BARTLE INTEGRATION IN LIE ALGEBRAS
ANDREAS BOUKAS AND PHILIP FEINSILVER

Received 29 December, 2015; accepted 15 February, 2016; published 17 May, 2016.

CENTRO VITO VOLterra, UNIVERSITÀ DI ROMA Tor Vergata, VIA COLUMBIA 2, 00133 ROMA, ITALY
andreasboukas@yahoo.com

DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, ILLINOIS 62901, USA
pfeinsil@math.siu.edu

ABSTRACT. Using Bartle’s bilinear vector integral we define stochastic integrals of bounded operator valued functions with respect to Stieltjes measures associated with the generators of the Heisenberg and Finite Difference Lie algebras. Our definition also covers the Square of White Noise and $\mathfrak{sl}(2)$ Lie algebras.

Key words and phrases: Bartle’s bilinear vector integral, Heisenberg Lie algebra, $\mathfrak{sl}(2, \mathbb{R})$, Square of White Noise Lie algebra, Finite Difference Lie algebra, Fock space, Quantum random variable, Bochner’s theorem.

2010 Mathematics Subject Classification Primary 28B05, 46G10, 81S25. Secondary 60H05, 17B15.
1. Introduction

Based on the notion of quantum stochastic process developed in [1], a first example of quantum stochastic calculus in Fock space [16] presented a noncommutative approach extending classical Itô calculus. Based on vector-valued integrators and integrands, stochastic integrals were defined via strong $L^2$-limits of matrix elements. Some generalizations of the Lebesgue integral from scalars to vectors are these:

- Bochner integral [5], [18] (Banach space valued integrand and scalar valued measure)
- Gelfand-Pettis integral [17], [14] (weak sense version of Bochner’s integral)
- Bartle-Dunford-Schwartz integral [4] (scalar valued integrand and vector valued measure)

In this paper, using the Bartle integral, we present a straightforward definition of integrals with respect to quantum stochastic processes, i.e., families of operators on Fock space, adapted in a certain sense, with respect to vector valued Stieltjes measures. We focus here on integrators defined in terms of the generators of the Heisenberg and Finite Difference Lie algebras in Fock space representation.

2. Vector valued measures and Bartle’s integral

Let $\mathcal{X}$ be a set, $\mathcal{F}$ a $\sigma$-field of subsets of $\mathcal{X}$ and $Y$ a normed linear space. A $Y$-valued measure on $\mathcal{X}$ is a countably additive set function $m : \mathcal{F} \rightarrow Y$. If $\mathcal{X} = [0, \infty)$ and $\mathcal{F}$ is the $\sigma$-field of the Borel measurable subsets of $\mathcal{X}$ then $m$ is called a Stieltjes measure. The semi-variation of $m$ is the extended real-valued set function $\|m\|$ defined on each $E \in \mathcal{F}$ by

$$\|m\|(E) = \sup \left\{ \sum_{i=1}^{k} x_i m(E_i) \right\}$$

where the supremum is over all finite partitions of $E$ into disjoint sets $\{E_i\}_{i=1}^{k} \subset \mathcal{F}$, $k \in \mathbb{N}$, and all finite collections of elements $\{x_i\}_{i=1}^{k} \subset B(Y)$, the space of bounded linear operators $T : Y \mapsto Y$, with $\|x_i\|_{B(Y)} \leq 1$. The variation of $m$ is the extended real-valued set function $|m|$ defined on each $E \in \mathcal{F}$ by

$$|m|(E) = \sup \left\{ \sum_{i \in J} m(E_i) \right\}$$

where the supremum is over all finite families $\{E_i : i \in J\}$ of disjoint subsets of $\mathcal{F}$ such that $E = \bigcup_{i \in J} E_i$. If $\mathcal{X}$ is a topological space and for each $E \in \mathcal{F}$ given $\varepsilon > 0$ there exist a compact set $K$ and an open set $G$ such that $K \subset E \subset G$ and for every $E' \in \mathcal{F}$ with $K \subset E' \subset G$, $\|m(E) - m(E')\|_Y < \varepsilon$ then $m$ is called regular. If $f : [0, \infty) \mapsto Y$ where $Y$ is a normed space and $[a, b] \subset [0, \infty)$ then the total variation $V^b_a(f)$ of $f$ on $[a, b]$ is defined by

$$V^b_a(f) = \sup \Pi V_{\Pi}(f)$$

where

$$\Pi = \{a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$$

is a partition of $[a, b]$ and

$$V_{\Pi}(f) = \sum_{i=0}^{n-1} \|f(t_{i+1}) - f(t_i)\|_Y$$
If $V_b^y(f) < \infty$ for all $[a, b] \subset [0, \infty]$ then $f$ is said to be with finite variation. To each function $f$ with finite variation there corresponds a unique countably additive regular Stieltjes measure $m_f$ with finite variation, such that for all $a, b \in \mathbb{R}$

$$m_f([a, b]) = f(b) - f(a), \quad m_f((a, b]) = f(b) - f(a^-) \quad m_f((a, b)) = f(b) - f(a^+)$$

If $Y$ is a Banach space and $m$ is a Stieltjes measure then a simple function $f : \mathcal{X} \mapsto B(Y)$ is a function of the form

$$f = \sum_{i=1}^{n} f_i \chi_{E_i}$$

where $n \in \mathbb{N}, E_i \in \mathcal{F}, \|m\|(E_i) < \infty, f_i \in B(Y)$ for each $i$ and $E_i \cap E_j = \emptyset$ if $i \neq j$. If $E \in \mathcal{F}$ and $f$ is a simple function then the integral of $f$ over $E$ is defined by

$$\int_E f(s) m(ds) = \sum_{i=1}^{n} f_i m(E \cap E_i)$$

A sequence $f_n : \mathcal{X} \mapsto B(Y)$ is said to converge in $m$-measure to a function $f : \mathcal{X} \mapsto B(Y)$ if for each $\epsilon > 0, \|m\|\{s \in \mathcal{X} / \|f_n(s) - f(s)\|_{B(Y)} \geq \epsilon\} \to 0$ as $n \to \infty$. A function $f : \mathcal{X} \mapsto B(Y)$ is $m$-measurable if it is the limit in $m$-measure of a sequence of simple functions. Such a function $f$ is Bartle $m$-integrable over $\mathcal{X}$ if there is a sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions such that:

- $f_n \to f$ in $m$-measure
- The sequence $\{\lambda_n\}_{n=1}^{\infty}$ of integrals

$$\lambda_n(E) = \int_E f_n(s) m(ds) \quad E \in \mathcal{F}$$

has the property that given $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that $\|m\|(E) < \delta$ implies $\|\lambda_n(E)\|_Y < \epsilon$

If $f$ is Bartle $m$-integrable over $\mathcal{X}$ then for each $E \in \mathcal{F}$ the sequence $\{\lambda_n(E)\}_{n=1}^{\infty}$ converges in $\|\cdot\|_Y$ and the integral of $f$ over $E$ is defined by

$$\int_E f(s) m(ds) = \lim_{n \to \infty} \int_E f_n(s) m(ds)$$

**Theorem 2.1.** Let $Y$ be a Banach space and let $f : [0, \infty) \mapsto B(Y)$ be Bartle integrable with respect to a countably additive regular $Y$-valued measure $m$ defined on the $\sigma$-field of Borel measurable subsets of $[0, \infty)$. Then, for any $\epsilon > 0$ there exists a step function $g : [0, \infty) \mapsto B(Y)$, i.e. $g = \sum_{i=1}^{n} a_i \chi_{I_i}$ where $a_i \in B(Y)$ and the $I_i$’s are pairwise disjoint bounded subintervals of $[0, \infty)$, such that

$$\| \int_{[0,\infty)} f(s) dm(s) - \int_{[0,\infty)} g(s) dm(s) \|_Y < \epsilon$$

**Proof.** By Theorem 1 of [3] there exists a simple function $\sigma : [0, \infty) \mapsto B(Y)$, say $\sigma = \sum_{j=1}^{m} b_j \chi_{B_j}$, where $b_j \in B(Y)$ and the $B_j$’s are pairwise disjoint Borel measurable subsets of $[0, \infty)$ with finite semivariation $\|m\|(B_j)$, thus $\|m(B_j)\|_Y < \infty$, such that

$$\| \int_{[0,\infty)} f(s) dm(s) - \int_{[0,\infty)} \sigma(s) dm(s) \|_Y < \frac{\epsilon}{2}$$

Now let $\delta = \frac{\epsilon}{4} \left(1 + \sum_{j=1}^{m} \|b_j\|_{B(Y)}\right)$ and fix $j \in \{1, 2, ... , m\}$. By the regularity of $m$ there exists an open set $U_j \subset [0, \infty)$ such that $B_j \subset U_j$ and $\|m(U_j) - m(B_j)\|_Y < \delta$, which
implies that \( \|m(U_j)\|_Y \leq \|m(B_j)\|_Y + \delta < \infty \). For each \( j \), \( U_j = \bigcup_{k=1}^{\infty} I_{j,k} \) where the \( I_{j,k} \)'s are pairwise disjoint open subintervals of \([0, \infty)\). By the countable additivity of \( m \) we have \( \| \sum_{k=1}^{\infty} m(I_{j,k})\|_Y = \|m(U_j)\|_Y < \infty \). Thus, there exists \( k_0 \) such that \( \| \sum_{k=k_0+1}^{\infty} m(I_{j,k})\|_Y = \| \sum_{k=0}^{k_0} m(I_{j,k}) - \sum_{k=0}^{k_0} m(I_{j,k})\|_Y < \delta \). Let \( V_j = \bigcup_{k=1}^{k_0} I_{j,k} \). Then

\[
\|m(V_j) - m(B_j)\|_Y = \|m(U_j) - \sum_{k=k_0+1}^{\infty} m(I_{j,k}) - m(B_j)\|_Y \\
\leq \|m(U_j) - m(B_j)\|_Y + \| \sum_{k=k_0+1}^{\infty} m(I_{j,k})\|_Y < 2\delta
\]

Now let \( g = \sum_{j=1}^{m} b_j \chi_{V_j} \). Since each \( V_j \) is a finite union of intervals, \( g \) is a step function. Also,

\[
\| \int_{[0, \infty)} \sigma(s) \, dm(s) - \int_{[0, \infty)} g(s) \, dm(s) \|_Y = \| \sum_{j=1}^{m} b_j m(B_j) - \sum_{j=1}^{m} b_j m(V_j) \|_Y \\
\leq \sum_{j=1}^{m} \| b_j \|_{B(Y)} \| m(B_j) - m(V_j) \|_Y \leq \sum_{j=1}^{m} \| b_j \|_{B(Y)} \| 2\delta \leq \left( 1 + \sum_{j=1}^{m} \| b_j \|_{B(Y)} \right) 2\delta \leq \frac{\epsilon}{2}
\]

by the definition of \( \delta \). Thus

\[
\| \int_{[0, \infty)} f(s) \, dm(s) - \int_{[0, \infty)} g(s) \, dm(s) \|_Y \\
\leq \| \int_{[0, \infty)} f(s) \, dm(s) - \int_{[0, \infty)} \sigma(s) \, dm(s) \|_Y + \| \int_{[0, \infty)} \sigma(s) \, dm(s) - \int_{[0, \infty)} g(s) \, dm(s) \|_Y \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

\( \square \)

### 3. Bochner’s Theorem and Quantum Stochastic Integrals

**Definition 3.1.** A continuous function \( f : \mathbb{R} \to \mathbb{C} \) is **positive definite** if

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} f(t-s)\phi(t)\tilde{\phi}(s) \, dt \, ds \geq 0
\]

for every continuous function \( \phi : \mathbb{R} \to \mathbb{C} \) with compact support.

Bochner’s theorem [18] states that such a function can be represented as

\[
f(t) = \int_{\mathbb{R}} e^{it\lambda} \, dv(\lambda)
\]

where \( v \) is a non-decreasing right-continuous bounded function. If \( f(0) = 1 \) then such a function \( v \) defines a probability measure on \( \mathbb{R} \) and Bochner’s theorem says that \( f \) is the Fourier transform of a probability measure, i.e., the characteristic function of a random variable that follows the probability distribution defined by \( v \). Moreover, the condition of positive definiteness of \( f \) is necessary and sufficient for such a representation. An example of such a positive definite function is provided by

\[
f(t) = \langle \Phi, e^{itX} \Phi \rangle
\]
where $\Phi$ is the normalized (i.e., $\|\Phi\| = 1$) vacuum vector of a Fock-Hilbert space $\mathcal{F}$ and $X$ is an observable (self-adjoint operator on $\mathcal{F}$) also called a quantum random variable in which case

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t-s)\phi(t)\bar{\phi}(s)\,dt\,ds = \| \int_{\mathbb{R}} e^{-iX}\phi(t)\,dt\,\Phi \|_{\mathcal{F}}^2 \geq 0$$

In this paper we consider quantum random variables $X$ acting on the Fock space $\mathcal{F}$ associated with the Heisenberg, $\mathfrak{sl}(2, \mathbb{R})$, Square of White Noise, and Finite Difference Lie algebras.

**Definition 3.2.** Let $\mathcal{S}$ be a test function space and for each $f \in \mathcal{S}$ let $\psi(f)$ be the associated exponential vector. In particular, $\Phi = \psi(0)$ is the vacuum vector. Denote $\mathcal{E} = \text{span}\{\psi(f) / f \in \mathcal{S}\}$. If

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \mapsto [0, \infty)$$

is an inner product on $\mathcal{E}$ then the **Fock space** $\mathcal{F}$ over $\mathcal{S}$ is the Hilbert space completion of $(\mathcal{E}, \langle \cdot, \cdot \rangle)$.

**Definition 3.3.** A family $\{X(t) / t \geq 0\}$ of quantum random variables is called a quantum stochastic process if it is adapted in the following sense: for each $t \geq 0$ and $\psi(f) \in \mathcal{E}$, $X(t)\psi(f) = X_t\psi(f_t) \otimes \psi(f_t)$, where $f_t = f_{[0,t]}$ and $f_t = f_{(-\infty,t]}$ [15], i.e., we have a decomposition $X(t) = X_t \otimes 1_t$ corresponding to the decomposition $\psi(f) = \psi(f_t) \otimes \psi(f_t)$. If for each $t \geq 0$, $X_t$ is a bounded linear operator then the process is said to be bounded.

Our goal is to define, in the sense of Bartle [3], stochastic integrals of the form

$$\int_E f(s)\,dX(s)$$

where $E$ is a measurable subset of $[0, \infty)$ and $f : [0, \infty) \mapsto B(\mathcal{F})$ where $B(\mathcal{F})$ is the space of bounded linear operators $T : \mathcal{F} \mapsto \mathcal{F}$. The integrators $X$ will in each case act on the Fock spaces $\mathcal{F}_H$ and $\mathcal{F}_{FD}$, associated with the Heisenberg [9, 11, 13, 15] and Finite Difference [6, 7, 11, 12, 13] Lie algebras respectively. The Square of White Noise [2] and $\mathfrak{sl}(2)$ integrators can be expressed as linear combinations of the Finite Difference integrators and so they are also covered. The $\mathcal{F}_H$ and $\mathcal{F}_{FD}$ Fock spaces have been essential in the development of quantum stochastic calculus.

### 4. Main Examples: The Basic Ingredients

The basic ingredients used to build a quantum stochastic process are these:

- the test function space
- inner product of exponential vectors
- action of the integrators
- commutation relations

We present these for our main examples, $\mathcal{F}_H$ and $\mathcal{F}_{FD}$.

1. **Heisenberg Fock space $\mathcal{F}_H$**: In this case $\mathcal{S} = (L^2 \cap L^\infty)([0, \infty), \mathbb{C})$ with inner product

$$\langle \psi(f), \psi(g) \rangle = e_{[0,\infty)} f(s)g(s)\,d\mu(s)$$

where $\mu$ is Lebesgue measure and the basic integrators are $A(f)$, $A^\dagger(f)$ with action

$$A(f) \psi(g) = \left( \int_0^\infty f(t) \bar{g}(t)\,dt \right) \psi(g) , \quad A^\dagger(f) \psi(g) = \frac{\partial}{\partial \epsilon}_{\epsilon=0} \psi(g + \epsilon f)$$
and commutation relations

\[ [A(f), A^\dagger(g)] = \int_0^\infty \bar{f}(t) g(t) \, dt \]

Moreover, using the notation \( X(t) = X(\chi_{[0,t]}) \) we obtain \[16\] the matrix elements

\[
<\psi(f), A(t)\psi(g)> = \int_0^t g(s) \, ds <\psi(f), \psi(g)>
\]

\[
<\psi(f), A^\dagger(t)\psi(g)> = \int_0^t \bar{f}(s) \, ds <\psi(f), \psi(g)>
\]

\[
<A(t)\psi(f), A(s)\psi(g)> = \int_0^t \bar{f}(z) \, dz \int_0^s g(w) \, dw <\psi(f), \psi(g)>
\]

\[
<A^\dagger(t)\psi(f), A^\dagger(s)\psi(g)> = \left( \int_0^s \bar{f}(z) \, dz \int_0^t g(w) \, dw + \min\{t, s\} \right) <\psi(f), \psi(g)>
\]

(2) Finite Difference Fock space \( \mathcal{F}_{FD} \): In this case

\[ \mathcal{S} = \{ f : [0, \infty) \mapsto \mathbb{R}, |f| < 1, f \text{ is a simple function} \} \]

with inner product

\[ \langle \psi(f), \psi(g) \rangle = e^{-\int_0^\infty \ln(1-f(s)g(s)) \, d\mu(s)} \]

where \( \mu \) is Lebesgue measure and the basic integrators are \( P(f), Q(f), T(f) \) with action

\[ Q(f) \, \psi(g) = \frac{\partial}{\partial \epsilon} \big|_{\epsilon=0} (\psi(g + \epsilon f) + \psi(e^{\epsilon} f g)) \]

\[ P(f) \, \psi(g) = \left( \int_0^\infty f(t) g(t) \, dt + Q(fg) \right) \psi(g) \]

\[ T(f) \, \psi(g) = \left( Q(f) + P(f) + \int_0^\infty f(t) g(t) \, dt \right) \psi(g) \]

commutation relations

\[ [P(f), Q(g)] = [P(f), T(g)] = [T(f), Q(g)] = T(fg) \]
where which are a trivial central extension of the \textit{sl}(2) of the Square of White Noise Lie algebra with charge \(c\delta\) of the Dirac delta function, and commutation relations

\[
[B_g, B_f^\dagger] = 2c \int_0^\infty (gf)(s) \, ds + 4N_{gf}, \quad [N_h, B_f^\dagger] = 2B_{h,f}, \quad [B_g, N_h] = -2B_{gh}
\]

which are a trivial central extension of the \textit{sl}(2) commutation relations, then the Square of White Noise operators are related to the Finite Difference operators through

\[
Q(f) = \frac{1}{2} \left( B_f^\dagger + N_f \right), \quad P(f) = \frac{1}{2} (B_f + N_f), \quad T(f) = \int_0^\infty f(s) \, ds \mathbf{1} + P(f) + Q(f)
\]

5. \textbf{The \textit{Stieltjes} measures in the \(\mathcal{F}_H\) and \(\mathcal{F}_{FD}\) cases}

\textbf{Proposition 5.1.} In the Heisenberg case, for each \(f \in S\) the functions

\[
F : t \in [0, +\infty) \mapsto A(t)\psi(f) \in \mathcal{F}_H
\]

and

\[
G : t \in [0, +\infty) \mapsto A(t)^\dagger \psi(f) \in \mathcal{F}_H
\]

where \(A(t) = A(\chi_{[0,t]})\) and \(A(t) = A^\dagger(\chi_{[0,t]})\), are with finite variation.
Proof. Let \( f \in (L^2 \cap L^\infty)([0, \infty), \mathbb{C}) \), let \( M = \text{ess sup} \ (f) \) and let \( \Pi = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\} \) a partition of \([a, b] \subseteq [0, +\infty)\). Then

\[
V_\Pi(F) = \sum_{i=0}^{n-1} \|F(t_{i+1}) - F(t_i)\|_{\mathcal{F}_H}
\]

\[
= \sum_{i=0}^{n-1} (\langle F(t_{i+1}), F(t_{i+1}) \rangle - \langle F(t_{i+1}), F(t_i) \rangle - \langle F(t_i), F(t_{i+1}) \rangle + \langle F(t_i), F(t_i) \rangle)^{1/2}
\]

\[
= \sum_{i=0}^{n-1} (\langle A(t_{i+1}) \psi(f), A(t_{i+1}) \psi(f) \rangle - \langle A(t_{i+1}) \psi(f), A(t_i) \psi(f) \rangle
\]

\[-\langle A(t_i) \psi(f), A(t_{i+1}) \psi(f) \rangle + \langle A(t_i) \psi(f), A(t_i) \psi(f) \rangle)^{1/2}
\]

\[
= \sum_{i=0}^{n-1} \left( \int_0^{t_{i+1}} \tilde{f}(s) \, ds \int_0^{t_{i+1}} f(s) \, ds - \int_0^{t_i} \tilde{f}(s) \, ds \int_0^{t_i} f(s) \, ds - \int_0^{t_i} \tilde{f}(s) \, ds \int_0^{t_i} f(s) \, ds \right)^{1/2} \|\psi(f)\|
\]

\[
= \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \tilde{f}(s) \, ds \int_{t_i}^{t_{i+1}} f(s) \, ds - \int_0^{t_i} \tilde{f}(s) \, ds \int_{t_i}^{t_{i+1}} f(s) \, ds \right)^{1/2} \|\psi(f)\|
\]

\[
= \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \tilde{f}(s) \, ds \int_{t_i}^{t_{i+1}} f(s) \, ds \right)^{1/2} \|\psi(f)\| = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(s) \, ds \|\psi(f)\| \leq M \sum_{i=0}^{n-1} (t_{i+1} - t_i) \|\psi(f)\| = M(b - a) \|\psi(f)\| < \infty
\]

Thus

\[
V_\Pi^b(F) = \sup_{\Pi} V_\Pi(F) \leq M (b - a) \|\psi(f)\| < \infty
\]

Similarly

\[
V_\Pi(G) = \sum_{i=0}^{n-1} \|G(t_{i+1}) - G(t_i)\|_{\mathcal{F}_H}
\]

\[
= \sum_{i=0}^{n-1} (\langle (G(t_{i+1}), G(t_{i+1}) \rangle - \langle G(t_{i+1}), G(t_i) \rangle - \langle G(t_i), G(t_{i+1}) \rangle + \langle G(t_i), G(t_i) \rangle)^{1/2}
\]

\[
= \sum_{i=0}^{n-1} (\langle A^\dagger(t_{i+1}) \psi(f), A^\dagger(t_{i+1}) \psi(f) \rangle - \langle A^\dagger(t_{i+1}) \psi(f), A^\dagger(t_i) \psi(f) \rangle
\]

\[-\langle A^\dagger(t_i) \psi(f), A^\dagger(t_{i+1}) \psi(f) \rangle + \langle A^\dagger(t_i) \psi(f), A^\dagger(t_i) \psi(f) \rangle)^{1/2}
\]

\[
= \sum_{i=0}^{n-1} \left( \int_0^{t_{i+1}} \tilde{f}(s) \, ds \int_0^{t_{i+1}} f(s) \, ds + t_{i+1} - \int_0^{t_i} \tilde{f}(s) \, ds \int_0^{t_{i+1}} f(s) \, ds - t_i
\]

\[- \int_0^{t_{i+1}} \tilde{f}(s) \, ds \int_0^{t_i} f(s) \, ds - t_i + \int_0^{t_i} \tilde{f}(s) \, ds \int_0^{t_{i+1}} f(s) \, ds + t_i \right)^{1/2} \|\psi(f)\|
\]
Bartle integration in Lie algebras

\[
= \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} f(s) \, ds \right)^2 + (t_{i+1} - t_i) \right)^{1/2} \|\psi(f)\| \\
\leq \sum_{i=0}^{n-1} \left( M^2 (t_{i+1} - t_i)^2 + (\sqrt{t_{i+1} - t_i})^2 \right)^{1/2} \|\psi(f)\|
\]

and using \( \sqrt{x^2 + y^2} \leq x + y \) for \( x, y \geq 0 \), we have

\[
V_\Pi(G) \leq M \sum_{i=0}^{n-1} (t_{i+1} - t_i) \|\psi(f)\| + \sum_{i=0}^{n-1} \sqrt{t_{i+1} - t_i} \|\psi(f)\| \\
= M(b - a) \|\psi(f)\| + \sum_{i=0}^{n-1} \sqrt{t_{i+1} - t_i} \|\psi(f)\|
\]

Since \( V_\Pi(G) \) increases as the partition \( \Pi \) becomes finer and we are interested in the supremum over all partitions \( \Pi \), we may assume that \( \sqrt{t_{i+1} - t_i} \leq 2^{-i} \) for all \( i = 0, 1, \ldots \). Thus

\[
V_\Pi(G) \leq \left( M(b - a) + \sum_{i=0}^{n} \frac{1}{2^i} \right) \|\psi(f)\| \leq (M(b - a) + 2) \|\psi(f)\| < \infty
\]

and so

\[
V_a^b(G) = \sup_{\Pi} V_\Pi(G) \leq (M(b - a) + 2) \|\psi(f)\| < \infty
\]

□

**Proposition 5.2.** In the Finite Difference case, for each \( f \in S \) the functions

\[
F : t \in [0, +\infty) \mapsto P(t)\psi(f) \in \mathcal{F}_{FD} \\
G : t \in [0, +\infty) \mapsto Q(t)\psi(f) \in \mathcal{F}_{FD}
\]

and

\[
H : t \in [0, +\infty) \mapsto T(t)\psi(f) \in \mathcal{F}_{FD}
\]

where \( P(t) = P(\chi_{[0,t]}) \), \( Q(t) = Q(\chi_{[0,t]}) \) and \( T(t) = T(\chi_{[0,t]}) \), are with finite variation.

**Proof.** Let \( f = \sum_{i=1}^{k} a_k \chi_{I_k} \) be a simple function and \( \Pi = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\} \) a partition of \( [a, b] \subseteq [0, +\infty) \). Let also \( \phi = \frac{f(1+x)}{1+x^2} \). Then \( \phi \) is also a simple function with \( M = \max\{\phi(x) : x \geq 0\} < \infty \). Therefore
\[ V_{\Pi}(F) = \sum_{i=0}^{n-1} \| F(t_{i+1}) - F(t_i) \|_{\mathcal{F}_P} \]

\[ = \sum_{i=0}^{n-1} (\langle F(t_{i+1}), F(t_{i+1}) \rangle - \langle F(t_i), F(t_i) \rangle) - \langle F(t_i), F(t_{i+1}) \rangle + \langle F(t_i), F(t_i) \rangle^{1/2} \]

\[ = \sum_{i=0}^{n-1} (\langle P(t_{i+1}) \psi(f), P(t_{i+1}) \psi(f) \rangle - \langle P(t_i) \psi(f), P(t_i) \psi(f) \rangle - \langle P(t_i) \psi(f), P(t_{i+1}) \psi(f) \rangle + \langle P(t_i) \psi(f), P(t_i) \psi(f) \rangle^{1/2} \]

\[ = \sum_{i=0}^{n-1} \left( \int_0^{t_{i+1}} \phi(s) \, ds \int_0^{t_{i+1}} \phi(s) \, ds + \int_0^{t_{i+1}} \phi^2(s) \, ds - \int_0^{t_{i+1}} \phi(s) \, ds \int_0^{t_{i+1}} \phi(s) \, ds - \int_0^{t_{i+1}} \phi^2(s) \, ds + \int_0^{t_{i+1}} \phi(s) \, ds \int_0^{t_{i+1}} \phi(s) \, ds \right) + \int_0^{t_{i+1}} \phi^2(s) \, ds \right)^{1/2} \| \psi(f) \| \]

\[ \leq M \sum_{i=0}^{n-1} \left( (t_{i+1} - t_i)^2 + (t_{i+1} - t_i) \right)^{1/2} \| \psi(f) \| \]

As in the proof of Proposition 5.1 we may assume that \((t_{i+1} - t_i)^2 \leq (t_{i+1} - t_i) \leq 2^{-2i} \) for all \(i = 0, 1, \ldots\). Thus

\[ V_{\Pi}(F) \leq \left( \sqrt{2} M \sum_{i=0}^{n} \frac{1}{2^i} \right) \| \psi(f) \| \leq 2\sqrt{2} M \| \psi(f) \| < \infty \]

and so

\[ V_a^h(F) = \sup_{\Pi} V_{\Pi}(F) \leq 2\sqrt{2} M \| \psi(f) \| < \infty \]

Similarly letting \(\sigma = \frac{1}{(1-x)^2}\), and letting \(K > 1\) be such that \(|\phi(x)| \leq K\) and \(|\sigma(x)| \leq K\) for all \(x \geq 0\) we have
Finally, letting \( \rho \) and so

\[ V_{\Pi}(G) = \sum_{i=0}^{n-1} \|G(t_{i+1}) - G(t_i)\|_F \]

\[ = \sum_{i=0}^{n-1} \langle G(t_{i+1}), G(t_i) \rangle - \langle G(t_i), G(t_{i+1}) \rangle + \langle G(t_i), G(t_i) \rangle^{1/2} \]

\[ \leq K \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 + (t_{i+1} - t_i) \rangle^{1/2} \|\psi(f)\| \leq 2\sqrt{2}K \|\psi(f)\| < \infty \]

and so

\[ V_{\alpha}^{\beta}(G) = \sup_{\Pi} V_{\Pi}(G) \leq 2\sqrt{2}K \|\psi(f)\| < \infty \]

Finally, letting \( \rho = \frac{1+f}{1-f} \) and \( L > 1 \) be such that \( |\rho(x)| \leq L \) for all \( x \geq 0 \) we have

\[ V_{\Pi}(H) = \sum_{i=0}^{n-1} \|H(t_{i+1}) - H(t_i)\|_F \]

\[ = \sum_{i=0}^{n-1} \langle H(t_{i+1}), H(t_i) \rangle - \langle H(t_i), H(t_{i+1}) \rangle + \langle H(t_i), H(t_i) \rangle^{1/2} \]

\[ \leq K \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 + (t_{i+1} - t_i) \rangle^{1/2} \|\psi(f)\| \leq 2\sqrt{2}L \|\psi(f)\| < \infty \]
Definition 6.1. For each measurable subset \( E \) of \([0, a] \), for \( X \in \{A, A^t, P, Q, T\} \) and for \( f : [0, a] \rightarrow B(\mathcal{F}) \), where \( \mathcal{F} = \mathcal{F}_H \) or \( \mathcal{F} = \mathcal{F}_{FD} \), integrable with respect to the Stieltjes measure \( m_{X,g} \) defined by \( X \) and the arbitrary test function \( g \), we define the integral of \( f \) with respect to \( X \) on the exponential domain \( E \) as the operator

\[
\int_E f(s) dX(s) : E \mapsto \mathcal{F}
\]

with

\[
\int_E f(s) dX(s) \psi(g) = \int_E f(s) dm_{X,g}(s)
\]

Proposition 6.1. Let \( X \in \{A, A^t, P, Q, T\} \) and let \( f : [0, a] \rightarrow B(\mathcal{F}) \), where \( \mathcal{F} = \mathcal{F}_H \) or \( \mathcal{F} = \mathcal{F}_{FD} \), be integrable with respect to the Stieltjes measure \( m_{X,g} \) defined by \( X \) and the arbitrary test function \( g \) and such that the map \( t \mapsto f(t) \) is continuous. Then, on each interval \([a, b]\) \( \in [0, a] \)

\[
\int_{[a, b]} f(s) dX(s) \psi(g) = \lim_{\Pi \rightarrow 0} \sum_{\lambda=0}^{n} f(s_\lambda) (\phi(s_{\lambda+1}) - \phi(s_\lambda))
\]

where \( \Pi = \{s_0 = a < s_1 < \ldots < s_n = b\} \) is any partition of \([a, b]\), \( \phi \in \{F, G, H\} \) where \( F, G, H \) are as in Propositions 5.1 and 5.2 and the limit is in the \( \mathcal{F} \) norm sense.

Proof. For each \( n \geq 1 \) define \( f_n : [0, a] \rightarrow B(\mathcal{F}) \) by

\[
\begin{cases} 
\sum_{\lambda=0}^{n} f(s_\lambda) \chi_{(s_\lambda, s_{\lambda+1})} & \text{if } s \in [a, b] \\
\chi_{(s, b)} & \text{otherwise}
\end{cases}
\]

Then

\[
\int_{[a, b]} f_n(s) dm_{X,g}(s) = \sum_{\lambda=0}^{n} f(s_\lambda) m_{X,g}( [s_\lambda, s_{\lambda+1}) )
\]

We will show that \( f_n \) converges to \( f \) uniformly on \([0, a]\). Since \( f_n \) agrees with \( f \) outside \([a, b]\), it suffices to show that \( f_n \) converges to \( f \) uniformly on \([a, b]\). Let \( \epsilon > 0 \) be given. Since \( f \) is uniformly continuous on \([a, b]\) there exists \( \delta > 0 \) such that \( \|f(s) - f(t)\|_{B(\mathcal{F})} < \epsilon \) for all \( s, t \in [a, b] \) with \( |s - t| < \delta \). Let \( n_0 \geq 1 \) be such that for all \( \lambda \in \{0, 1, 2, \ldots, n_0\} \), \( s_{\lambda+1} - s_\lambda < \delta \). Such an \( n_0 \) exists since the mesh of the partition tends to zero. Then, for all \( n \geq n_0 \) (corresponding to finer partitions) we also have that \( s_{\lambda+1} - s_\lambda < \delta \) for all \( \lambda \in \{0, 1, 2, \ldots, n\} \), and, by the definition of \( f_n \), for \( s \in [a, b] \)

\[
\|f(s) - f_n(s)\|_{B(\mathcal{F})} \leq \max_{0 \leq \lambda \leq n} ||f(s) - f(s_\lambda)||_{B(\mathcal{F})} < \epsilon
\]

since if \( s \in [s_\lambda, s_{\lambda+1}) \) then \( s - s_\lambda < s_{\lambda+1} - s_\lambda < \delta \). Thus (3), Theorem 8
\[
\int_{[a,b]} f(s) \, dm_{X,g}(s) = \lim_{n \to \infty} \int_{[a,b]} f_n(s) \, dm_{X,g}(s) = \lim_{n \to \infty} \sum_{\lambda=0}^n f(s_\lambda) \, m_{X,g}([s_\lambda, s_{\lambda+1}]) = \lim_{n \to \infty} \sum_{\lambda=0}^n f(s_\lambda) \, (\phi(s_{\lambda+1}) - \phi(s_\lambda)) = \lim_{\Pi \to 0} \sum_{\lambda=0}^n f(s_\lambda) \, (\phi(s_{\lambda+1}) - \phi(s_\lambda))
\]

\[\square\]

7. Computation of Matrix Elements

In the case of adapted processes, matrix elements

\[\langle \int_0^t f(s) \, dX(s) \, \psi(g), \psi(h) \rangle\]

and

\[\langle \int_0^t f_1(s) \, dX(s) \, \psi(g), \int_0^t f_2(s) \, dX_2(s) \, \psi(h) \rangle\]

are computed with the use of Theorem 6.1 as follows and result in the familiar formulas of quantum stochastic calculus [6, 12, 16]:

\[\langle \int_0^t f(s) \, dX(s) \, \psi(g), \psi(h) \rangle = \langle \lim_{\Pi \to 0} \sum_{\lambda=0}^n f(s_\lambda) \, (\phi(s_{\lambda+1}) - \phi(s_\lambda)), \psi(h) \rangle = \lim_{\Pi \to 0} \sum_{\lambda=0}^n \langle f(s_\lambda) \, (\phi(s_{\lambda+1}) - \phi(s_\lambda)), \psi(h) \rangle\]

and assuming decompositions \( f(t) = f_I \otimes 1_{(t)}, X(t) = X_I \otimes 1_{(t)}, \psi(g) = \psi(g_I) \otimes \psi(g_{(t)}), \psi(h) = \psi(h_I) \otimes \psi(h_{(t)}), \) as in Definition 3.3, we have

\[\langle \int_0^t f(s) \, dX(s) \, \psi(g), \psi(h) \rangle = \lim_{\Pi \to 0} \sum_{\lambda=0}^n \langle f(s_\lambda) \, (X(s_{\lambda+1}) - X(s_\lambda)), \psi(g), \psi(h) \rangle = \lim_{\Pi \to 0} \sum_{\lambda=0}^n \langle f(s_\lambda) \, (\psi(g_{(s_{\lambda+1})})), \psi(h_{(s_{\lambda+1})}) \rangle \int_{s_{\lambda}}^{s_{\lambda+1}} w_X(s) \, ds \langle \psi(g_{(s_{\lambda+1})}), \psi(h_{(s_{\lambda+1})}) \rangle \langle \psi(g_{(s_{\lambda+1})}), \psi(h_{(s_{\lambda+1})}) \rangle\]
\[
\begin{align*}
\lim_{\Pi \to 0} \sum_{\lambda=0}^{n} \int_{s_{\lambda}}^{s_{\lambda+1}} w_X(s) \, ds \, \langle f(s_{\lambda}) \psi(g_{s_{\lambda}}), \psi(h_{s_{\lambda}}) \rangle \langle \psi(g_{s_{\lambda}s_{\lambda+1}}), \psi(h_{s_{\lambda}s_{\lambda+1}}) \rangle \\
= \lim_{\Pi \to 0} \sum_{\lambda=0}^{n} \int_{s_{\lambda}}^{s_{\lambda+1}} w_X(s) \, ds \, \langle f(s_{\lambda}) \psi(g), \psi(h) \rangle
\end{align*}
\]

where \( w_X \) is a simple function. In fact we may assume \([8]\) that \( w_X \) is a step function and that the partition \( \Pi \) is so small that each interval \([s_{\lambda}, s_{\lambda} + 1]\) is contained in one of the disjoint intervals defining \( w_X \). Let \( w_X(s_{\lambda}) \) be the value of \( w_X \) in that interval. Then

\[
\langle \int_0^t f(s) \, dX(s) \psi(g), \psi(h) \rangle
\]

is similar.

**REFERENCES**


