The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 13, Issue 1, Article 21, pp. 1-14, 2016

STABILITY OF THE D-BAR RECONSTRUCTION METHOD FOR COMPLEX CONDUCTIVITIES

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Received 22 November, 2016; accepted 21 December, 2016; published 29 December, 2016.

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ABSTRACT. In 2000, Francini solved the inverse conductivity problem for twice-differentiable conductivities and permittivities. This solution was considered to be the first approach using D-bar methods with complex conductivities. In 2012, based on Francini's work, Hamilton introduced a reconstruction method of the conductivity distribution with complex values. The method consists of six steps. A voltage potential is applied on the boundary. Solving a D-Bar equation gives the complex conductivity. In this paper, the stability of the D-Bar equation is studied via two approximations, t^{exp} and t^{B}, for the scattering transform. The study is based on rewriting the reconstruction method in terms of continuous operators. The conductivity is considered to be non smooth.

Key words and phrases: D-Bar method; Scattering transform; Stability; Complex conductivity.

2000 *Mathematics Subject Classification.* Primary: 35R30.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The inverse conductivity problem can be mathematically expressed as the Dirichlet problem

$$(\mathbf{P1}) \qquad \begin{cases} \nabla \cdot \big(\gamma(z) \nabla u(z)\big) = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}$$

where $\gamma \in L^\infty(\Omega)$ is the electrical admittivity, $u(z) \in H^1(\Omega)$ represents the electrical potential, and Ω is an open smooth and bounded domain in $\mathbb{R}^2$.
The Dirichlet-to-Neumann map

$$\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$
$$\Lambda_\gamma f = \gamma \tfrac{\partial u}{\partial \nu}|_{\partial\Omega}$$

represents the current flux on the boundary $\partial\Omega$, where $\dfrac{\partial u}{\partial \nu}$ is the normal derivative, and ν is the outer unit vector. For complex values, the admittivity $\gamma(z)$ can be written as $\gamma(z) = \sigma(z)+i\varepsilon(z)$, where $\sigma(z)$ represents the conductivity and $\varepsilon(z)$ the admittivity.
An important application of the inverse admittivity problem, is the electrical impedance tomography (EIT). This technology allows us to determine the location of the tumors inside a body (cancer tissues). Experimentally, the cancer cells appear to be four times more conductive than the normal tissues. Also the (EIT) technology is considered to be less harmful for patients and more time consumable.
In 1980, Calderon introduced the inverse conductivity problem. He solved the problem by considering small perturbations of constant conductivity (see[4]).
Later, in 1996, Nachman solved the Calderon's problem using the D-Bar method. In this method he found the scattering transform $t(k)$ from the boundary values Λ_γ, then he reconstructed the real-valued function γ from the scattering transform $t(k)$. The method was of high regularity, where $\gamma \in C^2(\Omega)$, (see [18]).
Brown and Uhlmann, in 1997, transformed the problem into a first-order system. In this method, the problem was solved for weaker regularity, namely $\gamma \in C^1(\Omega)$ (see[2]). This method was reformulated as a reconstructive procedure by K. Knudsen and A. Tamasan in [14].
Later, the problem was solved for smooth conductivity γ, i.e. $\gamma \in L^\infty(\Omega)$. This was done by Astala and Paivarinta in 2006 in [1].
In 2003, an approximation for the scattering transform $t(k)$ was introduced by S. Siltanen, J. Mueller and D. Isaacson. The approximation was denoted by t^{exp}, where, in the scattering transform $t(k)$, the (CGO) solutions are replaced by the asymptotic behavior of the CGO solutions (for more details see [19]).
Another approximation for $t(k)$ was introduced by S. Siltanen and J. Mueller in [17]. This approximation is denoted by t^B, where an approximation for the CGO solutions is determined by solving a boundary integral equation. These approximations can be seen as regularization techniques.
In 2014, El Arwadi studied the stability of the reconstruction method via t^B approximation in [5], while Siltanen and Mueller in [10], studied the stability of the method via t^{exp} approximation.
The uniqueness of the solutions for the inverse conductivity problem for complex conductivity was proved by Francini in [7]. She used the CGO solutions with $\gamma = \sigma + i\omega\varepsilon$, where γ, ω and ε are respectively the electrical conductivity, frequency and electrical permittivity. Her work can be considered as an extension of the real-valued conductivity results to complex-valued including the electrical permittivity ε. Francini's work included the proof of the existence and

uniqueness of admittivity γ, in addition to introduce an approach to solve the problem using the D-Bar method.

Later in 2012, S. Hamilton and A. Von Herrmann introduced a six-step D-Bar reconstruction algorithm based on the D-Bar method, (see[8]), which is considered as a reformulation of Francini's work. The considered reconstruction problem is an ill-posed inverse problem, since small differences in the boundary measurements will cause changing in the internal conductivity.

In studying the stability of the reconstruction method, the boundary values are no more Λ_γ. Since due to the noise ε, the boundary data is denoted by $\Lambda_\gamma^\varepsilon$. Thus, as the noise ε tends to zero, then γ^ε tends to γ. Hence the problem is said to be stable.

In this paper, we present the six-step reconstruction method in section 2, then the stability of this method is studied in section 3, by considering the approximation t^{exp}. In section 4, we study the stability via the t^B approximation and we end with a concluding section.

2. DIRECT RECONSTRUCTION ALGORITHM

In this section, we present the direct reconstruction algorithm as mentioned in Hamilton's work in [9]. The algorithm consists of six steps, it starts with Dirichlet-to-Neumann map Λ_γ, which represents the current flux on the boundary and ending with the values of the admittivity γ. The six steps are as follows

$$\Lambda_\gamma \to u_1(z,k), u_2(z,k) \to \psi_{12}(z,k), \psi_{12}(z,k) \to S(k) \to M(z,k) \to Q(z) \to \gamma(z)$$

Throughout the whole algorithm, σ_0 and E are assumed to be positive such that

$$\sigma(z) > \sigma_0, \;\; z \in \Omega \subset \mathbb{R}^2 \tag{2.1}$$

and

$$\|\sigma\|_{W^{1,\infty}}, \|\varepsilon\|_{W^{1,\infty}} \leq E, \tag{2.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$.

The following two theorems relate the current flux Λ_γ on the boundary with the exponentially growing solutions u_1 and u_2.

Theorem 1. [22]. *Let $\gamma(z) = \sigma(z) + iw\varepsilon(z) \in W^{2,p}(\Omega)$, with $p > 1$ such that σ and ε satisfy* (2.1) *and* (2.2)*, and let $\gamma(z) - 1$ have a compact support in $W^{2,p}(\Omega)$. Suppose that $\gamma = 1$ in a neighborhood of $\partial\Omega$. Then for any non-exceptional $k \in \mathbb{C}\backslash\{0\}$, the trace of the exponentially growing solution $u_1(.,k)$ on $\partial\Omega$ is the unique solution to the boundary integral equation*

$$u_1(z,k) = \frac{e^{ikz}}{ik} - \int_{\partial\Omega} G_k(z-\zeta)(\Lambda_\gamma - \Lambda_1)u_1(\zeta,k)dS(\zeta) \tag{2.3}$$

Theorem 2. [8] *Let $\gamma(z) = \sigma(z) + iw\varepsilon(z) \in W^{2,p}(\Omega)$, with $p > 1$ such that σ and ε satisfy* (2.1) *and* (2.2)*, and let $\gamma(z) - 1$ have a compact support in $W^{2,p}(\Omega)$. Suppose that $\gamma = 1$ in a neighborhood of $\partial\Omega$. Then for any non-exceptional $k \in \mathbb{C}\backslash\{0\}$, the trace of the exponentially growing solution $u_2(.,k)$ on $\partial\Omega$ is the unique solution to the boundary integral equation*

$$u_2(z,k) = \frac{e^{-ik\bar{z}}}{-ik} - \int_{\partial\Omega} G_k(-\bar{z}+\bar{\zeta})(\Lambda_\gamma - \Lambda_1)u_2(\zeta,k)dS(\zeta) \tag{2.4}$$

These theorems result from the relationship between the exponentially growing solutions $\psi_s(z,k)$ to the Schrödinger equation

$$(-\Delta + q(z))\psi_s(z,k) = 0$$

and the CGO solutions $u_1(z,k)$ and $u_2(z,k)$, where $q = \dfrac{\Delta\gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}}$.

In the case of a real-valued function γ, the trace of the function $\psi_s(z,k)$ on $\partial\Omega$ satisfies the integral equation

$$\psi_s(z,k) = e^{ikz} - \int_{\partial\Omega} G_k(z-\zeta)(\Lambda_\gamma - \Lambda_1)\psi_s(z,k)dS(\zeta) \tag{2.5}$$

where $z \in \partial\Omega$ and $k \in \mathbb{C}\backslash\{0\}$.

The function $G_k(z)$ is the Faddeev's Green's function, and is defined by

$$G_k(z) = e^{ikz} g_k(z), \ -\Delta G_k = \delta$$

where

$$g_k(z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{iz\xi}}{\xi(\bar{\xi}+2k)}d\xi, \ (-\Delta - 4ik\bar{\partial}_z)g_k = \delta.$$

Equation (2.5) is a Fredholm equation of the second kind and is uniquely solvable in $H^{\frac{1}{2}}(\partial\Omega)$ for any $k \in \mathbb{C}\backslash\{0\}$.

The following lemmas state the relation between $\psi_s(z,k)$ and the CGO solutions $u_1(z,k)$ and $u_2(z,k)$

Lemma 1. [22]. *Let $\gamma(z) = \sigma(z) + iw\varepsilon(z) \in W^{2,p}(\Omega)$, with $p > 2$ such that σ and ε satisfy* (2.1) *and* (2.2)*, and let $\gamma(z) - 1$ have a compact support in $W^{1,p}(\Omega)$. Let $u_1(z,k)$ be the exponentially growing solution to the admittivity equation in (P1), and $\psi_s(z,k)$ be the exponentially growing solution to the Schrödinger equation* (2)*, when it exists, then*

$$iku_1(z,k) = \gamma^{-\frac{1}{2}}(z)\psi_s(z,k)$$

Lemma 2. [8]. *Let $\gamma(z) = \sigma(z) + iw\varepsilon(z) \in W^{2,p}(\Omega)$, with $p > 2$ such that σ and ε satisfy* (2.1) *and* (2.2)*, and let $\gamma(z) - 1$ have a compact support in $W^{1,p}(\Omega)$. Let $u_2(z,k)$ be the exponentially growing solution to the admittivity equation in (P1), and $\psi_s(z,k)$ be the exponentially growing solution to the Schrödinger equation* (2)*, when it exists, then*

$$-iku_2(z,k) = \gamma^{-\frac{1}{2}}(-\bar{z})\psi_s(-\bar{z},k)$$

If $u(z)$ is the solution of the Dirichlet problem (**P1**), and *f* is the electrical potential applied on the boundary, then the problem (**P1**) can be transformed into a system of first order as follows,

$$D\begin{pmatrix} u \\ v \end{pmatrix} - Q\begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{2.6}$$

where

$$D = \begin{pmatrix} \bar{\partial}_z & 0 \\ 0 & \partial_z \end{pmatrix}$$

The potential matrix Q is given by

$$Q = \begin{pmatrix} 0 & -\frac{1}{2}\partial_z \log\gamma(z) \\ -\frac{1}{2}\bar{\partial}_z \log\gamma(z) & 0 \end{pmatrix} \tag{2.7}$$

The operators ∂_z and $\bar{\partial}_z$ are the Cauchy-Riemann operators given by the formulas

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and
$$\bar{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$
Let
$$\begin{pmatrix} v \\ w \end{pmatrix} = \gamma(z)^{\frac{1}{2}} \begin{pmatrix} \partial_z u \\ \bar{\partial}_z u \end{pmatrix}$$
be a particular solution to the system in (2.6). $\mathbb{R}^2$ is represented with $\mathbb{C}$ so that $z = x + iy$,where $x, y \in \mathbb{R}$.

The second step of the algorithm is to relate the geometrical optics solutions $\Psi(z,k)$ with the exponentially growing solutions u_1 and u_2, such that
$$\begin{pmatrix} \Psi_{11} \\ \Psi_{21} \end{pmatrix} = \gamma^{\frac{1}{2}} \begin{pmatrix} \partial_z u_1 \\ \bar{\partial}_z u_1 \end{pmatrix}$$
and
$$\begin{pmatrix} \Psi_{12} \\ \Psi_{22} \end{pmatrix} = \gamma^{\frac{1}{2}} \begin{pmatrix} \partial_z u_2 \\ \bar{\partial}_z u_2 \end{pmatrix}.$$
Then system (2.6) takes the form
$$(D - Q)\,\Psi(z,k) = 0$$
where k is a complex parameter.
The solutions $\Psi(z,k)$ of the above system can be written in the form
$$\Psi(z,k) = M(z,k)\begin{pmatrix} e^{ikz} & 0 \\ 0 & e^{-ik\bar{z}} \end{pmatrix} = \begin{pmatrix} M_{11}e^{ikz} & M_{12}e^{ikz} \\ M_{21}e^{-ik\bar{z}} & M_{22}e^{-ik\bar{z}} \end{pmatrix} \tag{2.8}$$
where $M(z,k)$ is a complex-valued matrix function such that
$$\|M(z,k) - I\|_{L^p(R^2)} < \infty \tag{2.9}$$
for some $p > 2$. I is the 2×2 identity matrix.
In Theorem 4.1 in [7], Francini solves a D-Bar equation, which determines the complex matrix M. The theorem states the following:

Theorem 3. *Let σ and ε satisfy the conditions* (2.1) *and* (2.2) *and M be as in* (2.9)*. The map $k \longrightarrow M(.,k)$ is differentiable as a map into $L^r_{-\beta}$ and satisfies the equation*
$$\bar{\partial}_k M(z,k) = M(z,\bar{k})\begin{pmatrix} e_{\bar{k}}(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix} S(k)$$
where
$$S(k) = \frac{i}{\pi}\int_{\mathbb{R}^2} \begin{pmatrix} e_{-\bar{k}}(z) & 0 \\ 0 & -e_k(z) \end{pmatrix} (QM)^{off} d\mu(z)$$
Moreover, for every $p > 2$, $sup\|M(z,.) - I\|_{L^p(R^2)} \leq k_2$ where k_2 depends on E, σ_0 , Ω and p.

The matrix S is called the scattering matrix associated to the admittivity γ.
Using (2.8) in System (2.6), allows us to find a relation between the complex matrix M and the potential matrix Q as follows
$$\bar{\partial}_z M_{11}(z,k) - Q_{12}(z)M_{21}(z,k) = 0 \tag{2.10}$$
$$(\bar{\partial}_z - ik)M_{12}(z,k) - Q_{12}(z)M_{22}(z,k) = 0 \tag{2.11}$$
$$(\partial_z + ik)M_{21}(z,k) - Q_{21}(z)M_{11}(z,k) = 0 \tag{2.12}$$

$$\partial_z M_{22}(z,k) - Q_{21}(z)M_{12}(z,k) = 0 \tag{2.13}$$

and in matrix form

$$D_k M(z,k) - Q(z)M(z,k) = 0$$

where

$$D_k M = DM - ik \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^{\text{off}}$$

and M^{off} is the off-diagonal part of the matrix M.
The determination of $\Psi_{12}(z,k)$ and $\Psi_{21}(z,k)$ in terms of the CGO solutions $u_1(z,k)$ and $u_2(z,k)$ can be seen in the following theorem.

Theorem 4. [8] *The traces of the exponentially growing solutions $\Psi_{12}(z,k)$ and $\Psi_{21}(z,k)$ for $k \in C\backslash\{0\}$ can be determined by*

$$\Psi_{12}(z,k) = \int_{\partial\Omega} \frac{e^{i\bar{k}(z-\zeta)}}{4\pi(z-\zeta)}(\Lambda_\gamma - \Lambda_1)u_2(\zeta,k)dS(\zeta) \tag{2.14}$$

$$\Psi_{21}(z,k) = \int_{\partial\Omega} \overline{\left[\frac{e^{ik(z-\zeta)}}{4\pi(z-\zeta)}\right]}(\Lambda_\gamma - \Lambda_1)u_1(\zeta,k)dS(\zeta) \tag{2.15}$$

where $u_1(z,k)$ and $u_2(z,k)$ are determined using equations (2.3) *and* (2.4).

The third step of the algorithm is to determine the scattering transform $S(k)$ from the exponentially growing solutions $\Psi_{12}(z,k)$ and $\Psi_{21}(z,k)$. In theorem 3, the scattering transform matrices take the form

$$\begin{aligned} S(k) &= \frac{i}{\pi}\int_{\mathbb{R}^2} \begin{pmatrix} e_{-\bar{k}}(z) & 0 \\ 0 & -e_k(z) \end{pmatrix} (QM)^{off} d\mu(z) \\ &= \frac{i}{\pi}\int_{\mathbb{R}^2} \begin{pmatrix} e_{-\bar{k}}(z) & 0 \\ 0 & -e_k(z) \end{pmatrix} \begin{pmatrix} 0 & Q_{12}(z)M_{22}(z,k) \\ Q_{21}(z)M_{11}(z,k) & 0 \end{pmatrix} d\mu(z) \end{aligned}$$

The entries S_{12} and S_{21} are

$$S_{12}(k) = \frac{i}{2\pi}\int_{\partial\Omega} e^{-i\bar{k}z}\psi_{12}(z,k)\nu dS(z) \tag{2.16}$$

$$S_{21}(k) = \frac{-i}{2\pi}\int_{\partial\Omega} e^{i\bar{k}}\psi_{21}(z,k)\nu dS(z) \tag{2.17}$$

where ν represents the outer unit vector to the boundary $\partial\Omega$.

The complex matrix M is related to the scattering transform matrix S in the D-Bar equation stated in theorem 3. Thus the D-Bar equation takes the form

$$\bar{\partial}_k M(z,k) = M(z,\bar{k}) \begin{pmatrix} e_{\bar{k}}(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix} S(k) \tag{2.18}$$

This equation is solved inside the domain Ω.

The entries of the potential matrix Q can be determined in terms of the complex matrix M using equations (2.10), (2.11), (2.12) and (2.13) as follows

$$Q_{12}(z) = \frac{\bar{\partial}_z M_+(Q,z,0)}{M_-(Q,z,0)} \tag{2.19}$$

$$Q_{21}(z) = \frac{\bar{\partial}_z M_-(Q,z,0)}{M_+(Q,z,0)} \tag{2.20}$$

where

$$M_+(Q,z,k) = M_{11}(Q,z,k) + e(z,-k)M_{12}(Q,z,k) \tag{2.21}$$

$$M_-(Q,z,k) = M_{22}(Q,z,k) + e(z,k)M_{21}(Q,z,k). \tag{2.22}$$

The potential matrix Q is defined in terms of the admittivity γ mentioned in (2.7). Thus,

$$Q = \begin{pmatrix} 0 & -\frac{1}{2}\partial_z \log(\gamma(z)) \\ -\frac{1}{2}\bar{\partial}_z \log(\gamma(z)) & 0 \end{pmatrix} = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \tag{2.23}$$

then

$$Q_{12} = -\frac{1}{2}\partial_z \log\gamma(z) \tag{2.24}$$

$$Q_{21} = -\frac{1}{2}\bar{\partial}_z \log\gamma(z). \tag{2.25}$$

Therefore, the admittivity distribution γ is determined from the above relations, and we write

$$\log\gamma(z) = -\frac{2}{\pi}\int_C \frac{Q_{12}}{\bar{z}-\zeta}d\mu(\zeta) = -\frac{2}{\pi}\int_C \frac{Q_{21}}{z-\zeta}d\mu(\zeta). \tag{2.26}$$

3. STABILITY OF THE D-BAR EQUATION VIA t^{exp} APPROXIMATION

The t^{exp} regularization can be obtained by replacing the trace of the exponentially growing solutions $u_1(z,k)$ and $u_2(z,k)$ with $\dfrac{e^{ikz}}{ik}$ and $\dfrac{e^{-ik\bar{z}}}{-ik}$ respectively.
In this section we study the stability of the D-Bar reconstruction method with t^{exp} regularization for complex values of the admittivity $\gamma(z)$. For real conductivities, the stability was studied in [10]. In this section we follow a similar strategy. We start by reformulating each step of the reconstruction method as an operator. Thus we define,

$$u_1^{exp} = \frac{e^{ikz}}{ik} \text{ and } u_2^{exp} = \frac{e^{-ik\bar{z}}}{-ik}$$

which represents the approximation of step 1 in the reconstruction method.
Our target is to construct an operator T^{exp} such that $\gamma^{exp} = T^{exp}(\Lambda_\gamma - \Lambda_1)$. If δ represents the noise level due to external effects, then $\gamma_\delta^{exp} = T^{exp}(\Lambda_\gamma^\delta - \Lambda_1)$. The stability of the reconstruction method can be established by studying the difference

$$\|\gamma^{exp} - \gamma_\delta^{exp}\| = \left\|T^{exp}(\Lambda_\gamma - \Lambda_1) - T^{exp}(\Lambda_\gamma^\delta - \Lambda_1)\right\|$$

Thus, as δ tends to zero, γ_δ^{exp} must tend to γ. This result can be established by proving the continuity of the operator T^{exp}.
We denote by **M** $\subset \mathbb{L}(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega))$, the set of operators $L \in \mathbb{L}(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega))$, such that $\Lambda_\gamma 1 = \Lambda_1 1 = 0$ and $\int_{\partial\Omega} Lf d\sigma = 0$.
We introduce the following linear operators T_1^{exp} and $\widetilde{T}_1^{exp}$ from **M** to $H^{\frac{1}{2}}(\partial\Omega)$ by

$$(T_1^{exp}L)(z,k) = \frac{-1}{ik}\int_{\partial\Omega} \frac{e^{i\bar{k}(z-\zeta)}}{4\pi(z-\zeta)} L(e^{-ik\bar{\zeta}}-1)dS(\zeta)$$

and

$$(\widetilde{T}_1^{exp}L)(z,k) = \frac{1}{ik}\int_{\partial\Omega} \overline{\left[\frac{e^{ik(z-\zeta)}}{4\pi(z-\zeta)}\right]} L(e^{ik\zeta}-1)dS(\zeta)$$

Lemma 3. *The operators T_1^{exp} and $\widetilde{T}_1^{exp}$ are continuous from* **M** $\rightarrow H^{\frac{1}{2}}(\partial\Omega)$

Proof. Consider the operator

$$(T_1^{exp}L)(z,k) = \frac{-1}{ik}\int_{\partial\Omega} \frac{e^{i\bar{k}(z-\zeta)}}{4\pi(z-\zeta)} L(e^{ik\bar{\zeta}}-1)dS(\zeta)$$

from **M** $\rightarrow H^{\frac{1}{2}}(\partial\Omega)$. Then

$$\begin{aligned} |(T_1^{exp}L)(z,k)| &= \frac{1}{|k|}\left|\int_{\partial\Omega} \frac{e^{i\bar{k}(z-\zeta)}}{4\pi(z-\zeta)} L(e^{ik\bar{\zeta}}-1)dS(\zeta)\right| \\ &\leq \frac{1}{|k|}\left\|\frac{e^{i\bar{k}(z-\zeta)}}{4\pi(z-\zeta)}\right\|_{H^{\frac{1}{2}}(\partial\Omega)} \|L\|_{\frac{1}{2},-\frac{1}{2}} \left\|(e^{ik\bar{\zeta}}-1)\right\|_{H^{\frac{1}{2}}(\partial\Omega)} \end{aligned}$$

We recall some estimates from [6] and [16] for some terms appearing in (3.1)

$$\left\|e^{ikz}-1\right\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq |k|e^{|k|}. \tag{3.1}$$

and

$$\left\|\frac{e^{i\bar{k}(z-\zeta)}}{z-\xi}\right\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq e^{|k|} \tag{3.2}$$

then $|(T_1^{exp}L)(z,k)|$ could be bounded as follows

$$\begin{aligned} |(T_1^{exp}L)(z,k)| &\leq \frac{1}{|k|}e^{|k|}. \|L\|_{\frac{1}{2},-\frac{1}{2}} .|k|.e^{|k|} \\ &\leq C_1 \|L\|_{\frac{1}{2},-\frac{1}{2}} \end{aligned}$$

where C_1 is a constant depending on the parameter k. Then the operator T_1^{exp} is continuous. Similarly, we prove the continuity of the operator $\widetilde{T}_1^{exp}$. Thus Lemma 3 is proved. □

The scattering transform $S^{exp}(k)$ blows up as k tends to infinity. This allows us to use its truncation

$$S_R^{exp}(k) = \begin{cases} S^{exp}(k) & \text{if } |k| < R, \\ 0 & \text{otherwise}. \end{cases}$$

Define the operators T_2^{exp} and $\widetilde{T}_2^{exp}$ from **M** to $L_c^p(\mathbb{R}^2)$ by

$$(T_{R,2}^{exp}L)(z,k) = \frac{1}{2\pi}\int_{\partial\Omega} e^{-i\bar{k}z}(T_1^{exp}L)(z,k)\vec{\nu}dS(\zeta) \tag{3.3}$$

and

$$(\widetilde{T}_{R,2}^{exp}L)(z,k) = \frac{-1}{2\pi}\int_{\partial\Omega} e^{\overline{ikz}}(\widetilde{T}_1^{exp}L)(z,k)\vec{\nu}dS(\zeta)$$

Lemma 4. *The operators $T_{R,2}^{exp}$ and $\widetilde{T}_{R,2}^{exp}$ are continuous from **M** $\to L_c^p(\mathbb{R}^2)$*

Proof. Using (3.3) we write,

$$\left|(T_{R,2}^{exp}L)(k)\right| = \left|\frac{1}{2\pi}\int_{\partial\Omega} e^{-i\bar{k}z}(T_1^{exp}L)(\zeta,k)\vec{\nu}dS(\zeta)\right|$$

then,

$$\begin{aligned}\left|(T_{R,2}^{exp}L)(k)\right| &\leq \left\|e^{-i\bar{k}z}\right\|_{H^{\frac{1}{2}}(\partial\Omega)} \|T_1^{exp}L\|_{L^\infty(\mathbb{R}^2)} \\ &\leq C_2\,\|L\|_{\frac{1}{2},-\frac{1}{2}}\end{aligned}$$

where we used (3.1) and (3.1), and C_2 is a constant depending on *k*.
This shows that the operator $T_{R,2}^{exp}$ is continuous. Similarly, we prove that the operator $\widetilde{T}_{R,2}^{exp}$ is continuous, and this proves Lemma 4. □

The study of the stability of the D-Bar equation via t^{exp} approximation depends on the operator **S** as shown in [10]. The operator **S** : $L_c^p(\mathbb{R}^2) \to C^\infty(\bar{\Omega})$, relates the scattering data Φ to the matrix $M(z,k)$ by

$$S(\Phi(z)) = M(z,0) \tag{3.4}$$

where $L_c^p(\mathbb{R}^2)$ is the space of functions in $L^p(\mathbb{R}^2)$ with compact support. The continuity of **S** is proved in [10].
From theorem (6.2) of [7], the potential matrix Q can take the form,

$$Q(z) = \lim_{k_0\to\infty} \mu\left(B_{\rho(0)}\right)^{-1}\int_{k:|k-k_0|<\rho} D_kM(z,k)d\mu(k) \tag{3.5}$$

Since the region $\mathbb{R}^2$ is truncated for $|k| < R$, and a disc of radius $\rho < 1$ is considered, then the matrix Q takes the form

$$Q_R(z) = \int_{k:|k-k_0|<\rho} D_kM(z,k)d\mu(k) \tag{3.6}$$

Define the operator $T_{R,3}^{exp}$ from $C^\infty(\bar{\Omega})$ to $C^\infty(\Omega)$ by

$$T_{R,3}^{exp}M(z,k) = Q_R(z) \tag{3.7}$$

Lemma 5. *The operator $T_{R,3}^{exp}$ from $C^\infty(\bar{\Omega})$ to $C^\infty(\Omega)$ is continuous.*

Proof. The matrix $M \in C^\infty(\bar{\Omega})$, then M_{ij} are bounded for $i, j = 1, 2$. The operator D_k is a continuous operator (see [9]). Using (3.6) we write

$$\begin{aligned} \left\| T^{exp}_{R,3} M \right\|_{L^\infty(\Omega)} &= \left\| \int_{k:|k-k_0|<\rho} D_k M(z,k) d\mu(k) \right\|_{L^\infty(\Omega)} \\ &\leq \| D_k M \|_{L^\infty(\Omega)} \int_{k:|k-k_0|<\rho} d\mu(k) \\ &\leq C(R) \| D_k M \|_{L^\infty(\Omega)} \\ &\leq C_3 \| M \|_{L^\infty(\Omega)} . \end{aligned}$$

This proves Lemma 5, where C_3 is a constant depending on k and R. □

The admittivity γ^{exp}_R is related to the potential matrix Q_R by the following relations

$$\log \gamma^{exp}_R(z) = \bar{\partial}^{-1}_k Q_{R,12}(z) \tag{3.8}$$

The operator $\bar{\partial}^{-1}_k$ is bounded, see [7]. Then it is continuous.

The admittivity γ can be written as a composite of continuous linear operators as follows

$$\begin{aligned} \log \gamma^{exp}_R(z) &= \bar{\partial}^{-1} \left(T^{exp}_3 o \mathbf{S} o T^{exp}_2 o T^{exp}_1 L \right)(z) \\ &= (T^{exp} L)(z) \end{aligned}$$

where $T^{exp} = \bar{\partial}^{-1} \left(T^{exp}_3 o \mathbf{S} o T^{exp}_2 o T^{exp}_1 \right)$. If we let $L = \Lambda_\gamma - \Lambda_1$ then the reconstructed conductivity can be written in the form

$$\begin{aligned} \left\| \log \gamma^{exp}_R - \log \gamma^{exp}_{R,\varepsilon} \right\|_{L^\infty(\Omega)} &= \left\| T^{exp}(\Lambda_\gamma - \Lambda_1) - T^{exp}(\Lambda^\varepsilon_\gamma - \Lambda_1) \right\|_{L^\infty(\Omega)} \\ &= \left\| T^{exp}(\Lambda_\gamma - \Lambda^\varepsilon_\gamma) \right\|_{L^\infty(\Omega)} \\ &\leq |T^{exp}| \left\| \Lambda_\gamma - \Lambda^\varepsilon_\gamma \right\|_{\frac{1}{2},-\frac{1}{2}} \end{aligned}$$

Since the operator T^{exp} is continuous as shown above, then as δ tends to zero, $\log \gamma^{exp}_{R,\delta}$ tends to $\log \gamma^{exp}_R$. The logarithmic function is bijective, then $\gamma^{exp}_{R,\delta}$ tends to γ^{exp}_R and this shows that the reconstruction method is stable via t^{exp} regularization.

4. STABILITY OF THE D-BAR EQUATION VIA t^B APPROXIMATION

In [5], the stability of the D-Bar method was studied via the t^B approximation. The study was done for real values of the admittivity. In this section, we study the stability for complex values of the admittivity, following similar idea. The t^B approximation can be applied by replacing the Faddeev's Green's function $G_k(z)$ with the standard Green's functions $G_0 = -\frac{1}{2\pi}\log(z)$, in the boundary integral equations (2.3) and (2.4). Then

$$\begin{aligned} u^B_1(z,k) &= \frac{e^{ikz}}{ik} - \int_{\partial\Omega} G_0(z-\zeta)(\Lambda_\gamma - \Lambda_1) u^B_1(\zeta,k) dS(\zeta) \\ &= \frac{e^{ikz}}{ik} - S_0(\Lambda_\gamma - \Lambda_1) u^B_1(z,k) \end{aligned}$$

where S_0 is the single-layer operator with kernel G_0, see [16]. As $\Lambda_\gamma 1 = \Lambda 1 = 0$, and adding the term $\frac{-1}{ik}$ on both sides of (4.1), we write

$$u^B_1(z,k) - \frac{1}{ik} = \frac{e^{ikz}}{ik} - S_0(\Lambda_\gamma - \Lambda_1)\left(u^B_1(z,k) - \frac{1}{ik} \right) - \frac{1}{ik}$$

then

$$(I + S_0(\Lambda_\gamma - \Lambda_1)) \left(u^B_1(z,k) - \frac{1}{ik} \right) = \frac{1}{ik}\left(e^{ikz} - 1 \right)$$

then $u_1^B(z,k)$ can be written as

$$u_1^B(z,k) = \frac{1}{ik}\left[(I + S_0(\Lambda_\gamma - \Lambda_1))^{-1}\left(e^{ikz} - 1\right) + 1\right]$$

where the operator $I + S_0(\Lambda_\gamma - \Lambda_1)$ is invertible, (see [19]).

Applying the change of variable $-\bar{\zeta} = \xi$ and replacing $-\bar{z}$ by z in equation (2.4) we write

$$u_2^B(z,k) = -\frac{1}{ik}\left[(I + S_0(\Lambda_\gamma - \Lambda_1))^{-1}(e^{-ik\bar{z}} - 1) + 1\right]$$

Let $L \in M$, define the linear operators T_1^B and $\widetilde{T}_1^B$ from **M** to $H^{\frac{1}{2}}(\partial\Omega)$ as

$$(T_1^B L)(z,k) = \frac{1}{ik}\left[(I + S_0 L)^{-1}(e^{ikz} - 1) + 1\right]$$

and

$$(\widetilde{T}_1^B L)(z,k) = -\frac{1}{ik}\left[(I + S_0 L)^{-1}(e^{-ik\bar{z}} - 1) + 1\right]$$

Lemma 6. *The operators T_1^B and $\widetilde{T}_1^B$ are continuous from **M** $\rightarrow H^{\frac{1}{2}}(\partial\Omega)$*

Proof. For $L, \tilde{L} \in M$, let $A_{0,L} = (I + S_0 L)$, then

$$(T_1^B L)(z,k) = \frac{1}{ik}\left[A_{0,L}^{-1}(e^{ikz} - 1) + 1\right]$$

In Theorem (3.1) in [5], the operator $A_{0,L} = (I + S_0 L)$ was studied in details. From this theorem we can write

$$\left\|A_{0,L}^{-1} - A_{0,\tilde{L}}^{-1}\right\|_{\frac{1}{2},-\frac{1}{2}} \le C_4 \left\|L - \tilde{L}\right\|_{\frac{1}{2},-\frac{1}{2}}$$

where C_4 is a constant. Then,

$$\begin{aligned}
\left|(T_1^B L - T_1^B \tilde{L})(k)\right| &= \left|\left(A_{0,L}^{-1} - A_{0,\tilde{L}}^{-1}\right)(e^{ikz} - 1)\right| \\
&\le \left\|A_{0,L}^{-1} - A_{0,\tilde{L}}^{-1}\right\|_{\frac{1}{2},-\frac{1}{2}} \left\|e^{ikz} - 1\right\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\le C_5 \left\|L - \tilde{L}\right\|_{\frac{1}{2},-\frac{1}{2}}
\end{aligned}$$

Thus the operator T_1^B is continuous. Similarly, we prove the continuity of the operator $\widetilde{T}_1^B$. Thus the result of Lemma 6 is proved. □

Moreover, another two operators T_2^B and $\widetilde{T}_2^B$ can be defined from **M** to $H^{\frac{1}{2}}(\partial\Omega)$ by

$$(T_2^B L)(z,k) = \int_{\partial\Omega} \frac{e^{i\bar{k}(z-\zeta)}}{4\pi(z-\zeta)} L(\widetilde{T}_1^B L)(k)dS(\zeta)$$

and

$$(\widetilde{T}_2^B L)(z,k) = \int_{\partial\Omega} \overline{\left[\frac{e^{ik(z-\zeta)}}{4\pi(z-\zeta)}\right]} L(T_1^B L)(k)dS(\zeta)$$

The above operators are non linear operators due to the existence of the terms $L(\widetilde{T}_1^B L)$ and $L(T_1^B L)(k)$.

Lemma 7. *The operators T_2^B and $\widetilde{T}_2^B$ are continuous from **M** to $H^{\frac{1}{2}}(\partial\Omega)$*

Proof. Let $L, \tilde{L} \in M$, then

$$\left|T_2^B L(z,k) - T_2^B \tilde{L}(z,k)\right| = \left|\int_{\partial\Omega} \frac{e^{i\bar{k}(z-\zeta)}}{4\pi(z-\zeta)} \left[(L\widetilde{T}_1^B L)(z,k) - (\tilde{L}\widetilde{T}_1^B \tilde{L})(z,k)\right] dS(\zeta)\right| \tag{4.1}$$

Consider the term $(L\widetilde{T}_1^B L)(z,k) - (\tilde{L}\widetilde{T}_1^B \tilde{L})(z,k)$, then we write

$$\begin{aligned}(L\widetilde{T}_1^B L)(z,k) - (\tilde{L}\widetilde{T}_1^B \tilde{L})(z,k) &= (L-\tilde{L})(\widetilde{T}_1^B L)(z,k) + (\tilde{L}\widetilde{T}_1^B L)(z,k) - (\tilde{L}\widetilde{T}_1^B \tilde{L})(z,k) \\ &= (L-\tilde{L})(\widetilde{T_1^B} L)(z,k) + \tilde{L}((\widetilde{T}_1^B L)(z,k) - (\widetilde{T}_1^B \tilde{L})(z,k))\end{aligned}$$

then

$$\begin{aligned}\left|(L\widetilde{T}_1^B L)(z,k) - (\tilde{L}\widetilde{T}_1^B \tilde{L})(z,k)\right| &= \left|(L-\tilde{L})(\widetilde{T}_1^B L)(z,k) + \tilde{L}((\widetilde{T}_1^B L)(z,k) - (\widetilde{T}_1^B \tilde{L})(z,k))\right| \\ &\leq \left|(L-\tilde{L})\widetilde{T}_1^B L(z,k)\right| + \left|\tilde{L}((\widetilde{T}_1^B L)(z,k) - (\widetilde{T}_1^B \tilde{L})(z,k))\right|\end{aligned}$$

Using the facts that the operator $\widetilde{T}_1^B$ is continuous as proved in Lemma 6, and the operators L and $\tilde{L}$ are linear, then they are bounded. Hence we write

$$\left|L(\widetilde{T}_1^B L)(k) - \tilde{L}(\widetilde{T}_1^B \tilde{L})(k)\right| \leq C_7 \left\|L-\tilde{L}\right\|_{\frac{1}{2},-\frac{1}{2}} \tag{4.2}$$

Using (3.2) and (4.2) in (4.1) we get the proof of Lemma 7. □

The scattering transform $S^B(k)$ is truncated for $k < R$, then we write

$$S_R^B(k) = \begin{cases} S^B(k) & \text{if } \ |k| < R, \\ 0 & \text{otherwise}. \end{cases}$$

If we define the operators $T_{R,3}^B$ and $\widetilde{T}_{R,3}^B$ from **M** to $L_c^p(\mathbb{R}^2)$ by

$$(T_{R,3}^B L)(k) = \frac{1}{2\pi}\int_{\partial\Omega} e^{-i\bar{k}z}(T_2^B L)(z,k)\vec{\nu}\,dS(\zeta)$$

and

$$(\widetilde{T}_{R,3}^B L)(k) = \frac{-1}{2\pi}\int_{\partial\Omega} e^{\overline{ikz}}(\widetilde{T}_2^B L)(z,k)\vec{\nu}\,dS(\zeta)$$

The continuity of the operators $T_{R,3}^B$ and $\widetilde{T}_{R,3}^B$ can be proved similar to that done in Lemma 4. The stability of the D-Bar method via the t^B approximation, is studied in a similar way to that in the t^{exp} approximation. Thus an operator **S** is introduced, which transforms the scattering data Φ to the matrix M as in (3.4). The continuity of this operator is studied in details in [10]
The relation between the potential matrix Q to the matrix M is given in (3.6). Define the operator $T_{R,4}^B$ from $C^\infty(\bar{\Omega})$ to $C^\infty(\Omega)$ by

$$T_{R,4}^B M(z,k) = Q(z)$$

The continuity of the operator $T_{R,4}^B$ can be proved in a similar way to that in Lemma 5.
The last step in constructing the operators, is to write γ^B as a composite of the operators T_1^B, T_2^B, T_3^B, **S** and T_4^B. The relation between γ and the potential matrix Q exists in (3.8), then

$$\log\gamma^B(z) = \bar{\partial}^{-1}\left(T_4^B o\mathbf{S}oT_3^B oT_2^B oT_1^B L\right)(z) = T^B L(z)$$

where $T^B = \bar{\partial}^{-1}(T_4^B o\mathbf{S}oT_3^B oT_2^B oT_1^B)$.
The operator T^B is a composite of continuous operators, thus it is continuous. If δ represents the noise level, then $\log\gamma_\delta^B(z) = T^B(\Lambda_\gamma^\delta - \Lambda_1)(z)$ and

$\left\|\log\gamma^B - \log\gamma^B_\delta\right\| = \left\|T^B(\Lambda_\gamma - \Lambda_1) - T^B(\Lambda^\delta_\gamma - \Lambda_1)\right\|$. Thus as δ tends to zero, $\log\gamma^B_\delta$ tends to $\log\gamma^B$. Since the logarithmic function is a bijective function, then if $\log\gamma^B_\delta$ tends to $\log\gamma^B$, then this leads to the result that γ^B_δ tends to γ^B and this shows that the reconstruction method via t^B is stable.

5. CONCLUSION

In this paper, we presented the six-step D-Bar reconstruction algorithm. Then we studied the stability of the method by considering two approximations t^B and t^{exp}. The main step in studying the stability in both approximations, was to write the admittivity $\gamma^{exp\, or\, B}$ in the form of an operator acting on the term $\Lambda_\gamma - \Lambda_1$, i.e. $\gamma^{exp\, or\, B} = T(\Lambda_\gamma - \Lambda_1)$. If a noise δ appears due to any external effects, then the admittivty $\gamma^{exp\, or\, B}$ can be written as $\gamma^{exp\, or\, B}_\delta = T(\Lambda^\delta_\gamma - \Lambda_1)$. We showed that as the noise δ goes to zero, the admittivity $\gamma^{exp\, or\, B}_\delta$ tends to $\gamma^{exp\, or\, B}$. This result can be achieved by proving the continuity of the operator T. It appeared in both approximations that the operator T is a composite of continuous operators, and thus it is continuous. This allows us to deduce that the method is stable in both approximations t^B and t^{exp}.
An open problem appears after this study: What would happen if the admittivity σ is not compactly supported i.e σ is different from 1 near the boundary $\partial\Omega$? And another question is what would happen if we consider Ω not to be a unit disc?

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