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# A GENERALIZATION OF VIÉTE'S INFINITE PRODUCT AND NEW MEAN ITERATIONS

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ABSTRACT. In this paper, we generalize Viéte's infinite product formula by use of Chebyshev polynomials. Furthermore, the infinite product formula for the lemniscate sine is also generalized. Finally, we obtain new mean iterations by use of these infinite product formulas.

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## 1. INTRODUCTION

The infinite product formula

$$\frac{\sin\theta}{\theta} = \prod_{k=1}^{\infty} \cos\frac{\theta}{2^k} \tag{1.1}$$

attracted many researchers. Applying the half-angle formula for the cosine function on the right-hand side and setting $\theta = \pi/2$, we obtain the famous formula

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots ,$$

which was established by François Viéte [2].

In [6], similar identities for $\sin\theta/\theta$ were obtained by Fourier transform and their relation to probability was also discussed. Additionally, Levin generalized (1.1) and derived similar infinite product formulas in [4]. Furthermore, there exists an analog of (1.1) for the lemniscate sine [7, 8], that is,

$$\frac{\operatorname{sl} x}{x} = \prod_{k=1}^{\infty} \left(\operatorname{sl}' \frac{x}{(1+i)^k}\right)^{-1}. \tag{1.2}$$

In this paper, we establish generalizations of the infinite product formula (1.1), (1.2). Our approach relies on the Chebyshev polynomials and their analog for the lemniscate function. Recently, the author established the relation between these infinite product formulas and several mean iterations [7, 8]. Similarly, in the last section, we apply the infinite product formulas to deduce new mean iterations.

## 2. A GENERALIZED INFINITE PRODUCT FORMULA FOR THE SINE

In this section, we generalize the infinite product (1.1) by use of Chebyshev polynomials of the first and second kind, $T_n$ and $U_n$ respectively, defined by

$$\begin{aligned} T_n(\cos\theta) &= \cos n\theta, \\ U_{n-1}(\cos\theta) &= \frac{\sin n\theta}{\sin\theta}. \end{aligned}$$

Additionally, we define the function $t_n$ as

$$\begin{aligned} t_n(\cos\theta) &= \cos\frac{\theta}{n} \\ &= \frac{e^{i\theta/n} + e^{-i\theta/n}}{2} \\ &= \frac{(\cos\theta + i\sqrt{1-\cos^2\theta})^{1/n} + (\cos\theta + i\sqrt{1-\cos^2\theta})^{-1/n}}{2}. \end{aligned} \tag{2.1}$$

The following theorem generalizes the infinite product (1.1).

**Theorem 2.1.** *If $n \geq 2$, then*

$$\frac{\sin\theta}{\theta} = \frac{U_{n-1}\left(\cos\frac{\theta}{n}\right)}{n} \cdot \frac{U_{n-1}\left(\cos\frac{\theta}{n^2}\right)}{n} \cdot \frac{U_{n-1}\left(\cos\frac{\theta}{n^3}\right)}{n} \cdots \tag{2.2}$$

$$= \frac{U_{n-1}\left(t_n(\cos\theta)\right)}{n} \cdot \frac{U_{n-1}\left(t_n(t_n(\cos\theta))\right)}{n} \cdot \frac{U_{n-1}\left(t_n(t_n(t_n(\cos\theta)))\right)}{n} \cdots . \tag{2.3}$$

*Proof.* Note that

$$\begin{aligned}\frac{\sin\theta}{\theta} &= \frac{\sin\frac{\theta}{n^m}}{\frac{\theta}{n^m}} \cdot \frac{\sin\theta}{n\sin\frac{\theta}{n}} \cdot \frac{\sin\frac{\theta}{n}}{n\sin\frac{\theta}{n^2}} \cdot \frac{\sin\frac{\theta}{n^2}}{n\sin\frac{\theta}{n^3}} \cdots \frac{\sin\frac{\theta}{n^{m-1}}}{n\sin\frac{\theta}{n^m}} \\ &= \frac{\sin\frac{\theta}{n^m}}{\frac{\theta}{n^m}} \cdot \frac{U_{n-1}\left(\cos\frac{\theta}{n}\right)}{n} \cdot \frac{U_{n-1}\left(\cos\frac{\theta}{n^2}\right)}{n} \cdots \frac{U_{n-1}\left(\cos\frac{\theta}{n^{m-1}}\right)}{n} \\ &= \frac{U_{n-1}\left(\cos\frac{\theta}{n}\right)}{n} \cdot \frac{U_{n-1}\left(\cos\frac{\theta}{n^2}\right)}{n} \cdots\end{aligned}$$

as $\lim_{m\to\infty} \sin(\theta/n^m)/(\theta/n^m) = 1$. Applying (2.1) to each term of (2.2) results in (2.3). This completes the proof. ∎

The Chebyshev polynomials have the following remarkable properties.

$$U_n(x) = 2\sum_{j\,:odd}^{n} T_j(x), \tag{2.4}$$

$$U_n(x) = 2\sum_{j\,:even}^{n} T_j(x) - 1. \tag{2.5}$$

If $n$ is an odd number, applying these relations to (2.2),

$$\frac{\sin\theta}{\theta} = \prod_{k=1}^{\infty} \frac{1}{n}\left(\sum_{j\,:odd}^{n} 2\cos\frac{j\theta}{n^k}\right).$$

If $n$ is an even number,

$$\frac{\sin\theta}{\theta} = \prod_{k=1}^{\infty} \frac{1}{n}\left(1 + \sum_{j\,:even}^{n} 2\cos\frac{j\theta}{n^k}\right).$$

These results were obtained by use of Fourier transform in [6].

## 3. A GENERALIZED INFINITE PRODUCT FORMULA FOR THE LEMNISCATE SINE

The lemniscate sine $\operatorname{sl} x$ is given as the inverse function of the arclemniscate sine [8] which is defined by

$$\operatorname{arcsl} t = \int_0^t (1-u^4)^{-1/2} du \quad (t^2 \leq 1).$$

The lemniscate sine can be extended to a complex function $\operatorname{sl} z$ and has the following properties [8]:

$$\begin{aligned}\operatorname{sl}(-z) &= -\operatorname{sl} z, \operatorname{sl}'(-z) = \operatorname{sl}'(z), \\ \operatorname{sl}(iz) &= i \operatorname{sl} z, \operatorname{sl}'(iz) = \operatorname{sl}'(z),\end{aligned}$$

and

$$(\mathrm{sl}'\, z)^2 = 1 - \mathrm{sl}^4\, z. \tag{3.1}$$

We then define the following analogs of Chebyshev polynomials for the lemniscate functions:

$$V_\beta\,(\mathrm{sl}'\, z) = \mathrm{sl}'\, \beta z,$$
$$W_\beta\,(\mathrm{sl}'\, z) = \frac{\mathrm{sl}\, \beta z}{\mathrm{sl}\, z}.$$

The following recurrent formulas hold.

$$W_{\beta+1}(x) = \frac{2xW_\beta(x)}{1 + (1 - x^2)W_\beta^2(x)} - W_{\beta-1}(x), \tag{3.2}$$

$$W_{\beta+i}(x) = \frac{2xW_\beta(x)}{1 - (1 - x^2)W_\beta^2(x)} - W_{\beta-i}(x), \tag{3.3}$$

and

$$V_\beta(x) = xW_\beta(x) - W_{\beta-1}(x)(1 + (1 - x^2)W_\beta^2(x)). \tag{3.4}$$

These relations follow from the addition formula for the lemniscate:

$$\mathrm{sl}(v + w) = \frac{\mathrm{sl}\, v\, \mathrm{sl}'\, w + \mathrm{sl}'\, v\, \mathrm{sl}\, w}{1 + \mathrm{sl}^2\, v\, \mathrm{sl}^2\, w}. \tag{3.5}$$

Setting $v = \beta z, w = z$ and dividing both sides by $\mathrm{sl}\, z$,

$$W_{\beta+1}(\mathrm{sl}'\, z) = \frac{\mathrm{sl}((\beta + 1)z)}{\mathrm{sl}\, z} \tag{3.6}$$

$$= \frac{\mathrm{sl}'\, z \frac{\mathrm{sl}\, \beta z}{\mathrm{sl}\, z} + \mathrm{sl}'\, \beta z}{1 + \mathrm{sl}^2\, z\, \mathrm{sl}^2\, \beta z} \tag{3.7}$$

$$= \frac{\mathrm{sl}'\, z W_\beta^2(\mathrm{sl}'\, z) + \mathrm{sl}'\, \beta z}{1 + (1 - (\mathrm{sl}'\, x)^2)W_\beta^2(\mathrm{sl}'\, z)}. \tag{3.8}$$

Meanwhile, by setting $v = \beta z, w = -z$,

$$W_{\beta-1}(\mathrm{sl}'\, z) = \frac{\mathrm{sl}'\, z W_\beta^2(\mathrm{sl}'\, z) - \mathrm{sl}'\, \beta z}{1 + (1 - (\mathrm{sl}'\, x)^2)W_\beta^2(\mathrm{sl}'\, z)}. \tag{3.9}$$

Therefore, we get the relation (3.2) by adding each side of (3.8) and (3.9), respectively. The equation (3.3) can be shown in the same manner as above. Furthermore, (3.4) is obtained from (3.5) directly.

As the case of trigonometric functions, we need the function $v_\beta$ which is defined by

$$v_\beta(\mathrm{sl}'\, x) = \mathrm{sl}'\, \frac{x}{\beta}. \tag{3.10}$$

This equation was studied by Abel and the way to get the function $v_\beta(\mathrm{sl}'\, x)$ in terms of $\mathrm{sl}'\, x$ was shown in [1]. The following theorem generalizes the infinite product (1.2).

**Theorem 3.1.** *If* $|\beta| > 1$, *then*

$$\frac{\operatorname{sl} x}{x} = \frac{W_\beta\left(\operatorname{sl}' \frac{x}{\beta}\right)}{\beta} \cdot \frac{W_\beta\left(\operatorname{sl}' \frac{x}{\beta^2}\right)}{\beta} \cdot \frac{W_\beta\left(\operatorname{sl}' \frac{x}{\beta^3}\right)}{\beta} \cdots \tag{3.11}$$

$$= \frac{W_\beta\left(v_\beta(\operatorname{sl}' x)\right)}{\beta} \cdot \frac{W_\beta\left(v_\beta(v_\beta(\operatorname{sl}' x))\right)}{\beta} \cdot \frac{W_\beta\left(v_\beta(v_\beta(v_\beta(\operatorname{sl}' x)))\right)}{\beta} \cdots . \tag{3.12}$$

*Proof.* The proof proceeds in the same manner as Theorem 2.1. First, we have

$$\begin{aligned}\frac{\operatorname{sl} x}{x} &= \frac{\operatorname{sl} \frac{x}{\beta^m}}{\frac{x}{\beta^m}} \cdot \frac{\operatorname{sl} x}{\beta \operatorname{sl} \frac{x}{\beta}} \cdot \frac{\operatorname{sl} \frac{x}{\beta}}{\beta \operatorname{sl} \frac{x}{\beta^2}} \cdot \frac{\operatorname{sl} \frac{x}{\beta^2}}{\beta \operatorname{sl} \frac{x}{\beta^3}} \cdots \frac{\operatorname{sl} \frac{x}{\beta^{m-1}}}{\beta \operatorname{sl} \frac{x}{\beta^m}} \\ &= \frac{\operatorname{sl} \frac{x}{\beta^m}}{\frac{x}{\beta^m}} \cdot \frac{W_\beta\left(\operatorname{sl}' \frac{x}{\beta}\right)}{\beta} \cdot \frac{W_\beta\left(\operatorname{sl}' \frac{x}{\beta^2}\right)}{\beta} \cdots \frac{W_\beta\left(\operatorname{sl}' \frac{x}{\beta^m}\right)}{\beta} \\ &= \frac{W_\beta\left(\operatorname{sl}' \frac{x}{\beta}\right)}{\beta} \cdot \frac{W_\beta\left(\operatorname{sl}' \frac{x}{\beta^2}\right)}{\beta} \cdots\end{aligned}$$

as $\lim_{m\to\infty} \operatorname{sl}(x/\beta^m)/(x/\beta^m) = 1$. Using (3.10) to each term of (3.11), we obtain (3.12). This completes the proof. ∎

Setting $\beta = 1 + i$,

$$W_{1+i}(\operatorname{sl}' z) = (\operatorname{sl}' z)^{-1}, v_{1+i}(\operatorname{sl}' z) = \sqrt{\frac{2}{1 + \operatorname{sl}' z}}.$$

Therefore, we have

$$\begin{aligned}\frac{\operatorname{sl} x}{x} &= \left(\operatorname{sl}' \frac{x}{1+i}\right)^{-1} \cdot \left(\operatorname{sl}' \frac{x}{(1+i)^2}\right)^{-1} \cdot \left(\operatorname{sl}' \frac{x}{(1+i)^3}\right)^{-1} \cdots \\ &= \sqrt{\frac{1 + \operatorname{sl}' x}{2}} \cdot \sqrt{\frac{1 + \sqrt{\frac{2}{1+\operatorname{sl}' x}}}{2}} \cdot \sqrt{\frac{1 + \sqrt{\frac{2}{1+\sqrt{\frac{2}{1+\operatorname{sl}' x}}}}}{2}} \cdots .\end{aligned}$$

This infinite product is known as the lemniscatic analog of Viéte's infinite product (1.1) [7, 8].

The infinite product formula (1.1) follows from the double angle formula for sine, which is equivalent to the following relation

$$U_1(\cos x) = \frac{\sin 2x}{\sin x} = 2\cos x.$$

Similarly, as $\operatorname{sl}' x = \sqrt{1 - \operatorname{sl}^4 x}$,the following relation

$$W_2(\operatorname{sl}' x) = \frac{\operatorname{sl} 2x}{\operatorname{sl} x} = \frac{2 \operatorname{sl}' x}{2 - (\operatorname{sl}' x)^2} \tag{3.13}$$

is equivalent to Fagnano's double angle formula, that is,

$$\operatorname{sl} 2x = \frac{2 \operatorname{sl} x \sqrt{1 - \operatorname{sl}^4 x}}{1 + \operatorname{sl}^4 x}. \tag{3.14}$$

Therefore, using (3.13) in Theorem 3.1,

$$\frac{\operatorname{sl} x}{x} = \frac{\operatorname{sl}' \frac{x}{2}}{2 - (\operatorname{sl}' \frac{x}{2})^2} \cdot \frac{\operatorname{sl}' \frac{x}{2^2}}{2 - (\operatorname{sl}' \frac{x}{2^2})^2} \cdots .$$

This is an another analog of Viéte's infinite product (1.1).

## 4. NEW MEAN ITERATIONS

In [7, 8], the author showed the relation between the infinite products (1.1), (1.2) and the Schwab-Borchardt and lemniscatic means [3], respectively. In this section, we deduce the following new mean iterations from Theorem 2.1, 3.1.

**Theorem 4.1.** *Let* $0 < b_0 < c_0$ *where* $a_0 = b_0(1 + b_0/c_0)/2$ *and*

$$a_{n+1} = \frac{b_{n+1}}{2}\left(1 + \frac{b_{n+1}^{1/2}}{a_n^{1/2}}\right), b_{n+1} = \frac{a_n}{2}\left(1 + \frac{a_n^{1/2}}{b_n^{1/2}}\right).$$

*Then,*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{b_0\left(1 - \frac{b_0^2}{c_0^2}\right)}{\left(\arccos\frac{b_0}{c_0}\right)^2}.$$

*Proof.* First, we put

$$c_{n+1} = (b_{n+1}a_n)^{1/2}. \tag{4.1}$$

Then we have

$$\begin{aligned}\frac{a_{n+1}}{b_{n+1}} &= \frac{1}{2}\left(1 + \frac{b_{n+1}^{1/2}}{a_n^{1/2}}\right)\\ &= \frac{1}{2}\left(1 + \frac{b_{n+1}}{c_{n+1}}\right).\end{aligned}$$

By using this relation,

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{a_n}{2b_n}\left(1 + \frac{a_n^{1/2}}{b_n^{1/2}}\right)\\ &= \frac{1 + \frac{b_n}{c_n}}{2}\cdot\frac{1 + \sqrt{\frac{1+\frac{b_n}{c_n}}{2}}}{2}.\end{aligned}$$

Therefore, from (4.1),

$$\begin{aligned}\frac{b_{n+1}}{c_{n+1}} &= \left(\frac{b_n}{a_n}\cdot\frac{b_{n+1}}{b_n}\right)^{1/2}\\ &= \left(\frac{b_n}{a_n}\cdot\frac{1 + \frac{b_n}{c_n}}{2}\cdot\frac{1 + \sqrt{\frac{1+\frac{b_n}{c_n}}{2}}}{2}\right)^{1/2}\\ &= \sqrt{\frac{1 + \sqrt{\frac{1+\frac{b_n}{c_n}}{2}}}{2}}.\end{aligned} \tag{4.2}$$

Setting $n = 4$ in Theorem 2.1,

$$\frac{\sin^2\theta}{\theta^2} = \left(\frac{U_3(v_4(\cos\theta))}{4}\right)^2 \cdot \left(\frac{U_3(v_4(v_4(\cos\theta)))}{4}\right)^2 \cdots$$
$$= \left(\frac{1+\cos\theta}{2} \cdot \frac{1+\sqrt{\frac{1+\cos\theta}{2}}}{2}\right) \cdot \left(\frac{1+v_4(\cos\theta)}{2} \cdot \frac{1+\sqrt{\frac{1+v_4(\cos\theta)}{2}}}{2}\right) \cdots$$

where

$$v_4(\cos\theta) = \sqrt{\frac{1+\sqrt{\frac{1+\cos\theta}{2}}}{2}}.$$

Therefore, by setting $b_0/c_0 = \cos\theta$,

$$\frac{\left(1-\frac{b_0^2}{c_0^2}\right)}{\left(\arccos\frac{b_0}{c_0}\right)^2}$$
$$= \frac{\sin^2\theta}{\theta^2}$$
$$= \left(\frac{1+b_0/c_0}{2} \cdot \frac{1+\sqrt{\frac{1+b_0/c_0}{2}}}{2}\right) \cdot \left(\frac{1+v_4(b_0/c_0)}{2} \cdot \frac{1+\sqrt{\frac{1+v_4(b_0/c_0)}{2}}}{2}\right) \cdots$$
$$= \left(\frac{1+b_0/c_0}{2} \cdot \frac{1+\sqrt{\frac{1+b_0/c_0}{2}}}{2}\right) \cdot \left(\frac{1+b_1/c_1}{2} \cdot \frac{1+\sqrt{\frac{1+b_1/c_1}{2}}}{2}\right) \cdots$$
$$= \lim_{n\to\infty} \frac{b_1}{b_0} \cdot \frac{b_2}{b_1} \cdots \frac{b_n}{b_{n-1}}$$
$$= \lim_{n\to\infty} \frac{b_n}{b_0}.$$

This completes the proof. ∎

**Theorem 4.2.** *Let* $0 < a_0 < c_0$ *where* $b_0 = 2a_0c_0/(a_0+c_0)$ *and*

$$a_{n+1} = \frac{2a_n^{1/2}b_n}{a_n^{1/2}+b_n^{1/2}}, b_{n+1} = \frac{2a_{n+1}b_n^{1/2}}{{a_{n+1}}^{1/2}+{b_n}^{1/2}}. \tag{4.3}$$

*Then,*

$$0 < a_0 < a_1 < \cdots < a_n < \cdots < b_n < \cdots < b_1 < b_0 \tag{4.4}$$

*and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{a_0 \operatorname{arcsl}^2\left(1-\frac{a_0^2}{c_0^2}\right)^{1/4}}{\left(1-\frac{a_0^2}{c_0^2}\right)^{1/2}}.$$

*Proof.* First, note that

$$\begin{aligned} b_{n+1} - a_{n+1} &= \frac{a_n}{a_n^{1/2} + b_n^{1/2}} \cdot (b_n^{1/2} - a_n^{1/2}) > 0, \\ a_{n+1} - a_n &= \frac{a_n^{1/2}(a_n^{1/2} + 2b_n^{1/2})}{a_n^{1/2} + b_n^{1/2}} \cdot (b_n^{1/2} - a_n^{1/2}) > 0, \\ b_{n+1} - b_n &= \frac{b_n^{1/2}(2a_{n+1}^{1/2} + b_n^{1/2})}{a_{n+1}^{1/2} + b_n^{1/2}} \cdot (a_n^{1/2} - b_n^{1/2}) < 0, \end{aligned}$$

since $b_0^{1/2} - a_0^{1/2} > 0$. This shows (4.4).

Next, we put

$$c_{n+1} = (a_{n+1} b_n)^{1/2}. \tag{4.5}$$

We then have

$$\frac{b_{n+1}}{a_{n+1}} = \frac{2b_n^{1/2}}{{a_{n+1}}^{1/2} + {b_n}^{1/2}} = \frac{2c_{n+1}}{a_{n+1} + c_{n+1}}. \tag{4.6}$$

Therefore,

$$\begin{aligned} \frac{a_n}{a_{n+1}} &= a_n \cdot \frac{a_n^{1/2} + b_n^{1/2}}{2a_n^{1/2} b_n} \\ &= a_n \cdot \frac{a_n^{1/2} + (2a_n c_n/(a_n + c_n))^{1/2}}{2a_n^{1/2} \cdot 2a_n c_n/(a_n + c_n)} \\ &= \frac{1 + \frac{a_n}{c_n}}{2} \cdot \frac{1 + \sqrt{\frac{2}{1 + \frac{a_n}{c_n}}}}{2}. \end{aligned}$$

Thus, from (4.5),

$$\begin{aligned} \frac{a_{n+1}}{c_{n+1}} &= \left( \frac{a_n}{b_n} \cdot \frac{a_{n+1}}{a_n} \right)^{1/2} \\ &= \left( \frac{a_n}{b_n} \cdot \frac{2c_n}{a_n + c_n} \cdot \frac{2}{1 + \sqrt{\frac{2}{1 + \frac{a_n}{c_n}}}} \right)^{1/2} \\ &= \sqrt{\frac{2}{1 + \sqrt{\frac{2}{1 + a_n/c_n}}}}. \end{aligned}$$

Setting $\beta = 2$ in Theorem 3.1, we obtain

$$\begin{aligned} \frac{\mathrm{sl}^2 x}{x^2} &= \left( \frac{\mathrm{sl}' \frac{x}{2}}{2 - (\mathrm{sl}' \frac{x}{2})^2} \right)^2 \cdot \left( \frac{\mathrm{sl}' \frac{x}{2^2}}{2 - (\mathrm{sl}' \frac{x}{2^2})^2} \right)^2 \cdots \\ &= \left\{ \frac{1 + \mathrm{sl}' x}{2} \cdot \frac{1 + \sqrt{\frac{2}{1 + \mathrm{sl}' x}}}{2} \right\} \cdot \left\{ \frac{1 + v(\mathrm{sl}' x)}{2} \cdot \frac{1 + \sqrt{\frac{2}{1 + v_2(\mathrm{sl}' x)}}}{2} \right\} \cdots \end{aligned}$$

where

$$v_2(\mathrm{sl}'\,x) = \sqrt{\frac{2}{1+\sqrt{\frac{2}{1+\mathrm{sl}'\,x}}}}.$$

Then, by setting $a_0/c_0 = \mathrm{sl}'\,x = \sqrt{1-\mathrm{sl}^4\,x}$, we obtain the assertion in the same manner as Theorem 4.1. ∎

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