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PROPERTIES OF q-GAMMA AND q-BETA FUNCTIONS DERIVED FROM THE q-GAUSS-PÓLYA INEQUALITIES

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ABSTRACT. We consider log-convexity and other properties of several functions related to q-gamma and q-beta functions. These properties are consequences of the general inequality, so-called q-analogue of the Gauss-Pólya inequality. Various inequalities involving these special functions are also given.

Key words and phrases: the Jackson integral, log-convexity, q-beta function, q-gamma function, q-Gauss-Pólya inequalities.

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1. INTRODUCTION

Twenty years ago a unified treatment of several inequalities related to moments of distribution was given in papers [4] and [5]. Precisely, the following theorem was proved:

Theorem 1.1. *Let $f : [a,b] \to \mathbf{R}$, $b \in \langle a, \infty]$, be a non-negative non-decreasing function. Let $p_i \ \ (i = 1, \dots, n)$ be positive real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. If $a_i \ (i = 1, \dots, n)$ are real numbers such that $a_i > -\frac{1}{p_i}$, then*

$$\int_a^b x^{a_1+\cdots+a_n} f(x)dx \geq \frac{\prod_{i=1}^{n}(a_i p_i + 1)^{\frac{1}{p_i}}}{1+\sum_{i=1}^{n} a_i} \prod_{i=1}^{n} \left(\int_a^b x^{a_i p_i} f(x)dx \right)^{\frac{1}{p_i}} \tag{1.1}$$

provided that all integrals exist. If $a = 0$ and f is non-increasing, then the reverse inequality in (1.1) holds.

This theorem generalizes some results from the famous book "Problems and Theorems in Analysis" by Pólya and Szegö. Also, as a consequence of it, the Gauss-Winckler inequality is arised :

$$((s+1)m_s)^{1/s} \leq ((r+1)m_r)^{1/r}, \ \ s \leq r$$

where $m_r = \int_0^\infty x^r f(x)dx$, f is a non-negative continuous non-increasing function on $[0, \infty\rangle$ such that $\int_0^\infty f(x)dx = 1$. For particular values: $s = 2$, $r = 4$ this inequality was given by Gauss as an inequality between the second and the fourth moments. Nowadays inequalities similar to (1.1) are called inequalities of Gauss-Pólya type. In this paper we give a q-analogue of Theorem 1.1 and apply it for getting new results for the q-gamma and q-beta functions.

We use notation and definitions from the book [2] and the article [1]. Let $q \in \langle 0, 1\rangle$ be a fixed real number. The q-derivative $D_q f(x)$ of a function f at point x is a quotient

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

D_q is a linear operator. The definite q-integral of a function $f : [0, c\rangle \to \mathbb{R}$, $c \in \langle 0, \infty]$ is defined as

$$\int_0^b f(x) d_q x = (1-q)b \sum_{k=0}^{\infty} q^k f(bq^k)$$

for $b \in \langle 0, c\rangle$, provided that the series on the right-hand side converges. This kind of integral was defined by F.H. Jackson at the beginning of the XXth century and very often it is called the Jackson integral. If we are interested only on integral $\int_0^b$ then the interval $[0, c\rangle$ as a domain of a function f can be substituted with a set $\mathbb{T}_b = \{q^n b \ : \ n \in \mathbb{N}_0\} \cup \{0\}$.

The q-analogue $[a]_q$ of a real number a is defined by

$$[a]_q = \frac{1-q^a}{1-q}.$$

If $t \in \mathbb{R}$, then $D_q x^t = [t]_q x^{t-1}$. Formula for integration by parts is based on the q-analogue of the Leibniz rule $D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x)$ and it has a form

$$\int_0^b f(x) D_q g(x) d_q x + \int_0^b g(qx) D_q f(x) d_q x = f(b)g(b) - \lim_{n\to\infty} (fg)(q^n b). \tag{1.2}$$

The structure of this paper is the following: after these introductory section we give two q-Gauss-Pólya inequalities. In the third section log-convexity of some functions is discussed and several general inequalities are given. The last section is devoted to applications for q-gamma and q-beta functions.

2. THE q-GAUSS-PÓLYA INEQUALITIES

Before the main result let us give the weighted Hölder inequality for q-integral which is one of main tools using in the proof of the main theorem.

Proposition 2.1. *(The weighted Hölder inequality for q-integral.) Let $p_i \quad (i = 1, \ldots, n)$ be positive real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Let $w, f_i : \mathbb{T}_b \to \mathbf{R}$ be non-negative functions such that integrals $\int_0^b f_1(x) \cdot \ldots \cdot f_n(x) w(x) d_q x$, $\int_0^b f_i^{p_i}(x) w(x) d_q x$, $(i = 1, \ldots, n)$ exist. Then*

$$\int_0^b f_1(x) \cdot \ldots \cdot f_n(x) w(x) d_q x \leq \prod_{i=1}^{n} \left(\int_0^b f_i^{p_i}(x) w(x) d_q x \right)^{\frac{1}{p_i}}. \tag{2.1}$$

Proof. The proof is based on the following discrete Hölder inequality: If $p_i \quad (i = 1, \ldots, n)$ are positive real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ and $(w_k)_k, (f_{ik})_k \quad (i = 1, \ldots, n)$ are non-negative sequences such that sums $\sum_{k=0}^{\infty} w_k f_{ik}^{p_i}$, $\sum_{k=0}^{\infty} w_k f_{1k} \ldots f_{nk}$ are finite, then

$$\sum_{k=0}^{\infty} w_k f_{1k} \ldots f_{nk} \leq \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} w_k f_{ik}^{p_i} \right)^{\frac{1}{p_i}}. \tag{2.2}$$

It is a consequence of more general results given in [3, p.135, p.143]. Putting in (2.2):

$$w_k = q^k w(bq^k),\ f_{ik} = f_i(bq^k),\ (i = 1, \ldots, n)$$

and multiplying with $(1-q)b$ we get

$$(1-q)b \sum_{k=0}^{\infty} q^k w(bq^k) f_1(bq^k) \ldots f_n(bq^k) \leq \prod_{i=1}^{n} \left((1-q)b \sum_{k=0}^{\infty} q^k w(bq^k) f_i^{p_i}(bq^k) \right)^{\frac{1}{p_i}}$$

which is equivalent to (2.1). ∎

The first theorem is devoted to a function f which q-derivative has one sign on $\mathbb{T}_b$, while in the second theorem we deal with a function f which the second q-derivative is non-positive.

Theorem 2.2. *Let $f : \mathbb{T}_b \to \mathbf{R}$ be a non-negative function, $b > 0$. Let $p_i \quad (i = 1, \ldots, n)$ be positive real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ and let $a_i \ (i = 1, \ldots, n)$ be real numbers such that $a_i > -\frac{1}{p_i}$.*

If q-derivative $D_q f \geq 0$ on $\mathbb{T}_b$, then

$$\int_0^b x^{a_1+\cdots+a_n} f(x) d_q x \geq \frac{\prod_{i=1}^{n} [a_i p_i + 1]_q^{\frac{1}{p_i}}}{[1 + a_1 + \cdots + a_n]_q} \prod_{i=1}^{n} \left(\int_0^b x^{a_i p_i} f(x) d_q x \right)^{\frac{1}{p_i}} \tag{2.3}$$

provided that all above-mentioned integrals exist.

If $D_q f \leq 0$ on $\mathbb{T}_b$, then the reverse inequality in (2.3) holds.

Proof. Suppose that $D_q f \geq 0$. The inequality (2.3) reduces to an equality for $f \equiv 0$ and thus we may, without loss of generality, assume that $f(b) > 0$. Since $D_q(x^{\alpha+1}) = [\alpha]_q x^{\alpha}$ applying integration by parts we conclude

$$\left[1 + \sum_{i=1}^{n} a_i \right]_q \int_0^b x^{a_1+\cdots+a_n} f(x) d_q x = \int_0^b D_q(x^{a_1+\cdots+a_n+1}) f(x) d_q x$$

$$\begin{aligned}
&= f(b)b^{a_1+\cdots+a_n+1} - \int_0^b (qx)^{a_1+\cdots+a_n+1} D_q f(x) d_q x \\
&= f(b)b^{a_1+\cdots+a_n+1} - \int_0^b \prod_{i=1}^n (qx)^{\frac{a_i p_i+1}{p_i}} D_q f(x) d_q x \\
&\geq f(b)b^{a_1+\cdots+a_n+1} - \prod_{i=1}^n \left(\int_0^b (qx)^{a_i p_i+1} D_q f(x) d_q x \right)^{1/p_i},
\end{aligned}$$

where in the last inequality we use the q-weighted Hölder inequality (2.1) with a non-negative weight $D_q f(x)$ and functions $f_i(x) = (qx)^{\frac{a_i p_i+1}{p_i}}$, $(i = 1, \ldots, n)$.

Let us consider the Popoviciu inequality [3, p. 118]:

$$\sum_{j=1}^m w_j a_{j1} \cdot \ldots \cdot a_{jn} \geq \prod_{i=1}^n (\sum_{j=1}^m w_j a_{ji}^{p_i})^{1/p_i},$$

where $w_1 > 0, w_2, \ldots, w_m \leq 0, a_{ji} \geq 0$ for $j = 1, \ldots, m, i = 1, \ldots, n$, $p_i > 0$ such that $\sum_{i=1}^n \frac{1}{p_i} = 1$ and $\sum_{j=1}^m w_j a_{ji}^{p_i} \geq 0$ for $i = 1, \ldots, n$.
Set:

$$m = 2, \quad w_1 = f(b) > 0, \quad w_2 = -1, \quad a_{1i} = b^{\frac{a_i p_i+1}{p_i}},$$

$$a_{2i} = \left(\int_0^b (qx)^{a_i p_i+1} D_q f(x) d_q x \right)^{1/p_i} \quad \text{for } i = 1, \ldots, n.$$

Using the Popoviciu inequality and integration by parts we obtain

$$\begin{aligned}
&f(b)b^{a_1+\ldots+a_n+1} - \prod_{i=1}^n \left(\int_0^b (qx)^{a_i p_i+1} D_q f(x) d_q x \right)^{1/p_i} \\
&\geq \prod_{i=1}^n \left(f(b)b^{a_i p_i+1} - \int_0^b (qx)^{a_i p_i+1} D_q f(x) d_q x \right)^{1/p_i} \\
&= \prod_{i=1}^n \left([a_i p_i + 1]_q \int_0^b x^{a_i p_i} f(x) d_q x \right)^{1/p_i}
\end{aligned}$$

and so, (2.3) is proven.

If $D_q f \leq 0$, then the first inequality is obtained by applying the Hölder inequality with the non-negative weight $w(x) = -D_q f(x) \geq 0$, and the second inequality is obtained by applying the Hölder inequality for discrete case instead of the Popoviciu inequality and the proof is similar to the previous one. ∎

Theorem 2.3. *Let $f : \mathbb{T}_1 \to \mathbf{R}$ be a non-negative function such that $D_q^2 f \leq 0$. Let p_i $(i = 1, \ldots, n)$ be positive real numbers such that $\sum_{i=1}^n \frac{1}{p_i} = 1$ and let a_i $(i = 1, \ldots, n)$ be real numbers such that $a_i > -\frac{1}{p_i}$.*

If $\int_0^1 x^{a_i p_i} f(x) d_q x$, $(i = 1, \ldots, n)$ exist, then

$$\int_0^1 x^{a_1+\cdots+a_n} f(x) d_q x \geq \frac{\prod_{i=1}^n [a_i p_i + 2]_q^{\frac{1}{p_i}} [a_i p_i + 1]_q^{\frac{1}{p_i}}}{[2 + a_1 + \cdots + a_n]_q [1 + a_1 + \cdots + a_n]_q} \prod_{i=1}^n \left(\int_0^1 x^{a_i p_i} f(x) d_q x \right)^{\frac{1}{p_i}}.$$

Proof. Using the inequality between the geometric mean and the arithmetic mean and then integration by parts twice, we get

$$\begin{aligned}
\prod_{i=1}^{n} & [a_ip_i+2]_q^{\frac{1}{p_i}}[a_ip_i+1]_q^{\frac{1}{p_i}} \prod_{i=1}^{n}\left(\int_0^1 x^{a_ip_i} f(x) d_qx\right)^{\frac{1}{p_i}} \\
& \le \sum_{i=1}^{n} \frac{1}{p_i}[a_ip_i+2]_q[a_ip_i+1]_q \int_0^1 x^{a_ip_i} f(x) d_qx \\
& = \sum_{i=1}^{n} \frac{1}{p_i}[a_ip_i+2]_q \left(f(1) - \int_0^1 (qx)^{a_ip_i+1} D_q f(x) d_qx\right) \\
& = f(1)\sum_{i=1}^{n} \frac{1}{p_i}[a_ip_i+2]_q - \sum_{i=1}^{n} \frac{1}{p_i} q^{a_ip_i+1}\left(D_qf(1) - \int_0^1 (qx)^{a_ip_i+2} D_q^2 f(x) d_qx\right). \qquad (2.4)
\end{aligned}$$

Using a definition of $[a_ip_i+2]_q$ and inequality between the geometric mean and the arithmetic mean we get

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{p_i}[a_ip_i+2]_q & = \sum_{i=1}^{n} \frac{1}{p_i}\frac{1-q^{a_ip_i+2}}{1-q} = \frac{1}{1-q}\left(\sum_{i=1}^{n}\frac{1}{p_i} - \sum_{i=1}^{n}\frac{1}{p_i}q^{a_ip_i+2}\right) \\
& \le \frac{1}{1-q}\left(1 - \prod_{i=1}^{n} q^{\frac{a_ip_i+2}{p_i}}\right) = \frac{1}{1-q}\left(1 - q^{2+a_1+\cdots+a_n}\right) \\
& = [2+a_1+\cdots+a_n]_q.
\end{aligned}$$

So, (2.4) is less than or equal to the following

$$\begin{aligned}
& f(1)[2+\sum_{i=1}^{n} a_i]_q - D_qf(1)q^{1+\sum_{i=1}^n a_i} + \int_0^1 \frac{1}{q}\left(\sum_{i=1}^{n}\frac{1}{p_i}(q^2x)^{a_ip_i+2}\right) D_q^2 f(x) d_qx \\
& \le f(1)[2+\sum_{i=1}^{n} a_i]_q - D_qf(1)q^{1+\sum_{i=1}^n a_i} + \int_0^1 \frac{1}{q}\left(\prod_{i=1}^{n}(q^2x)^{\frac{a_ip_i+2}{p_i}}\right) D_q^2 f(x) d_qx \\
& = f(1)[2+\sum_{i=1}^{n} a_i]_q - D_qf(1)q^{1+\sum_{i=1}^n a_i} + \int_0^1 q^{\sum_{i=1}^n a_i+1}(qx)^{\sum_{i=1}^n a_i+2} D_q^2 f(x) d_qx \\
& = [2+a_1+\cdots+a_n]_q[1+a_1+\cdots+a_n]_q \int_0^1 x^{a_1+\cdots+a_n} f(x) d_qx,
\end{aligned}$$

where in the first inequality we again use the inequality between the geometric mean and the arithmetic mean and in the last line we use integration by parts. ∎

3. LOG-CONVEXITY AND CONSEQUENCES

In this section we consider functions m_r, F and G defined as

$$r \mapsto m_r = \int_0^b x^r f(x) d_qx$$

$$F(r) = [r+1]_q m_r = [r+1]_q \int_0^b x^r f(x) d_qx$$

and

$$G(r) = [r+2]_q[r+1]_q m_r = [r+2]_q[r+1]_q \int_0^1 x^r f(x) d_qx.$$

By I we denote a domain of definition of these functions. Under certain assumptions these functions are log-convex or log-concave and since that, several inequalities hold for them.

Theorem 3.1. *a) If f is non-negative on $\mathbb{T}_b$, then the function m_r is log-convex.*

b) If f is non-negative such that $D_q f \leq 0$ on $\mathbb{T}_b$, then the function F is log-convex. If $D_q f \geq 0$, then F is a log-concave function.

c) If f is non-negative such that $D_q^2 f \leq 0$ on $\mathbb{T}_1$, then the function G is log-concave.

Proof. (a) Let $\lambda \in \langle 0, 1\rangle$, $r_1, r_2 \in I$. Using the weighted q-Hölder inequality (2.1) for $n = 2$, $f_1(x) = x^{\lambda r_1}$, $f_2(x) = x^{(1-\lambda) r_2}$, $p_1 = \frac{1}{\lambda}$, $p_2 = \frac{1}{1-\lambda}$ and $w = f$ we get

$$\int_0^b x^{\lambda r_1 + (1-\lambda) r_2} f(x) d_q x \leq \left(\int_0^b x^{r_1} f(x) d_q x \right)^{\lambda} \left(\int_0^b x^{r_2} f(x) d_q x \right)^{1-\lambda}$$

i.e.

$$m_{\lambda r_1 + (1-\lambda) r_2} \leq m_{r_1}^{\lambda} \cdot m_{r_2}^{1-\lambda}. \tag{3.1}$$

If $\lambda = 0$ or $\lambda = 1$, then (3.1) holds with equality. So we conclude that m_r is log-convex.

b) Let $\lambda \in \langle 0, 1\rangle$, $r_1, r_2 \in I$ and $D_q f \leq 0$. Setting in the reversed (2.3):

$$n = 2,\ p_1 = \frac{1}{\lambda},\ p_2 = \frac{1}{1-\lambda},\ a_1 = \lambda r_1,\ a_2 = (1-\lambda) r_2,$$

we get

$$[1 + \lambda r_1 + (1-\lambda) r_2]_q \int_0^b x^{\lambda r_1 + (1-\lambda) r_2} f(x) d_q x$$

$$\leq [r_1 + 1]_q^{\lambda} [r_2 + 1]_q^{1-\lambda} \left(\int_0^b x^{r_1} f(x) d_q x \right)^{\lambda} \left(\int_0^b x^{r_2} f(x) d_q x \right)^{1-\lambda}$$

i.e.

$$F(\lambda r_1 + (1-\lambda) r_2) \leq F(r_1)^{\lambda} \cdot F(r_2)^{1-\lambda}$$

what means that F is log-convex.

c) We prove it in the similar manner as in part b) using Theorem 2.3. ∎

Theorem 3.2. *Let f is non-negative, $p, r, t, u, v, w \in I$. Then the first inequality holds in each of the parts a), b), c), d) and e) below. If additionally, $D_q f \leq 0$ on $\mathbb{T}_b$, then the second inequality holds in each of the parts below. If f is non-negative and $D_q^2 f \leq 0$ on $\mathbb{T}_1$, then the third inequality holds in each of the parts below with $b = 1$ as an upper bound of integral m_r.*

a) If $p > u > r$, then

$$m_u^{p-r} \leq m_r^{p-u} m_p^{u-r}, \tag{3.2}$$

$$([u+1]_q m_u)^{p-r} \leq ([r+1]_q m_r)^{p-u} ([p+1]_q m_p)^{u-r}, \tag{3.3}$$

$$([u+2]_q [u+1]_q m_u)^{p-r} \geq ([r+2]_q [r+1]_q m_r)^{p-u} ([p+2]_q [p+1]_q m_p)^{u-r}.$$

b) If $p \geq u$, $r \geq v$ and $p \neq r$, $u \neq v$, then

$$\left(\frac{m_p}{m_r} \right)^{\frac{1}{p-r}} \geq \left(\frac{m_u}{m_v} \right)^{\frac{1}{u-v}}, \tag{3.4}$$

$$\left(\frac{[p+1]_q m_p}{[r+1]_q m_r} \right)^{\frac{1}{p-r}} \geq \left(\frac{[u+1]_q m_u}{[v+1]_q m_v} \right)^{\frac{1}{u-v}}, \tag{3.5}$$

$$\left(\frac{[p+2]_q [p+1]_q m_p}{[r+2]_q [r+1]_q m_r} \right)^{\frac{1}{p-r}} \leq \left(\frac{[u+2]_q [u+1]_q m_u}{[v+2]_q [v+1]_q m_v} \right)^{\frac{1}{u-v}}. \tag{3.6}$$

c) If $p \geq u$, $p, u \neq 0$, $0 \in I$, then

$$\left(\frac{m_p}{m_0}\right)^{1/p} \geq \left(\frac{m_u}{m_0}\right)^{1/u},$$

$$\left(\frac{[p+1]_q m_p}{m_0}\right)^{1/p} \geq \left(\frac{[u+1]_q m_u}{m_0}\right)^{1/u}, \tag{3.7}$$

$$\left(\frac{[p+2]_q[p+1]_q m_p}{[2]_q m_0}\right)^{1/p} \leq \left(\frac{[u+2]_q[u+1]_q m_u}{[2]_q m_0}\right)^{1/u}. \tag{3.8}$$

d) If $r_1, \dots, r_n > 0$, $r_1 + \cdots + r_n + r, r_i + r \in I$ then

$$m_r^{n-1} m_{r_1+\cdots+r_n+r} \geq m_{r_1+r} \cdot \ldots \cdot m_{r_n+r}, \tag{3.9}$$

$$\begin{aligned}&([r+1]_q m_r)^{n-1}[r_1+\cdots+r_n+r+1]_q m_{r_1+\cdots+r_n+r}\\ &\geq [r_1+r+1]_q \ldots [r_n+r+1]_q m_{r_1+r} \ldots m_{r_n+r},\end{aligned} \tag{3.10}$$

$$\begin{aligned}&([r+2]_q[r+1]_q m_r)^{n-1}[r_1+\cdots+r_n+r+2]_q[r_1+\cdots+r_n+r+1]_q m_{r_1+\cdots+r_n+r}\\ &\leq [r_1+r+2]_q \ldots [r_n+r+2]_q[r_1+r+1]_q \ldots [r_n+r+1]_q m_{r_1+r} \ldots m_{r_n+r}.\end{aligned} \tag{3.11}$$

e) If $u > v > r > p$, $p \leq w \leq u$, then

$$m_p^{\frac{u-w}{u-p}} m_u^{\frac{w-p}{u-p}} \geq m_r^{\frac{v-w}{v-r}} m_v^{\frac{w-r}{v-r}},$$

$$([p+1]_q m_p)^{\frac{u-w}{u-p}}([u+1]_q m_u)^{\frac{w-p}{u-p}} \geq ([r+1]_q m_r)^{\frac{v-w}{v-r}}([v+1]_q m_v)^{\frac{w-r}{v-r}}, \tag{3.12}$$

$$\begin{aligned}&([p+2]_q[p+1]_q m_p)^{\frac{u-w}{u-p}}([u+1]_q[u+2]_q m_u)^{\frac{w-p}{u-p}}\\ &\leq ([r+2]_q[r+1]_q m_r)^{\frac{v-w}{v-r}}([v+2]_q[v+1]_q m_v)^{\frac{w-r}{v-r}}.\end{aligned}$$

If $D_q f \geq 0$, then reverse inequalities in (3.3), (3.5), (3.7), (3.10), (3.12) hold.

Proof. Function m_r is log-convex and if $D_q f \leq 0$, then the function F is log-convex also. If $D_q^2 f \leq 0$, then the function G is log-concave. So, we use various inequalities for convex/concave function to get considered results.

a) The following inequality holds for a convex function H ([3, p.1])

$$\begin{vmatrix} H(p) & H(u) & H(r) \\ p & u & r \\ 1 & 1 & 1 \end{vmatrix} \geq 0 \quad \text{for } p > u > r.$$

Setting in the above inequality $H(r) = \log m_r$ we obtain

$$(p-r)\log m_u \leq (p-u)\log m_r + (u-r)\log m_p$$

from which we get (3.2). The second inequality appears if we put $H = \log F$. The third inequality appears if we put $H = \log G$.

b) For any convex function H the following inequality

$$\frac{H(p)-H(r)}{p-r} \geq \frac{H(u)-H(v)}{u-v}$$

holds for $p \geq u$ and $r \geq v$, $u \neq v, p \neq r$, [3, p.2]. Therefore, (3.4) and (3.5) are simple consequences of previous inequality if we set $H = \log m_r$ and $H = \log F$. If H is concave, then inequality for H holds with reversed sign and we get (3.6) putting $H = \log G$.

c) Setting in (3.4), (3.5) and (3.6): $r = v = 0$ we get desired inequalities.

d) Putting in (3.4): $p = r_1 + \ldots + r_n + r$, $u = r_i + r$, $v = r$ we get

$$\left(\frac{m_{r_1+\ldots+r_n+r}}{m_r}\right)^{\frac{1}{r_1+\ldots+r_n}} \geq \left(\frac{m_{r_i+r}}{m_r}\right)^{\frac{1}{r_i}}$$

i.e.

$$\left(\frac{m_{r_1+\ldots+r_n+r}}{m_r}\right)^{\frac{r_i}{r_1+\ldots+r_n}} \geq \frac{m_{r_i+r}}{m_r}$$

for $i = 1, 2, \ldots, n$. Multiplying all these inequalities we get (3.9). Inequalities (3.10), (3.11) are proved in the similar manner using inequalities (3.7), (3.8) respectively.

e) This is a consequence of Narumi's inequality [4]:

$$\frac{u-w}{u-p}H(p) + \frac{w-p}{u-p}H(u) \geq \frac{v-w}{v-r}H(r) + \frac{w-r}{v-r}H(v)$$

where H is convex and $u > v > r > p$, $p \leq w \leq u$. ∎

Remark 3.1. Let us point out that (3.7) is a q-analogue of the Gauss-Winckler inequality. Inequalities (3.9), (3.10) and (3.11) are inequalities of Chebyshev type.

4. INEQUALITIES FOR q-GAMMA AND q-BETA FUNCTIONS

In this section we apply Theorem 3.1 on q-gamma and q-beta function. Here we use the following notation:

$$\begin{aligned} (1+a)_q^\infty &= \prod_{j=0}^{\infty}(1+q^j a), \\ (1+a)_q^t &= \frac{(1+a)_q^\infty}{(1+aq^t)_q^\infty}. \end{aligned}$$

A q-analogue Γ_q of the gamma function ([1]) is given as

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q^{-qx} d_q x, \quad t > 0,$$

where E_q^x is a q-analogue of the exponential function:

$$E_q^x = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]!} = (1+(1-q)x)_q^\infty.$$

The q-beta function $B_q(t,s)$ is defined as

$$B_q(t,s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}, \quad t, s > 0,$$

and has the following integral representation

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x.$$

Theorem 4.1. *(i) Functions $t \mapsto \Gamma_q(t+1)$ and $t \mapsto B_q(t+1,s)$ for any fixed $s > 0$ are log-convex.*

(ii) If $s \geq 1$, then the function $t \mapsto [t+1]_q B_q(t+1,s)$ is log-convex and if $0 < s < 1$ then this function is log-concave.

(iii) If $s \in [1,2]$, then $t \mapsto [t+2]_q[t+1]_q B_q(t+1,s)$ is log-concave.

Proof. The function E_q^{-qx} is non-negative, so applying Theorem 3.1(a) we obtain that $t \mapsto \int_0^{\frac{1}{1-q}} x^t E_q^{-qx} d_qx$ is log-convex, i.e. $\Gamma_q(t+1)$ is log-convex. The same argument holds for the function $t \mapsto B_q(t+1,s)$ where s is a fixed positive number.

Let us calculate a sign of q-derivative of the function $(1-qx)_q^{s-1}$. Using the following property ([1, Lemma 2.4])

$$D_q(1+\beta x)_q^t = [t]_q\beta(1+\beta qx)_q^{t-1}$$

we get

$$D_q(1-qx)_q^{s-1} = -q[s-1]_q(1-q^2x)_q^{s-2}.$$

It is non-positive iff $-[s-1]_q = -\frac{1-q^{s-1}}{1-q} \le 0$, i.e. $s \ge 1$. By Theorem 3.1(b), if $s \ge 1$, then the function $t \mapsto [t+1]_q B_q(t+1,s)$ is log-convex while if $s < 1$ then this function is log-concave.

For the last statement let us calculate a sign of the second q-derivative of the function $(1-qx)_q^{s-1}$. Using the same property from [1, Lemma 2.4] we get

$$\begin{aligned} D_q^2(1-qx)_q^{s-1} &= D_q(D_q(1-qx)_q^{s-1}) = D_q(-q[s-1]_q(1-q^2x)_q^{s-2}) \\ &= q^3[s-1]_q[s-2]_q(1-q^3x)_q^{s-3}. \end{aligned}$$

It is non-positive iff $s \in [1,2]$. ∎

Corollary 4.2. *a) If $p > u > r$, then*

$$\Gamma_q(u+1)^{p-r} \le \Gamma_q(r+1)^{p-u}\Gamma_q(p+1)^{u-r},$$

$$B_q(u+1,s)^{p-r} \le B_q(r+1,s)^{p-u}B_q(p+1,s)^{u-r},$$

$$([u+1]_qB_q(u+1,s))^{p-r} \le ([r+1]_qB_q(r+1,s))^{p-u}([p+1]_qB_q(p+1,s))^{u-r},\ s \ge 1, \tag{4.1}$$

$$\begin{gathered} ([u+2]_q[u+1]_qB_q(u+1,s))^{p-r} \ge ([r+2]_q[r+1]_qB_q(r+1,s))^{p-u}\times \\ \times([p+2]_q[p+1]_qB_q(p+1,s))^{u-r},\ s \in [1,2]. \end{gathered}$$

If $0 < s < 1$, then (4.1) is reversed.

b) If $p \ge u$, $r \ge v$ and $p \ne r$, $u \ne v$, then

$$\left(\frac{\Gamma_q(p+1)}{\Gamma_q(r+1)}\right)^{\frac{1}{p-r}} \ge \left(\frac{\Gamma_q(u+1)}{\Gamma_q(v+1)}\right)^{\frac{1}{u-v}}$$

$$\left(\frac{B_q(p+1,s)}{B_q(r+1,s)}\right)^{\frac{1}{p-r}} \ge \left(\frac{B_q(u+1,s)}{B_q(v+1,s)}\right)^{\frac{1}{u-v}}$$

$$\left(\frac{[p+1]_qB_q(p+1,s)}{[r+1]_qB_q(r+1,s)}\right)^{\frac{1}{p-r}} \ge \left(\frac{[u+1]_qB_q(u+1,s)}{[v+1]_qB_q(v+1,s)}\right)^{\frac{1}{u-v}},\ s \ge 1 \tag{4.2}$$

$$\left(\frac{[p+2]_q[p+1]_qB_q(p+1,s)}{[r+2]_q[r+1]_qB_q(r+1,s)}\right)^{\frac{1}{p-r}} \le \left(\frac{[u+2]_q[u+1]_qB_q(u+1,s)}{[v+2]_q[v+1]_qB_q(v+1,s)}\right)^{\frac{1}{u-v}},\ s \in [1,2].$$

If $0 < s < 1$, then (4.2) is reversed.

Proof. These inequalities follows from results of Proposition 4.1 and Theorem 3.2 a) and b). ∎

Other inequalities which are consequences of Theorem 3.2 c), d) and e) are also can be stated.

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