^{\sharp}T\|_A=\|TT^{\sharp}\|_A$ and $\|S^{\sharp}S\|_B=\|SS^{\sharp}\|_B$ it follows that $d=1$.

Hence T is A-normal and S is B-normal.

(2) Assume that T is A-hyponormal and S is B-hyponormal operators. Then

$$\begin{aligned}\big(T\otimes S\big)^{\sharp}\big(T\otimes S\big) &= \big(T^{\sharp}\otimes S^{\sharp}\big)\big(T\otimes S\big)\\ &= T^{\sharp}T\otimes S^{\sharp}S\\ &\geq_{A\otimes B} TT^{\sharp}\otimes SS^{\sharp}=\big(T\otimes S\big)\big(T\otimes S\big)^{\sharp}.\end{aligned}$$

Which implies that $T\otimes S$ is $A\otimes B$-hyponormal operator.

Conversely, assume that $T\otimes S$ is $A\otimes B$-hyponormal operator.We aim to show that T is A-hyponormal and S is B-hyponormal. Since $T\otimes S$ is a $A\otimes A$-hyponormal operator, we obtain

$$\begin{aligned}(T\otimes S)\text{ is }(A\otimes B)-\text{hyponormal} &\iff (T\otimes S)^{\sharp}(T\otimes S)\geq_{A\otimes B}(T\otimes S)(T\otimes S)^{\sharp}\\ &\iff T^{\sharp}T\otimes SS^{\sharp}\geq_{A\otimes B}TT^{\sharp}\otimes SS^{\sharp}.\end{aligned}$$

By Proposition 3.2 we have that there exists $d>0$ such that

$$\begin{cases} d\,T^{\sharp}T\geq_A TT^{\sharp}\\ \text{and}\\ d^{-1}S^{\sharp}S\geq_B SS^{\sharp}\end{cases}$$

a simple computation shows that $d=1$ and hence

$$T^{\sharp}T\geq_A TT^{\sharp}\quad\text{and}\quad S^{\sharp}S\geq_B SS^{\sharp}.$$

Therefore, T is A-hyponormal and S is B-hyponormal.

(3) Assume that T is A-quasi-hyponormal and S is B-quasi-hyponormal operators. Then

$$\begin{aligned} \big(T\otimes S\big)^{\sharp 2}\big(T\otimes S\big)^2 &= \big(T^{\sharp 2}\otimes S^{\sharp 2}\big(T^2\otimes S^2\big) \\ &= T^{\sharp 2}T^2\otimes S^{\sharp 2}S^2 \\ &\geq_{A\otimes B} \big(T^{\sharp}T\big)^2\big(S^{\sharp}S\big)^2 \;\; \big(\text{by Lemma 3.1}\big) \\ &\geq_{A\otimes B} \big(T^{\sharp}T\otimes S^{\sharp}S\big)^2 \\ &\geq_{A\otimes B} \Big(\big(T\otimes S\big)^{\sharp}\big(T\otimes S\big)\Big)^2. \end{aligned}$$

Which implies that $T\otimes S$ is $A\otimes B$-quasi-hyponormal operator.

Conversely, assume that $T\otimes S$ is $A\otimes B$-quasi-hyponormal operator.We aim to show that T is A-quasi-hyponormal and S is B-quasi-hyponormal. Since $T\otimes S$ is a $A\otimes A$-quasi-hyponormal operator, we have

$$\begin{aligned} & T\otimes S \;\text{ is } A\otimes B-\text{quasi-hyponormal} \\ \Longleftrightarrow\;& \big(T\otimes S\big)^{\sharp 2}\big(T\otimes S\big)^2 \geq_{A\otimes B} \big((T\otimes S)^{\sharp}(T\otimes S)\big)^2 \\ \Longleftrightarrow\;& \big(T^{\sharp 2}T^2\otimes S^{\sharp 2}S^2\big) \geq_{A\otimes B} \big((T^{\sharp}T)^2\otimes(S^{\sharp}S)^2\big) \end{aligned}$$

from Proposition 3.2 it follows that there exists $d>0$ such that

$$\begin{cases} d\, T^{\sharp 2}T^2 \geq_A \big(T^{\sharp}T\big)^2 \\ \text{and} \\ d^{-1}S^{\sharp 2}S^2 \geq_B \big(S^{\sharp}S\big)^2 \end{cases}$$

We need only to prove that $d=1$. In fact it is clear that

$$\Big\|\big(T^{\sharp}T\big)^2\Big\|_A \leq d\left\|T^{\sharp 2}T^2\right\|_A \leq d\left\|T\right\|_A^4 = d\left\|T^{\sharp}T\right\|_A^2. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned} \left\|T^{\sharp}Tu\right\|_A^2 &= \left\langle T^{\sharp}Tu \mid T^{\sharp}Tu\right\rangle_A = \left\langle \big(T^{\sharp}T\big)^*AT^{\sharp}Tu \mid u\right\rangle \\ &= \Big\langle \big(T^{\sharp}T\big)^2u \mid u\Big\rangle_A \qquad \Big(\text{since }\; T^{\sharp}T \text{ is } A-\text{selfadjoint}\Big) \\ &\leq \left\|(T^{\sharp}T)^2u\right\|_A\left\|u\right\|_A. \end{aligned}$$

This implies that

$$\sup_{\|u\|_A=1}\left\|T^{\sharp}Tu\right\|_A^2 \leq \sup_{\|u\|_A=1}\left\|(T^{\sharp}T)^2u\right\|_A$$

and hence

$$\left\|T^{\sharp}T\right\|_A^2 \leq \left\|(T^{\sharp}T)^2\right\|_A. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain that

$$\left\|(T^{\sharp}T)^2\right\|_A \leq d\left\|(T^{\sharp}T)^2\right\|_A.$$

Hence $d\geq 1$. A similar argument shows that

$$\left\|(S^{\sharp}S)^2\right\|_A \leq d^{-1}\left\|(S^{\sharp}S)^2\right\|_A.$$

Thus, $d=1$ and hence T is A-quasi-hyponormal and S is B-quasi-hyponormal. ∎

From above theorem, we can get the corollary, its proof easy can be omitted.

Corollary 3.4. *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. If $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_B(\mathcal{H})$ are nonzero operators, then $T \otimes S \in \mathcal{B}_{A\otimes B}(\mathcal{H}\overline{\otimes}\mathcal{H})$ is $(A \otimes B, k)$-is quasi-hyponormal if and only if T is (A, k)-quasi-hyponormal and S is (B, k)-quasi-hyponormal operator.*

Proposition 3.5. *If $T, S \in \mathcal{B}_A(\mathcal{H})$ are A-normal, then $TS \otimes T, TS \otimes S, ST \otimes T$ and $ST \otimes S \in \mathcal{B}_{A\otimes A}(\mathcal{H}\overline{\otimes}\mathcal{H})$ are $A \otimes A$-normal if one of the following assertions holds:*

(1) $STT^\sharp = TT^\sharp S$ *and* $TSS^\sharp = SS^\sharp T$,

(2) $ST^\sharp T = T^\sharp TS$ *and* $TS^\sharp S = S^\sharp ST$.

Proof. If we assume that the condition (1) is satisfied, then the desired results follows form Theorem 2.5 and Theorem 3.3 (1).

If the condition (2) is satisfied, we have by a simple computation that

$$\begin{aligned}
\big(TS \otimes T\big)^\sharp\big(TS \otimes T\big) &= \big(S^\sharp T^\sharp \otimes T^\sharp\big)\big(TS \otimes T\big)\\
&= \big(S^\sharp T^\sharp TS\big) \otimes \big(T^\sharp T\big)\\
&= S^\sharp ST^\sharp T \otimes T^\sharp T\\
&= SS^\sharp TT^\sharp \otimes TT^\sharp\\
&= TSS^\sharp T^\sharp \otimes TT^\sharp\\
&= \big(TS \otimes T\big)\big(TS \otimes T\big)^\sharp.
\end{aligned}$$

Hence, $TS \otimes T$ must be $A \otimes A$-normal operator. A similar argument shows that $TS \otimes S, ST \otimes T$ and $ST \otimes S$ are also $A \otimes A$-normal operators. The proof is complete. ∎

Proposition 3.6. *If $T, S \in \mathcal{B}_A(\mathcal{H})$ are A-hyponormal, then $TS \otimes T, TS \otimes S, ST \otimes T$ and $ST \otimes S \in \mathcal{B}_{A\otimes A}(\mathcal{H}\overline{\otimes}\mathcal{H})$ are $A \otimes A$-hyponormal if the following assertions hold:*

(1) $STT^\sharp = TT^\sharp S$.

(2) $TS^\sharp S = S^\sharp ST$.

Proof. Assume that the conditions (1) and (2) are hold. Since T and S are A-hyponormal, we have

$$\big(TS \otimes T\big)^\sharp\big(TS \otimes T\big) = \big(S^\sharp T^\sharp \otimes T^\sharp\big)\big(TS \otimes T\big) = \big(S^\sharp T^\sharp TS\big) \otimes \big(T^\sharp T\big).$$

Since $T^\sharp T \geq_A TT^\sharp$ it follows from Lemma 2.1 that

$$S^\sharp T^\sharp TS \geq_A S^\sharp TT^\sharp S = S^\sharp STT^\sharp = TS^\sharp ST^\sharp \geq_A TSS^\sharp T^\sharp = TS(TS)^\sharp$$

Thus,

$$\begin{cases} S^\sharp T^\sharp TS \geq_A TS(TS)^\sharp \geq_A 0 \\ \text{and} \\ T^\sharp T \geq_A TT^\sharp \geq_A 0 \end{cases}$$

Lemma 3.1 implies that

$$\big(TS \otimes T\big)^\sharp\big(TS \otimes T\big) \geq_{A\otimes A} TS(TS)^\sharp \otimes TT^\sharp = \big(TS \otimes T\big)\big(TS \otimes T\big)^\sharp.$$

In the same way, we may deduce the $A \otimes A$-hyponormality of $TS \otimes S, ST \otimes T$ and $ST \otimes S$. ∎

Proposition 3.7. *If $T, S \in \mathcal{B}_A(\mathcal{H})$ are A-quasi-hyponormal, then $TS \otimes T, TS \otimes S \in \mathcal{B}_{A\otimes A}(\mathcal{H}\overline{\otimes}\mathcal{H})$ are $A \otimes A$-quasi-hyponormal if the following assertions hold:*

(1) $ST = TS$,

(2) $STT^\sharp = TT^\sharp S$,

(3) $TS^\sharp S = S^\sharp ST$.

Proof. Assume that the conditions (1),(2) and (3) are hold. Since T and S are A-quasi-hyponormal,we have

$$\begin{aligned}\big(TS\otimes T\big)^{\sharp 2}\big(TS\otimes T\big)^2 &= \big((TS)^{\sharp 2}\otimes T^{\sharp 2}\big)\big((TS)^2\otimes T^2\big)\\ &= \big(S^{\sharp 2}T^{\sharp 2}\otimes T^{\sharp 2}\big)\big(S^2T^2\otimes T^2\big)\\ &= S^{\sharp 2}T^{\sharp 2}T^2S^2\otimes T^{\sharp 2}T^2\end{aligned}$$

Since

$$T^{\sharp 2}T^2 \geq_A \big(T^\sharp T\big)^2 \tag{3.3}$$

it follows form Lemma 2.1 that

$$S^{\sharp 2}T^{\sharp 2}T^2S^2 \geq_A S^{\sharp 2}\big(T^\sharp T\big)^2S^2. \tag{3.4}$$

From equalities (3.3) and (3.4) we deduce by using Lemma 2.1 that

$$\begin{aligned}S^{\sharp 2}T^{\sharp 2}T^2S^2\otimes T^{\sharp 2}T^2 &\geq_{A\otimes A} S^{\sharp 2}\big(T^\sharp T\big)^2S^2\otimes\big(T^\sharp T\big)^2\\ &\geq_{A\otimes A} T^\sharp T\big(S^{\sharp 2}S^2\big)T^\sharp T\otimes\big(T^\sharp T\big))2\\ &\geq_{A\otimes A} T^\sharp T\big(S^\sharp S\big)^2T^\sharp T\otimes\big(T^\sharp T\big)^2 \quad \text{(by Lemma 2.1 (3)}\\ &\geq_{A\otimes A} \big(TS\big)^{\sharp 2}\big(TS\big)^2\otimes\big(T^\sharp T\big)^2\\ &\geq_{A\otimes A} \big((TS)^\sharp(TS)\big)^2\otimes\big(T^\sharp T\big)^2\\ &\geq_{A\otimes A} \big((TS\otimes T)^\sharp(TS\otimes T)\big)^2.\end{aligned}$$

Hence $TS\otimes T$ must be a $A\otimes A$-quasi-hyponromal operator. A similar argument shows that $TS\otimes S$ is also $A\otimes A$-quasi-hyponromal operator. The proof is complete. ∎

Proposition 3.8. *If T and $S\in\mathcal{B}_A(\mathcal{H})$ are A-quasinormal, then $TS\otimes T, TS\otimes S, ST\otimes T$ and $ST\otimes S\in\mathcal{B}_{A\otimes A}(\mathcal{H}\overline{\otimes}\mathcal{H})$ are $A\otimes A$-quasinormal if the following assertions hold:*

(1) $STT^\sharp = TT^\sharp S$.

(2) $TS^\sharp S = S^\sharp ST$.

Proof. Assume that the conditions (1) and (2) are hold. Since T and S are A-quasinormal,we have

$$\begin{aligned}\big(TS\otimes T\big)\big(TS\otimes T\big)^\sharp\big(TS\otimes T\big) &= \big(TS\otimes T\big)\big((TS)^\sharp TS\otimes T^\sharp T\big)\\ &= TSS^\sharp T^\sharp TS\otimes TTT^\sharp\\ &= SS^\sharp TT^\sharp TS\otimes TT^\sharp T\\ &= SS^\sharp T^\sharp T^2S\otimes TT^\sharp T\\ &= SS^\sharp ST^\sharp T^2\otimes T^\sharp T^2\\ &= S^\sharp S^2T^\sharp T^2\otimes T^\sharp T^2\\ &= (TS)^\sharp S^2T^2\otimes T^\sharp T^2\\ &= \big(TS\otimes T\big)^\sharp\big(TS\otimes T\big)^2.\end{aligned}$$

∎

Definition 3.1. ([15]) Let $T, S \in \mathcal{B}(\mathcal{H})$.The tensor sum of T and S is the transformation $T \boxplus S : \mathcal{H}\overline{\otimes}\mathcal{H} \longrightarrow \mathcal{H}\overline{\otimes}\mathcal{H}$ defined by

$$T \boxplus S = (T \otimes I) + (I \otimes S)$$

which is an operator in $\mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})$.

Lemma 3.9. *Let $T \in \mathcal{B}_A(\mathcal{H})$. The following statements hold:*

(1) T is A-normal if and only if $T \otimes I$-is $(A \otimes I))$-normal or $(I \otimes T)$ is $(I \otimes A)$-normal.

(2) T is A-hyponormal if and only if $T \otimes I$-is $(A \otimes I))$-hyponormal or $(I \otimes T)$ is $(I \otimes A)$-hyponormal.

(3) T is A-quasi-hyponormal if and only if $T \otimes I$-is $(A \otimes I))$-quasi-hyponormal or $(I \otimes T)$ is $(I \otimes A)$-quasi-hyponormal.

Basic operations with tensor sum of Hilbert space operators are summarized in the next proposition. For its proof see ([15]).

Proposition 3.10. *Let $T, S, T_k,\ S_k \in \mathcal{B}(\mathcal{H})$ $k = 1, 2$ and $\alpha,\ \beta \in \mathbb{C}$. The following properties hold:*

(1) $(\alpha + \beta)(T \boxplus S) = \alpha T \boxplus \beta S + \beta T \boxplus \alpha S$

(2) $(T_1 + T_2) \boxplus (S_1 + S_2) = T_1 \boxplus S_1 + T_2 \boxplus S_2$

(3) $(T_1 \boxplus S_1)(T_2 \boxplus S_2) = T_1 \otimes S_2 + T_2 \otimes S_1 + T_1T_2 \boxplus S_1S_2$

(4) $(T \boxplus S)^* = T^* \boxplus S^*$.

(5) $\|T \boxplus S\| \leq \|T\| + \|S\|$.

In the following proposition we generalized the normality of $T \boxplus S$ proved in [15] to A-normality,A-hyponormality and A-quasi-hyponormality.

Theorem 3.11. *If $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(A)$ is invariant for T and S. The following properties hold:*

(1) if T and S are A-normal then $T \boxplus S$ is $(A \otimes A)$-normal.

(2) if T is and S is A-hyponormal then $T \boxplus S$ is $(A \otimes A)$-hyponormal.

(3) if T and S are A-quasi-hyponormal then $T \boxplus S$ is $(A \otimes A)$-quasi-hyponormal.

Proof. (1) Assume that T and S are A-normal. In view of the fact that

$$T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$$

we have

$$\begin{aligned}
(T \boxplus S)(T \boxplus S)^\sharp &= \big(T \otimes I + I \otimes S\big)\big(T \otimes I + I \otimes S\big)^\sharp \\
&= \big(T \otimes I + I \otimes S\big)\big(T^\sharp \otimes P_{\overline{\mathcal{R}(A)}} + P_{\overline{\mathcal{R}(A)}} \otimes S^\sharp\big) \\
&= T^\sharp T \otimes P_{\overline{\mathcal{R}(A)}} + TP_{\overline{\mathcal{R}(A)}} \otimes S^\sharp + T^\sharp \otimes SP_{\overline{\mathcal{R}(A)}} + P_{\overline{\mathcal{R}(A)}} \otimes SS^\sharp \\
&= TT^\sharp \otimes P_{\overline{\mathcal{R}(A)}} + TP_{\overline{\mathcal{R}(A)}} \otimes S^\sharp + T^\sharp \otimes SP_{\overline{\mathcal{R}(A)}} + P_{\overline{\mathcal{R}(A)}} \otimes S^\sharp S \\
&= \big(T \otimes I + I \otimes S\big)^\sharp\big(T \otimes I + I \otimes S\big) \\
&= (T \boxplus S)^\sharp(T \boxplus S)
\end{aligned}$$

It follows that $T \boxplus S$ is $A \otimes A$-hyponormal.

(2) Firstly, observe that if $T^\sharp T \geq_A TT^\sharp$ and $S^\sharp S \geq_A SS^\sharp$ then we have following inequalities

$$\big(T\otimes I\big)^\sharp\big(T\otimes I\big) \geq_{A\otimes A} \big(T\otimes I\big)\big(T\otimes I\big)^\sharp$$

and

$$\big(S\otimes I\big)^\sharp\big(S\otimes I\big) \geq_{A\otimes A} \big(S\otimes I\big)\big(S\otimes I\big).$$

Taking into account that $TP_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}T$ and $SP_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}S$ we infer

$$\begin{array}{cl} & (T\boxplus S)^\sharp(T\boxplus S)\\ = & \big(T\otimes I+I\otimes S\big)^\sharp\big(T\otimes I+I\otimes S\big)\\ = & \big(T\otimes I\big)^\sharp\big(T\otimes I\big)+\big(T\otimes I\big)^\sharp\big(I\otimes S\big)+\big(I\otimes S\big)^\sharp\big(T\otimes I\big)+\big(I\otimes S\big)^\sharp\big(I\otimes S\big)\\ \geq_{A\otimes A} & \big(T\otimes I\big)\big(T\otimes I\big)^\sharp+\big(I\otimes S\big)\big(T\otimes I\big)^\sharp+\big(T\otimes\,\big)\big(I\otimes S\big)^\sharp+\big(I\otimes S\big)\big(I\otimes S\big)^\sharp\\ \geq_{A\otimes A} & \big(T\otimes I+I\otimes S\big)\big(T\otimes I+I\otimes S\big)^\sharp\\ \geq_{A\otimes A} & \big(T\boxplus S\big)\big(T\boxplus S\big)^\sharp, \end{array}$$

we obtain the desired inequality.

(3)

$$\begin{array}{rcl} (T\boxplus S)^{\sharp 2}(T\boxplus S)^2 & = & \big(T\otimes I+I\otimes S\big)^{\sharp 2}\big(T\otimes I+I\otimes S\big)^2\\ & = & \Big(\big(T\otimes I\big)^{\sharp 2}+2\big(T\otimes I\big)^\sharp\big(I\otimes S\big)^\sharp+\big(I\otimes S\big)^{\sharp 2}\Big)\\ & & \Big(\big(T\otimes I\big)^2+2\big(T\otimes I\big)\big(I\otimes S\big)+\big(I\otimes S\big)^2\Big). \end{array}$$

Since

$$\big(T\otimes I\big)^{\sharp 2}\big(T\otimes I\big)^2 \geq_{A\otimes A} \Big(\big(T\otimes I\big)^\sharp\big(T\otimes I\big)\Big)^2$$

and

$$\big(I\otimes S\big)^{\sharp 2}\big(I\otimes S\big)^2 \geq_{A\otimes A} \Big(\big(I\otimes S\big)^\sharp\big(I\otimes S\big)\Big)^2$$

we deduce that

$$\begin{array}{rcl} (T\boxplus S)^{\sharp 2}(T\boxplus S)^2 & = & \Big(\big(T\otimes I\big)^{\sharp 2}+2\big(T\otimes I\big)^\sharp\big(I\otimes S\big)^\sharp+\big(I\otimes S\big)^{\sharp 2}\Big)\\ & & \Big(\big(T\otimes I\big)^2+2\big(T\otimes I\big)\big(I\otimes S\big)+\big(I\otimes S\big)^2\Big)\\ & \geq_{A\otimes A} & \Big(\Big(\big(T\otimes I\big)^\sharp+\big(I\otimes S\big)^\sharp\Big)\Big(T\otimes I\big)+\big(I\otimes S\big)\Big)\Big)^2\\ & \geq_{A\otimes A} & \Big(\big(T\boxplus S\big)^\sharp\big(T\boxplus S\big)\Big)^2. \end{array}$$

∎

The proof is complete.

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