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INEQUALITIES FOR THE AREA BALANCE OF FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. We introduce the *area balance* function associated to a Lebesgue integrable function $f : [a,b] \to \mathbb{C}$ by

$$AB_f(a,b,\cdot) : [a,b] \to \mathbb{C},\ AB_f(a,b,x) := \frac{1}{2}\left[\int_x^b f(t)\,dt - \int_a^x f(t)\,dt\right].$$

Several sharp bounds for functions of bounded variation are provided. Applications for Lipschitzian and convex functions are also given.

Key words and phrases: Functions of bounded variation, Lipschitzian functions, Convex functions, Integral inequalities.

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1. INTRODUCTION

For a *Lebesgue integrable* function $f:[a,b]\to\mathbb{C}$ and a number $x\in(a,b)$ we can naturally ask how far the integral $\int_x^b f(t)\,dt$ is from the integral $\int_a^x f(t)\,dt$. If f is nonnegative and continuous on $[a,b]$, then the above question has the geometrical interpretation of comparing the area under the curve generated by f at the right of the point x with the area at the left of x. The point x will be called a *median point,* if

$$\int_x^b f(t)\,dt=\int_a^x f(t)\,dt.$$

Due to the above geometrical interpretation, we can introduce the *area balance* function associated to a Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ defined as

$$AB_f(a,b,\cdot):[a,b]\to\mathbb{C},\ AB_f(a,b,x):=\frac{1}{2}\left[\int_x^b f(t)\,dt-\int_a^x f(t)\,dt\right].$$

Utilising the *cumulative function* notation $F:[a,b]\to\mathbb{C}$ given by

$$F(x):=\int_a^x f(t)\,dt$$

then we observe that

$$AB_f(a,b,x)=\frac{1}{2}F(b)-F(x),\ x\in[a,b].$$

If f is a *probability density*, i.e. f is nonnegative and $\int_a^b f(t)\,dt=1$, then

$$AB_f(a,b,x)=\frac{1}{2}-F(x),\ x\in[a,b].$$

In this paper we obtain some inequalities concerning the area balance for functions of bounded variation and Lipschitzian functions. Applications for differentiable functions and convex functions are provided. Bounds for the *Jensen difference*

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)$$

with sharps constants are also established.

Jensen difference is closely related to the Hermite-Hadamard type inequalities where various bounds for the quantities

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)$$

are provided, see [1]-[6] and [8]-[18].

2. PRELIMINARY RESULTS

The following representation result holds:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then we have the representation*

$$AB_f(a,b,x)=\left(\frac{a+b}{2}-x\right)f(x)+\frac{1}{2}\left[\int_a^x(t-a)\,df(t)+\int_x^b(b-t)\,df(t)\right] \tag{2.1}$$

and

$$AB_f(a,b,x)=\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-\frac{1}{2}\int_a^b|t-x|\,df(t) \tag{2.2}$$

for any $x\in[a,b]$, where the integrals in the right hand side are taken in the Riemann-Stieltjes sense.

Proof. We observe that since f is of bounded variation, then the Riemann-Stieltjes integrals involved in (2.1) and (2.2) exist.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}&\int_a^x(t-a)\,df(t)+\int_x^b(b-t)\,df(t)\\&=(t-a)f(t)\big|_a^x-\int_a^x f(t)\,dt+(b-t)f(t)\big|_x^b+\int_x^b f(t)\,dt\\&=(x-a)f(x)-\int_a^x f(t)\,dt-(b-x)f(x)+\int_x^b f(t)\,dt\\&=(2x-a-b)f(x)+2AB_f(a,b,x)\end{aligned} \tag{2.3}$$

for any $x\in[a,b]$.

Dividing (2.3) by 2 and rearranging the equation, we deduce (2.1).

Integrating by parts, we also have

$$\begin{aligned}\int_a^b|t-x|\,df(t)&=\int_a^x(x-t)\,df(t)+\int_x^b(t-x)\,df(t)\\&=(x-t)f(t)\big|_a^x+\int_a^x f(t)\,dt+(t-x)f(t)\big|_x^b-\int_x^b f(t)\,dt\\&=-(x-a)f(a)+(b-x)f(b)-2AB_f(a,b,x)\\&=bf(b)+af(a)-[f(b)+f(a)]\,x-2AB_f(a,b,x)\end{aligned} \tag{2.4}$$

for any $x\in[a,b]$.

Dividing (2.4) by 2 and rearranging the equation, we deduce (2.2). ∎

Corollary 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$. Then*

$$\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x\ge AB_f(a,b,x)\ge\left(\frac{a+b}{2}-x\right)f(x) \tag{2.5}$$

for any $x\in[a,b]$.

In particular,

$$\frac{1}{4}(b-a)[f(b)-f(a)] \geq AB_f\left(a,b,\frac{a+b}{2}\right) \geq 0. \tag{2.6}$$

The constant $\frac{1}{4}$ *is a best possible constant in the sense that it cannot be replaced by a smaller quantity.*

Proof. The inequalities (2.5) follow from the representations (2.1) and (2.2) by taking into account that f is monotonic nondecreasing.

The inequality (2.6) follows by (2.5) for $x=\frac{a+b}{2}$.

Consider the function $f:[a,b]\to\mathbb{R}$ given by

$$f(t):=\begin{cases} 0 & \text{if } x\in\left[a,\frac{a+b}{2}\right) \\ \\ 1 & \text{if } x\in\left[\frac{a+b}{2},b\right]. \end{cases}$$

This function is monotonic nondecreasing on $[a,b]$,

$$\frac{1}{4}(b-a)[f(b)-f(a)]=\frac{1}{4}(b-a)$$

and

$$\begin{aligned} AB_f\left(a,b,\frac{a+b}{2}\right) &= \frac{1}{2}\left[\int_{\frac{a+b}{2}}^{b} f(t)\,dt-\int_a^{\frac{a+b}{2}} f(t)\,dt\right] \\ &= \frac{1}{2}\left[\left(b-\frac{a+b}{2}\right)-0\right]=\frac{1}{4}(b-a), \end{aligned}$$

which shows that the equality case is realized in the first inequality in (2.6). That proves the sharpness of the constant $\frac{1}{4}$. ∎

Remark 2.1. If $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and nonnegative (nonpositive) on $[a,b]$ then $AB_f(a,b,x)\geq 0$ for $x\in\left[a,\frac{a+b}{2}\right]\left(\left[\frac{a+b}{2},b\right]\right)$.

If $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, $f(b)\neq -f(a)$ and

$$\frac{bf(b)+af(a)}{f(b)+f(a)}\in[a,b] \tag{2.7}$$

then

$$AB_f\left(a,b,\frac{bf(b)+af(a)}{f(b)+f(a)}\right)\leq 0. \tag{2.8}$$

If $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and $f(a)>0$, then (2.7) holds and the inequality (2.8) is valid.

3. Bounds for Functions of Bounded Variation

For a function of bounded variation $v:[a,b]\to\mathbb{C}$ we define the *Cumulative Variation Function* (CVF) $V:[a,b]\to[0,\infty)$ by

$$V(t):=\bigvee_a^t(v),$$

the total variation of v on the interval $[a,t]$ with $t\in[a,b]$.

It is know that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous in a point $c \in [a,b]$ if and only if the generating function v is continuing in that point. If v is Lipschitzian with the constant $L > 0$, i.e.

$$|v(t) - v(s)| \le L|t - s| \text{ for any } t, s \in [a,b]$$

then V is also Lipschitzian with the same constant.

The following lemma is of interest in itself as well, see [7] for a simple proof and related results.

Lemma 3.1. *Let $f, u : [a,b] \to \mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)\,du(t)\right| \le \int_a^b |f(t)|\, d\left(\bigvee_a^t (u)\right) \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u). \tag{3.1}$$

We can state the first results as follows:

Theorem 3.2. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned} &\left|AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right) f(x)\right| \\ &\le AB_{\bigvee_a^{\cdot}(f)}(a,b,x) - \left(\frac{a+b}{2} - x\right)\bigvee_a^x (f) \\ &= \frac{1}{2}\left[\int_a^x \left(\bigvee_t^x (f)\right) dt + \int_x^b \left(\bigvee_x^t (f)\right) dt\right] \\ &\le \frac{1}{2}\left[(x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f)\right] \\ &\le \frac{1}{2} \times \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right](b-a), \end{cases} \end{aligned} \tag{3.2}$$

for any $x \in [a,b]$.

Proof. From the equality (2.1) and by Lemma 3.1 we have

$$\begin{aligned} &\left|AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right) f(x)\right| \\ &\le \frac{1}{2}\left|\int_a^x (t-a)\,df(t) + \int_x^b (b-t)\,df(t)\right| \\ &\le \frac{1}{2}\left[\left|\int_a^x (t-a)\,df(t)\right| + \left|\int_x^b (b-t)\,df(t)\right|\right] \\ &\le \frac{1}{2}\left[\int_a^x (t-a)\, d\left(\bigvee_a^t (f)\right) + \int_x^b (b-t)\, d\left(\bigvee_x^t (f)\right)\right] \end{aligned} \tag{3.3}$$

for any $x \in [a,b]$.

Since for $t \ge x$ we have $\bigvee_x^t (f) = \bigvee_a^t (f) - \bigvee_a^x (f)$, then

$$\int_x^b (b-t)\, d\left(\bigvee_x^t (f)\right) = \int_x^b (b-t)\, d\left(\bigvee_a^t (f)\right)$$

and by (3.3) we have

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right| \leq \frac{1}{2}\left[\int_a^x (t-a)\,d\left(\bigvee_a^t(f)\right)+\int_x^b (b-t)\,d\left(\bigvee_a^t(f)\right)\right] \tag{3.4}$$

for any $x\in[a,b]$.

Now, on utilizing the representation (2.1) for the CVF $\bigvee_a^{\cdot}(f)$ we have

$$\frac{1}{2}\left[\int_a^x (t-a)\,d\left(\bigvee_a^t(f)\right)+\int_x^b (b-t)\,d\left(\bigvee_a^t(f)\right)\right] = AB_{\bigvee_a^{\cdot}(f)}(a,b,x)-\left(\frac{a+b}{2}-x\right)\bigvee_a^x(f) \tag{3.5}$$

for any $x\in[a,b]$, we deduce from (3.4) the first inequality in (3.2).

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_a^x (t-a)\,d\left(\bigvee_a^t(f)\right) = (t-a)\bigvee_a^t(f)\Big|_a^x-\int_a^x\left(\bigvee_a^t(f)\right)dt = (x-a)\bigvee_a^x(f)-\int_a^x\left(\bigvee_a^t(f)\right)dt = \int_a^x\left(\bigvee_a^x(f)-\bigvee_a^t(f)\right)dt=\int_a^x\left(\bigvee_t^x(f)\right)dt \tag{3.6}$$

and

$$\int_x^b (b-t)\,d\left(\bigvee_x^t(f)\right) = (b-t)\bigvee_x^t(f)\Big|_x^b+\int_x^b\left(\bigvee_x^t(f)\right)dt = \int_x^b\left(\bigvee_x^t(f)\right)dt \tag{3.7}$$

for any $x\in[a,b]$.

Then

$$\frac{1}{2}\left[\int_a^x (t-a)\,d\left(\bigvee_a^t(f)\right)+\int_x^b (b-t)\,d\left(\bigvee_x^t(f)\right)\right] = \frac{1}{2}\left[\int_a^x\left(\bigvee_t^x(f)\right)dt+\int_x^b\left(\bigvee_x^t(f)\right)dt\right],$$

which proves the equality in (3.2).

Since $\bigvee_t^x(f)\leq\bigvee_a^x(f)$ for $t\in[a,x]$ and $\bigvee_x^t(f)\leq\bigvee_x^b(f)$ for $t\in[x,b]$, then

$$\int_a^x\left(\bigvee_t^x(f)\right)dt+\int_x^b\left(\bigvee_x^t(f)\right)dt\leq(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)$$

for any $x\in[a,b]$, which proves the second inequality in (3.2).

The last part is obvious by the max properties and the fact that for $c, d \in \mathbb{R}$ we have

$$\max\{c, d\} = \frac{c + d + |c - d|}{2}.$$

The details are omitted. ∎

Corollary 3.3. *With the assumptions of Theorem 3.2 we have the inequality*

$$\left|AB_f\left(a, b, \frac{a+b}{2}\right)\right| \le AB_{\bigvee_a^{\cdot}(f)}\left(a, b, \frac{a+b}{2}\right) \tag{3.8}$$
$$= \frac{1}{2}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f)\right)dt + \int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f)\right)dt\right]$$
$$\le \frac{1}{4}(b-a)\bigvee_a^b(f).$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *are best possible in (3.8).*

Proof. Consider the function $f : [a, b] \to \mathbb{R}$ given by

$$f(t) := \begin{cases} 0 & \text{if } x \in \left[a, \frac{a+b}{2}\right) \\ 1 & \text{if } x \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

This function is of bounded variation on $[a, b]$, $\bigvee_a^b(f) = 1$,

$$\bigvee_t^{\frac{a+b}{2}}(f) = 1 \text{ for any } t \in \left[a, \frac{a+b}{2}\right),$$

$$\bigvee_{\frac{a+b}{2}}^t(f) = 0 \text{ for any } t \in \left[\frac{a+b}{2}, b\right],$$

$$AB_f\left(a, b, \frac{a+b}{2}\right) = \frac{1}{2}\left[\int_{\frac{a+b}{2}}^b f(t)\,dt - \int_a^{\frac{a+b}{2}} f(t)\,dt\right]$$
$$= \frac{1}{2}\left[\left(b - \frac{a+b}{2}\right) - 0\right] = \frac{1}{4}(b-a),$$

and

$$\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f)\right)dt + \int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f)\right)dt = \frac{1}{2}(b-a).$$

Replacing this function in the inequality (3.8) we obtain in all terms the same quantity $\frac{1}{4}(b-a)$. ∎

Theorem 3.4. *Let* $f:[a,b]\to\mathbb{C}$ *be a function of bounded variation on* $[a,b]$*. Then*

$$
\begin{aligned}
(3.9)\qquad & \left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \\
&\leq \frac{1}{2}(b-x)\bigvee_a^b(f)-AB_{\bigvee_a^{\cdot}(f)}(a,b,x) \\
&=\frac{1}{2}\left[\int_a^x\left(\bigvee_a^t(f)\right)dt+\int_x^b\left(\bigvee_t^b(f)\right)dt\right] \\
&\leq \frac{1}{2}\left[(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)\right] \\
&\leq \frac{1}{2}\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f), \\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right](b-a),\end{cases}
\end{aligned}
$$

for any $x\in[a,b]$.

Proof. Taking the modulus in the equality (2.2) and utilizing Lemma 3.1 we have

$$
\begin{aligned}
(3.10)\qquad & \left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \\
&=\frac{1}{2}\left|\int_a^b|t-x|\,df(t)\right|\leq\frac{1}{2}\int_a^b|t-x|\,d\left(\bigvee_a^t(f)\right) \\
&=\frac{1}{2}\left[\int_a^x(x-t)\,d\left(\bigvee_a^t(f)\right)+\int_x^b(t-x)\,d\left(\bigvee_a^t(f)\right)\right]
\end{aligned}
$$

for any $x\in[a,b]$.

Utilising the identity (2.2) for the CVF $\bigvee_a^{\cdot}(f)$ we also have

$$
\int_a^b|t-x|\,d\left(\bigvee_a^t(f)\right)=\frac{1}{2}(b-x)(f)-AB_{\bigvee_a^{\cdot}(f)}(a,b,x)\geq 0
$$

and the first inequality in (3.9) is proved.

Integrating by parts in the Riemann-Stieltjes integral we have

$$
\begin{aligned}
(3.11)\qquad \int_a^x(x-t)\,d\left(\bigvee_a^t(f)\right) &=(x-t)\left(\bigvee_a^t(f)\right)\Bigg|_a^x+\int_a^x\left(\bigvee_a^t(f)\right)dt \\
&=\int_a^x\left(\bigvee_a^t(f)\right)dt
\end{aligned}
$$

and

$$\int_x^b (t-x)\, d\left(\bigvee_a^t (f)\right) = (t-x)\left(\bigvee_a^t (f)\right)\Bigg|_x^b - \int_x^b \left(\bigvee_a^t (f)\right) dt$$
$$= (b-x)\left(\bigvee_a^b (f)\right) - \int_x^b \left(\bigvee_a^t (f)\right) dt$$
$$= \int_x^b \left(\bigvee_a^b (f) - \bigvee_a^t (f)\right) dt = \int_x^b \left(\bigvee_t^b (f)\right) dt \tag{3.12}$$

for any $x \in [a,b]$.

Making use of (3.11) and (3.12) we get the equality case in (3.9).

Since $\bigvee_a^{\cdot}$ is monotonic nondecreasing on $[a,b]$ while $\bigvee_{\cdot}^b$ is nonincreasing in the same interval, we have

$$\int_a^x \left(\bigvee_a^t (f)\right) dt \le (x-a) \bigvee_a^x (f) \text{ and } \int_x^b \left(\bigvee_t^b (f)\right) dt \le (b-x) \bigvee_x^b (f),$$

for any $x \in [a,b]$, which gives the second inequality in (3.9).

Using the properties of the maximum, we have

$$(x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f)$$
$$\le \begin{cases} \max\{x-a, b-x\} \bigvee_a^b (f) \\ \max\left\{\bigvee_a^x (f), \bigvee_x^b (f)\right\}(b-a) \end{cases}$$
$$= \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right](b-a) \end{cases}$$

for any $x \in [a,b]$, and the proof is complete. ∎

Corollary 3.5. *With the assumptions of Theorem 3.4 we have*

$$\left|\frac{1}{4}(b-a)[f(b)-f(a)] - AB_f\left(a,b,\frac{a+b}{2}\right)\right|$$
$$\le \frac{1}{4}(b-a)\bigvee_a^b (f) - AB_{\bigvee_a^{\cdot}(f)}\left(a,b,\frac{a+b}{2}\right)$$
$$= \frac{1}{2}\left[\int_a^{\frac{a+b}{2}} \left(\bigvee_a^t (f)\right) dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_t^b (f)\right) dt\right]$$
$$\le \frac{1}{4}(b-a)\bigvee_a^b (f). \tag{3.13}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible in (3.13).

Proof. Consider the function $f:[a,b]\to\mathbb{R}$ given by

$$f(t):=\begin{cases}0 & \text{if } x=a\\ 1 & \text{if } x\in\left(\frac{a+b}{2},b\right].\end{cases}$$

This function is of bounded variation on $[a,b]$, $\bigvee_a^b(f)=1$,

$$\bigvee_a^t(f)=1 \text{ for any } t\in\left[a,\frac{a+b}{2}\right),$$

$$\bigvee_t^b(f)=0 \text{ for any } t\in\left[\frac{a+b}{2},b\right],$$

$$\begin{aligned}AB_f\left(a,b,\frac{a+b}{2}\right)&=\frac{1}{2}\left[\int_{\frac{a+b}{2}}^b f(t)\,dt-\int_a^{\frac{a+b}{2}}f(t)\,dt\right]\\&=\frac{1}{2}\left[\left(b-\frac{a+b}{2}\right)-\left(\frac{a+b}{2}-a\right)\right]=0,\end{aligned}$$

and

$$\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(f)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(f)\right)dt\right]=\frac{1}{2}(b-a).$$

Replacing this function in the inequality (3.13) we obtain in all terms the same quantity $\frac{1}{4}(b-a)$. ∎

4. Bounds for Lipschitzian Functions

If v is *Lipschitzian* with the constant $L>0$, i.e.

$$|v(t)-v(s)|\le L|t-s| \text{ for any } t,s\in[a,b]$$

then, it is well known that for any Riemann integrable function $g:[a,b]\to\mathbb{C}$ the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and

$$\left|\int_a^b g(t)\,dv(t)\right|\le L\int_a^b|g(t)|\,dt. \tag{4.1}$$

Theorem 4.1. *If $f:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[a,b]$, then*

$$\begin{aligned}&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right|\\&\le\frac{1}{2}L\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\end{aligned}\tag{4.2}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)\right|\le\frac{1}{8}L(b-a)^2. \tag{4.3}$$

The constant $\frac{1}{8}$ is best possible in (4.3).

Proof. Taking the modulus in the equality (2.1) and utilizing the property (4.1) we have

$$\begin{aligned}
&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right| \\
&\le \frac{1}{2}\left|\int_a^x (t-a)\,df(t)+\int_x^b (b-t)\,df(t)\right| \\
&\le \frac{1}{2}\left[\left|\int_a^x (t-a)\,df(t)\right|+\left|\int_x^b (b-t)\,df(t)\right|\right] \\
&\le \frac{1}{2}L\left[\int_a^x (t-a)\,dt+\int_x^b (b-t)\,dt\right] \\
&= \frac{1}{4}L\left[(x-a)^2+(b-x)^2\right]
\end{aligned} \tag{4.4}$$

for any $x\in[a,b]$.

Since

$$\frac{1}{2}\left[(x-a)^2+(b-x)^2\right]=\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2$$

for any $x\in[a,b]$, then by (4.4) we deduce the desired inequality (4.2).

Consider the function $f:[a,b]\to\mathbb{R}$, $f(t)=t$. The function f is Lipschitzian with the constant $L=1$ and

$$\begin{aligned}
AB_f\left(a,b,\frac{a+b}{2}\right) &= \frac{1}{2}\left[\int_{\frac{a+b}{2}}^b tdt-\int_a^{\frac{a+b}{2}} tdt\right] \\
&= \frac{1}{2}\left[\frac{b^2-\left(\frac{a+b}{2}\right)^2}{2}-\frac{\left(\frac{a+b}{2}\right)^2-a^2}{2}\right] \\
&= \frac{1}{4}\left[b^2+a^2-2\left(\frac{a+b}{2}\right)^2\right] \\
&= \frac{1}{8}(b-a)^2.
\end{aligned}$$

If we replace this function in (4.3), then we obtain in both sides the same quantity $\frac{1}{8}(b-a)^2$. ∎

The following result also holds:

Theorem 4.2. *If $f:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \\
&\le \frac{1}{2}L\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]
\end{aligned} \tag{4.5}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|\frac{1}{4}(b-a)\left[f(b)-f(a)\right]-AB_f\left(a,b,\frac{a+b}{2}\right)\right|\le\frac{1}{8}L(b-a)^2. \tag{4.6}$$

The constant $\frac{1}{8}$ is best possible in (4.6).

Proof. Taking the modulus in the equality (2.2) and utilizing the property (4.1) we have

$$\begin{aligned}
&\left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \\
&=\frac{1}{2}\left|\int_a^b |t-x|\,df(t)\right|\le\frac{1}{2}L\int_a^b|t-x|\,dt \\
&=\frac{1}{2}L\left[\int_a^x (x-t)\,dt+\int_x^b (t-x)\,dt\right] \\
&=\frac{1}{4}L\left[(x-a)^2+(b-x)^2\right] \\
&=\frac{1}{2}L\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]
\end{aligned}$$

for any $x\in[a,b]$ and the inequality (4.5) is proved.

Consider the function $f:[a,b]\to\mathbb{R}$, $f(t)=t$. The function f is Lipschitzian with the constant $L=1$ and, utilizing the calculation in Theorem 4.1 we have

$$\begin{aligned}
&\frac{1}{4}(b-a)[f(b)-f(a)]-AB_f\left(a,b,\frac{a+b}{2}\right) \\
&=\frac{1}{4}(b-a)^2-\frac{1}{8}(b-a)^2=\frac{1}{8}(b-a)^2.
\end{aligned}$$

Replacing this function (4.6) we get in both sides the same quantity $\frac{1}{8}(b-a)^2$. ∎

5. Applications for Differentiable Functions

The following approximation for differentiable functions can be stated:

Proposition 5.1. *Let $g:[a,b]\to\mathbb{C}$ be a differentiable function and such that the derivative g' is of locally bounded variation on (a,b). Then we have the representation*

$$\begin{aligned}
g(x)&=\frac{g(a)+g(b)}{2}+\left(x-\frac{a+b}{2}\right)g'(x) \\
&\quad-\frac{1}{2}\left[\int_a^x (t-a)\,dg'(t)+\int_x^b (b-t)\,dg'(t)\right]
\end{aligned}\tag{5.1}$$

and the bound

$$\left|g(x)-\frac{g(a)+g(b)}{2}-\left(x-\frac{a+b}{2}\right)g'(x)\right| \tag{5.2}$$
$$\leq AB_{\bigvee_a^{\cdot}(g')}(a,b,x)-\left(\frac{a+b}{2}-x\right)\bigvee_a^x(g')$$
$$=\frac{1}{2}\left[\int_a^x\left(\bigvee_t^x(g')\right)dt+\int_x^b\left(\bigvee_x^t(g')\right)dt\right]$$
$$\leq\frac{1}{2}\left[(x-a)\bigvee_a^x(g')+(b-x)\bigvee_x^b(g')\right]$$
$$\leq\frac{1}{2}\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(g'),\\ \left[\frac{1}{2}\bigvee_a^b(g')+\frac{1}{2}\left|\bigvee_a^x(g')-\bigvee_x^b(g')\right|\right](b-a).\end{cases}$$

If g' is Lipschitzian with the constant $K>0$ on (a,b), then we also have

$$\left|g(x)-\frac{g(a)+g(b)}{2}-\left(x-\frac{a+b}{2}\right)g'(x)\right| \tag{5.3}$$
$$\leq\frac{1}{2}K\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right].$$

Proof. Since $AB_f(a,b,x)=\frac{1}{2}F(b)-F(x)$, where $F(x):=\int_a^x f(t)\,dt$, then by (2.1) we have

$$F(x)=\frac{1}{2}F(b)-\left(\frac{a+b}{2}-x\right)f(x) \tag{5.4}$$
$$-\frac{1}{2}\left[\int_a^x(t-a)\,df(t)+\int_x^b(b-t)\,df(t)\right]$$

for any $x\in[a,b]$.

If we choose in (5.4) $f=g'$ and perform the required calculations, we get the representation (5.1).

The inequality (5.2) follows from (3.2) while (5.3) follows from (4.2). ∎

Remark 5.1. If g is a differentiable function and such that the derivative g' is of locally bounded variation on (a,b), then by the inequality (5.2) we have

$$\left|\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)\right| \tag{5.5}$$
$$\leq AB_{\bigvee_a^{\cdot}(g')}\left(a,b,\frac{a+b}{2}\right)$$
$$=\frac{1}{2}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(g')\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(g')\right)dt\right]$$
$$\leq\frac{1}{4}(b-a)\bigvee_a^b(g').$$

The constant $\frac{1}{2}$ is best possible in the first inequality (5.5).

Indeed, if we consider the function $g : [a, b] \to \mathbb{R}$, $g(t) = t^2$ then $g'(t) = 2t$ and

$$\left|\frac{g(a)+g(b)}{2} - g\left(\frac{a+b}{2}\right)\right| = \frac{(b-a)^2}{4},$$

$$\bigvee_{t}^{\frac{a+b}{2}}(g') = 2\left(\frac{a+b}{2} - t\right), \quad \bigvee_{\frac{a+b}{2}}^{t}(g') = 2\left(t - \frac{a+b}{2}\right)$$

while

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left(\bigvee_{t}^{\frac{a+b}{2}}(g')\right)dt + \int_{\frac{a+b}{2}}^{b}\left(\bigvee_{\frac{a+b}{2}}^{t}(g')\right)dt \\
&= 2\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2} - t\right)dt + 2\int_{\frac{a+b}{2}}^{b}\left(t - \frac{a+b}{2}\right)dt \\
&= \frac{(b-a)^2}{4} + \frac{(b-a)^2}{4} = \frac{(b-a)^2}{2}.
\end{aligned}$$

Replacing these values in the first inequality in (5.5) we get in both sides the same quantity $\frac{(b-a)^2}{4}$.

Remark 5.2. If g' is Lipschitzian with the constant $K > 0$ on (a, b), then we also have

$$\left|\frac{g(a)+g(b)}{2} - g\left(\frac{a+b}{2}\right)\right| \le \frac{1}{8}K(b-a)^2. \tag{5.6}$$

The constant $\frac{1}{8}$ is best possible in (5.6).

Indeed, if we take $g : [a, b] \to \mathbb{R}$, $g(t) = t^2$, then $g'(t) = 2t$ which is Lipschitzian with the constant $K = 2$. Moreover,

$$\left|\frac{g(a)+g(b)}{2} - g\left(\frac{a+b}{2}\right)\right| = \frac{(b-a)^2}{4}$$

and replacing in (5.6) we get in both sides the same quantity $\frac{(b-a)^2}{4}$.

Proposition 5.2. *Let $g : [a, b] \to \mathbb{C}$ be a differentiable function and such that the derivative g' is of locally bounded variation on (a, b). Then we have the representation*

$$\begin{aligned}
g(x) = {} & \frac{g(a)+g(b)}{2} - \frac{bg'(b)+ag'(a)}{2} + \frac{g'(b)+g'(a)}{2}x \\
& + \frac{1}{2}\int_a^b |t-x|\, dg'(t)
\end{aligned} \tag{5.7}$$

and the bound

$$\left|g(x)-\frac{g(a)+g(b)}{2}+\frac{bg'(b)+ag'(a)}{2}-\frac{g'(b)+g'(a)}{2}x\right| \tag{5.8}$$
$$\leq \frac{1}{2}(b-x)\bigvee_a^b(g')-AB_{\bigvee_a^{\cdot}(g')}(a,b,x)$$
$$=\frac{1}{2}\left[\int_a^x\left(\bigvee_a^t(g')\right)dt+\int_x^b\left(\bigvee_t^b(g')\right)dt\right]$$
$$\leq \frac{1}{2}\left[(x-a)\bigvee_a^x(g')+(b-x)\bigvee_x^b(g')\right]$$
$$\leq \frac{1}{2}\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(g'),\\[2mm] \left[\frac{1}{2}\bigvee_a^b(g')+\frac{1}{2}\left|\bigvee_a^x(g')-\bigvee_x^b(g')\right|\right](b-a).\end{cases}$$

If g' is Lipschitzian with the constant $K>0$ on (a,b), then we also have

$$\left|g(x)-\frac{g(a)+g(b)}{2}+\frac{bg'(b)+ag'(a)}{2}-\frac{g'(b)+g'(a)}{2}x\right| \tag{5.9}$$
$$\leq \frac{1}{2}K\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]$$

for any $x\in[a,b]$.

Proof. By the equality (2.2) we have

$$F(x)=\frac{1}{2}F(b)-\frac{bf(b)+af(a)}{2}+\frac{f(b)+f(a)}{2}x \tag{5.10}$$
$$+\frac{1}{2}\int_a^b|t-x|\,f'(t)\,dt$$

for any $x\in[a,b]$.

If we choose in (5.10) $f=g'$ and perform the required calculations, we get the representation (5.7).

The rest follows from (3.9) and (4.5). ∎

Remark 5.3. If g is a differentiable function and such that the derivative g' is of locally bounded variation on (a,b), then by the inequality (5.8) we have

$$\left|\frac{1}{4}(b-a)[g'(b)-g'(a)]-\frac{g(a)+g(b)}{2}+g\left(\frac{a+b}{2}\right)\right| \tag{5.11}$$
$$\leq \frac{1}{4}(b-a)\bigvee_a^b(g')-AB_{\bigvee_a^{\cdot}(g')}\left(a,b,\frac{a+b}{2}\right)$$
$$=\frac{1}{2}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(g')\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(g')\right)dt\right]$$
$$\leq \frac{1}{4}(b-a)\bigvee_a^b(g').$$

The constant $\frac{1}{2}$ is best possible in the first inequality in (5.11).

Indeed, if we consider the function $g : [a,b] \to \mathbb{R}$, $g(t) = t^2$ we have

$$\begin{aligned}&\left|\frac{1}{4}(b-a)[g'(b)-g'(a)] - \frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right| \\ &= \frac{1}{4}(b-a)^2\end{aligned}$$

and

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t (g')\right)dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_t^b (g')\right)dt \\ &= 2\int_a^{\frac{a+b}{2}}(t-a)\,dt + 2\int_{\frac{a+b}{2}}^b (b-t)\,dt \\ &= \frac{1}{4}(b-a)^2 + \frac{1}{4}(b-a)^2 = \frac{1}{2}(b-a)^2,\end{aligned}$$

which gives in the both sides of the first inequality in (5.11) the same quantity $\frac{1}{4}(b-a)^2$.

Remark 5.4. If g' is Lipschitzian with the constant $K > 0$ on (a,b), then we also have

$$\begin{aligned}&\left|\frac{1}{4}(b-a)[g'(b)-g'(a)] - \frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{8}K(b-a)^2.\end{aligned} \tag{5.12}$$

The constant $\frac{1}{8}$ is best possible in (5.12).

We observe that the equality is realized in (5.12) if we take the function $g : [a,b] \to \mathbb{R}$, $g(t) = t^2$. The details are omitted.

6. Applications for Convex Functions

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \quad \text{for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \quad \text{for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

Utilising these notations, we can state, for a convex function $f : I \to \mathbb{R}$ and $a, b \in \mathring{I}$ with $a < b$, the following identities

$$f(x) = \frac{f(a) + f(b)}{2} + \left(x - \frac{a+b}{2}\right) \varphi(x) - \frac{1}{2}\left[\int_a^x (t-a)\, d\varphi(t) + \int_x^b (b-t)\, d\varphi(t)\right] \tag{6.1}$$

and

$$f(x) = \frac{f(a) + f(b)}{2} - \frac{b\varphi(b) + a\varphi(a)}{2} + \frac{\varphi(b) + \varphi(a)}{2} x + \frac{1}{2}\int_a^b |t-x|\, d\varphi(t). \tag{6.2}$$

If f is differentiable and convex on $\mathring{I}$, then we can replace φ by f'.

We have the following inequalities for a convex function $f : I \to \mathbb{R}$ and $a, b \in \mathring{I}$ with $a < b$ and $\varphi \in \partial f$:

$$0 \le \frac{f(a) + g(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\int_a^b \left|\varphi(t) - \varphi\left(\frac{a+b}{2}\right)\right| dt \le \frac{1}{4}(b-a)\left[\varphi(b) - \varphi(a)\right], \tag{6.3}$$

and

$$0 \le \frac{1}{4}(b-a)\left[\varphi(b) - \varphi(a)\right] - \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[\int_a^{\frac{a+b}{2}} \left[\varphi(t) - \varphi(a)\right] dt + \int_{\frac{a+b}{2}}^b \left[\varphi(b) - \varphi(t)\right] dt\right]. \tag{6.4}$$

The constant $\frac{1}{2}$ is best possible in (6.3) and (6.4).

If φ is Lipschitzian with the constant $K > 0$, then

$$0 \le \frac{f(a) + g(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{1}{8} K (b-a)^2, \tag{6.5}$$

and

$$0 \le \frac{1}{4}(b-a)\left[\varphi(b) - \varphi(a)\right] - \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \le \frac{1}{8} K (b-a)^2. \tag{6.6}$$

The constant $\frac{1}{8}$ is best possible in both inequalities (6.5) and (6.6).

7. APPLICATIONS FOR MEANS

Consider the function $f_p : [a,b] \to (0,\infty)$ defined by $f_p(t) = t^p$ with $p \in \mathbb{R}\backslash\{-1\}$. Then

$$\begin{aligned} AB_{f_p}(a,b,x) &= \frac{1}{2}\left(\int_x^b t^p dt - \int_a^x t^p dt\right) \\ &= \frac{1}{2}\left(\frac{b^{p+1} - x^{p+1}}{p+1} - \frac{x^{p+1} - a^{p+1}}{p+1}\right) \\ &= \frac{1}{p+1}\left[A\left(b^{p+1}, a^{p+1}\right) - x^{p+1}\right] \end{aligned}$$

for $x \in [a,b]$, where $A(c,d) := \frac{c+d}{2}$ is the *arithmetic-mean* of the nonnegative numbers $c, d.$

If $f_{-1} : [a,b] \to (0,\infty)$ is defined by $f_{-1}(t) = t^{-1}$, then

$$\begin{aligned} AB_{f_{-1}}(a,b,x) &= \frac{1}{2}\left(\int_x^b \frac{1}{t} dt - \int_a^x \frac{1}{t} dt\right) \\ &= \frac{1}{2}\left[\ln\left(\frac{b}{x}\right) - \ln\left(\frac{x}{a}\right)\right] = \ln\left[\frac{G(a,b)}{x}\right], \end{aligned}$$

for $x \in [a,b]$, where $A(c,d) := \sqrt{cd}$ is the *geometric-mean* of the positive numbers $c, d.$

For $p \geq 1$ we have $f_p'(t) = pt^{p-1}$ and since

$$\sup_{t\in[a,b]} \left|f_p'(t)\right| = pb^{p-1}$$

then f_p' is Lipschitzian with the constant $L_p = pb^{p-1}.$

From the inequality (4.2) we get

$$\begin{aligned} &\left|\frac{1}{p+1}\left[A\left(b^{p+1}, a^{p+1}\right) - x^{p+1}\right] - \left[A(a,b) - x\right] x^p\right| \\ &\leq \frac{1}{2} pb^{p-1}\left[\frac{1}{4}(b-a)^2 + \left[x - A(a,b)\right]^2\right], \end{aligned} \tag{7.1}$$

while from (4.5) we have

$$\begin{aligned} &\left|A\left(b^{p+1}, a^{p+1}\right) - A\left(b^p, a^p\right) x - \frac{1}{p+1}\left[A\left(b^{p+1}, a^{p+1}\right) - x^{p+1}\right]\right| \\ &\leq \frac{1}{2} pb^{p-1}\left[\frac{1}{4}(b-a)^2 + \left[x - A(a,b)\right]^2\right] \end{aligned} \tag{7.2}$$

for any $x \in [a,b]$.

Similar inequalities may be obtained for $p \in (0,1)\setminus\{-1\}$.

If we take $x = A(a,b)$ in (7.1) and (7.2), then we get

$$0 \leq A\left(b^{p+1}, a^{p+1}\right) - A^{p+1}(a,b) \leq \frac{1}{8} p(p+1) b^{p-1}(b-a)^2 \tag{7.3}$$

and

$$\begin{aligned} 0 &\leq A\left(b^{p+1}, a^{p+1}\right) - A\left(b^p, a^p\right) A(a,b) \\ &\quad - \frac{1}{p+1}\left[A\left(b^{p+1}, a^{p+1}\right) - A^{p+1}(a,b)\right] \\ &\leq \frac{1}{8} pb^{p-1}(b-a)^2. \end{aligned} \tag{7.4}$$

We also have $f'_{-1}(t) = -t^{-2}$ and since

$$\sup_{t\in[a,b]} \left|f'_{-1}(t)\right| = \frac{1}{a^2}$$

then from the inequality (4.2) we get

$$\left|\ln\left[\frac{G(a,b)}{x}\right] - [A(a,b) - x]\,x^{-1}\right| \leq \frac{1}{2a^2}\left[\frac{1}{4}(b-a)^2 + [x - A(a,b)]^2\right] \tag{7.5}$$

while from (4.5) we have

$$\left|1 - H^{-1}(a,b)\,x^{-1} - \ln\left[\frac{x}{G(a,b)}\right]\right| \leq \frac{1}{2a^2}\left[\frac{1}{4}(b-a)^2 + [x - A(a,b)]^2\right] \tag{7.6}$$

for any $x \in [a,b]$. Here $H(a,b) := \frac{2ab}{a+b}$ denotes the *harmonic-mean* of the positive numbers $a, b > 0.$

If we take $x = A(a,b)$ in (7.5) and (7.6), then we get

$$0 \leq \ln\left[\frac{A(a,b)}{G(a,b)}\right] \leq \frac{1}{8}\left(\frac{b}{a} - 1\right)^2 \tag{7.7}$$

and

$$0 \leq 1 - H^{-1}(a,b)\,A^{-1}(a,b) - \ln\left[\frac{A(a,b)}{G(a,b)}\right] \leq \frac{1}{8}\left(\frac{b}{a} - 1\right)^2. \tag{7.8}$$

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