ON THE CONSTANT IN A TRANSFERENCE INEQUALITY FOR THE VECTOR-VALUED FOURIER TRANSFORM

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ABSTRACT. The standard proof of the equivalence of Fourier type on \( \mathbb{R}^d \) and on the torus \( \mathbb{T}^d \) is usually stated in terms of an implicit constant, defined as the minimum of a sum of powers of sinc functions. In this note we compute this minimum explicitly.

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1. Introduction

The motivation of this paper comes from a well-known transference result for the vector-valued Fourier transform. Let $X$ be a complex Banach space. The Fourier transform of a function $f \in L^1(\mathbb{R}^d; X)$ is defined by

$$\mathcal{F}_{\mathbb{R}^d}f(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) \, dx, \quad \xi \in \mathbb{R}^d.$$ 

Likewise, the Fourier transform of a function $f \in L^1(\mathbb{T}^d; X)$ is defined by

$$\mathcal{F}_{\mathbb{T}^d}f(k) := \int_{\mathbb{T}^d} e^{-2\pi i k \cdot t} f(t) \, dt, \quad k \in \mathbb{Z}^d.$$ 

Proposition 1.1. Let $X$ be a complex Banach space, fix $d \geq 1$ and $p \in (1, 2]$, and let $\frac{1}{p} + \frac{1}{q} = 1$. The following assertions are equivalent:

(i) $\mathcal{F}_{\mathbb{R}^d}$ extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ into $L^q(\mathbb{R}^d; X)$;
(ii) $\mathcal{F}_{\mathbb{T}^d}$ extends to a bounded operator from $L^p(\mathbb{T}^d; X)$ into $\ell^q(\mathbb{Z}^d; X)$.

In this situation, denoting the norms of these extensions by $\varphi_{p,X}(\mathbb{R}^d)$ and $\varphi_{p,X}(\mathbb{T}^d)$, we have

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq C_q^{-d/q} \varphi_{p,X}(\mathbb{R}^d),$$

where $C_q$ is the global minimum of the periodic function $x \mapsto \sum_{m \in \mathbb{Z}} \frac{|\sin(\pi(x + m))|}{\pi(x + m)}^q$, $x \in \mathbb{R}$.

This function, as well as several others considered below, have removable singularities. It is understood that we will always be working with their unique continuous extensions.

A complex Banach space $X$ which has the equivalent properties (i) and (ii) is said to have Fourier type $p$; this notion has been introduced in [5]. Proposition 1.1 goes back to [4]; in its stated form the result can be found in [2,3]. Related results may be found in [1]. These references do not comment on the location of the global minimum. A quick computer plot (see Figure 1) suggests that the minimum is taken in the points $\frac{1}{2} + \mathbb{Z}$. To actually prove this turns out to be surprisingly difficult. This is the modest objective of the present note:

Proposition 1.2. For every real number $r \geq 1$, the function $f_r : [0, 1] \to \mathbb{R}$ defined by

$$f_r(x) := \sum_{m \in \mathbb{Z}} \left| \frac{\sin(\pi(x + m))}{\pi(x + m)} \right|^{2r}, \quad x \in [0, 1],$$

has a global minimum at $x = \frac{1}{2}$.

Our proof has developed essentially by trial and error. We believe it is perfectly possible that a truly pedestrian proof can be given, but we failed to find one despite many hours of efforts. As a consequence of Proposition 1.2, we obtain the explicit estimate

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq \frac{\pi^d}{(2(2^q - 1)\zeta(q))^{d/q}} \varphi_{p,X}(\mathbb{R}^d),$$

noting that

$$\sum_{m \in \mathbb{Z}} \frac{1}{(\frac{1}{2} + m)^q} = 2(2^q - 1)\zeta(q).$$

For even integers $q = 2n$, the constant on the right-hand side may be evaluated explicitly in terms of the Bernoulli numbers. To further estimate this constant, recall that for any $x \in \ell^2(\mathbb{Z})$ the function $q \mapsto ||x||_q := (\sum_{m \in \mathbb{Z}} |x_m|^q)^{1/q}$ is decreasing on $[2, \infty)$ and $\lim_{q \to \infty} ||x||_q =$
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Figure 1: A plot of $f_r$, where $r = 1.02^k$ for $k = 1, 2, 4, \ldots, 256$.

sup$_{t \in \mathbb{Z}} |x_i|$. Taking $x_m := \frac{1}{2} + m^{-1}$ we find $(\sum_{m \in \mathbb{Z}} |\frac{1}{2} + m^{-1}|^{1/q}) \geq 2$ for every $q \geq 2$, and hence in particular

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq (\frac{1}{2}\pi)^d \varphi_{p,X}(\mathbb{R}^d).$$

2. THE MAIN RESULT

The proof of the proposition is based on the following lemmas. The main idea is contained in the first lemma.

Lemma 2.1. Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing convex function, and let $x_1, \ldots, x_n \in \mathbb{R}_+$ and $y_1, \ldots, y_n \in \mathbb{R}_+$ be such that

(i) $x_1 + \cdots + x_n \geq y_1 + \cdots + y_n$;

(ii) there exists $t \in \mathbb{R}_+$ such that

$\bullet$ $x_i \leq y_i$ if $y_i < t$;

$\bullet$ $x_i \geq y_i$ if $y_i \geq t$.

Then $g(x_1) + \cdots + g(x_n) \geq g(y_1) + \cdots + g(y_n)$.

Proof. We will prove the lemma by induction on $n$. The case $n = 1$ is clear: $x_1 \geq y_1$ implies that $g(x_1) \geq g(y_1)$ since $g$ is non-decreasing. Suppose now that the lemma has been proved for $n = 1, \ldots, m - 1$.

If $x_i = y_i$ for some index $1 \leq i \leq m$, then we may remove $x_i$ and $y_i$ and apply the induction hypothesis.

If $x_i \geq y_i$ for every index $1 \leq i \leq m$, then again the result is immediate since $g$ is non-decreasing. Therefore, we may assume that $x_i < y_i$ for some index $1 \leq i \leq m$. Then, by the first condition in the lemma, there is also an index $j$ for which $x_j > y_j$. By the second condition in the lemma we then have $x_i < y_i < t \leq y_j < x_j$.

Let $t := \min(y_i - x_i, x_j - y_j)$ and define $x'_i := x_i + t$, $x'_j := x_j - t$, and $x'_k := x_k$ for all other indices. Then $x'_1, \ldots, x'_m, y_1, \ldots, y_m$ satisfy the conditions in the lemma (with the same $t$) and $x'_i = y_i$ or $x'_j = y_j$. Hence, by the induction hypothesis, we have

$$g(x'_1) + \cdots + g(x'_m) \geq g(y_1) + \cdots + g(y_m).$$
In view of parts (i) and (ii) of Lemma 2.3 we have

\[ x_i \leq x'_i \leq x'_j \leq x_j, \]

we can write \( x'_i = \lambda x_i + (1 - \lambda)x_j \) for some \( \lambda \in [0, 1] \). Since \( x'_j = x_i + x_j - x'_i \), we have \( x'_j = (1 - \lambda)x_i + \lambda x_j \). By the convexity of \( g \) it follows that

\[
g(x'_i) + g(x'_j) \leq (\lambda g(x_i) + (1 - \lambda)g(x_j)) + ((1 - \lambda)g(x_i) + \lambda g(x_j)) = g(x_i) + g(x_j).
\]

Combining inequalities (2.1) and (2.2) we obtain the lemma for \( n = m \), thus completing the induction step. \( \square \)

In order to apply this lemma we need a number of technical facts. The first (cf. \( [2, (6.14)] \)) is elementary and is left as an exercise.

**Lemma 2.2.** \( f_1(x) = 1 \) for all \( x \in [0, 1] \).

Let \( h : \mathbb{R} \to \mathbb{R} \) be defined by

\[
h(x) := \sin^2(\pi x) = \left(\frac{\sin(\pi x)}{\pi x}\right)^2, \quad x \in \mathbb{R}.
\]

**Lemma 2.3.** Let \( r \geq 1 \). The following assertions hold on the interval \( [0, 1] \):

(i) the function \( h(x) + h(x - 1) \) has a global minimum at \( x = \frac{1}{2} \);

(ii) for all \( m = 1, 2, 3, \ldots \), \( h(x + m) + h(x - (m + 1)) \) has a global maximum at \( x = \frac{1}{2} \);

(iii) the function

\[
h(x) + h(x - 1) - (h(x)^r + h(x - 1)^r)^{1/r}
\]

has a global maximum at \( x = \frac{1}{2} \);

(iv) for all \( m = 1, 2, 3, \ldots \) and \( r \geq 1 \),

\[
(h(x + m) + h(x - (m + 1)))^r - h(x + m)^r - h(x - (m + 1))^r
\]

has a global maximum at \( x = \frac{1}{2} \).

Assuming the lemmas for the moment, let us first show how the proposition can be deduced from them.

**Proof of Proposition** \( \square \)

Fix \( r \geq 1 \) and set, for \( x \in [0, 1] \),

\[
s_m(x) := h(x + m) + h(x - (m + 1)) \quad (m = 0, 1, 2, \ldots)
\]

and

\[
\tilde{s}_0(x) := ((h(x))^r + (h(x - 1))^r)^{1/r}.
\]

In view of part (iv) of Lemma 2.3 it suffices to prove that

\[
\tilde{s}_0^r + s_1^r + s_2^r + \cdots
\]

has a global minimum at \( x = \frac{1}{2} \).

Fix an arbitrary \( x \in [0, 1] \) and set

\[
x_m := s_m(x), \quad y_m := s_m(1/2) \quad (m = 0, 1, 2, \ldots)
\]

and

\[
\tilde{x}_0 := ((h(x))^r + h(x - 1)^r)^{1/r}, \quad \tilde{y}_0 := ((h(1/2))^r + h(-1/2)^r)^{1/r}.
\]

In view of parts (i) and (ii) of Lemma 2.3 we have

\[
(2.3) \quad x_0 \geq y_0, \quad x_i \leq y_i \quad (i = 1, 2, \ldots)
\]

Lemma 2.2 implies

\[
(2.4) \quad x_0 + x_1 + x_2 + \cdots = y_0 + y_1 + y_2 + \cdots
\]

By (2.3) and (2.4),

\[
(2.5) \quad x_0 + x_1 + \cdots + x_n \geq y_0 + y_1 + \cdots + y_n \quad (n = 0, 1, 2, \ldots)
\]
Part (iii) of Lemma 2.3 implies
\[ x_0 - x_0 \geq \tilde{y}_0 - y_0. \]

By (2.5) and (2.6),
\[ x_0 + x_1 + \cdots + x_n \geq \tilde{y}_0 + y_1 + \cdots + y_n \quad (n = 0, 1, 2, \ldots) \]
Finally, by (2.3) and (2.6),
\[ x_0 \geq \tilde{y}_0. \]

A simple calculation shows that \( \tilde{y}_0 > \frac{4}{2\pi} \) and \( y_i < \frac{4}{2\pi} \) for \( i = 1, 2, \ldots \). Taking \( t = \frac{4}{2\pi} \) in Lemma 2.1 and \( g(x) := x^r \) now implies, by virtue of (2.3), (2.7), and (2.8), that
\[ \tilde{x}_0^r + x_1^r + \cdots + x_n^r \geq \tilde{y}_0^r + y_1^r + \cdots + y_n^r \]
holds for every \( n \). Taking limits for \( n \to \infty \) completes the proof. \( \blacksquare \)

3. PROOF OF LEMMA 2.3

This section is devoted to the proof of Lemma 2.3, which is based on the following observations:

**Lemma 3.1.** On the interval \([0, 1]\):

(i) \( \frac{\cos(\frac{1}{2}\pi x)}{1 - x^2} \) takes a global maximum at \( x = 0 \);

(ii) \( \frac{(x^2 + 1)\cos^2(\frac{1}{2}\pi x)}{(1 - x^2)^2} \) takes a global minimum at \( x = 0 \).

**Proof.** We start by showing that
\[ \sqrt{2} \sin(\frac{1}{4}\pi x) \geq x \quad \text{for all} \quad x \in [0, 1]. \]

To this end, consider \( f(x) := \sqrt{2} \sin(\frac{1}{4}\pi x) - x \). Observe that \( f'(x) = \frac{\pi\sqrt{2}}{4} \cos(\frac{1}{2}\pi x) - 1 \) is decreasing on \([0, 1]\), hence \( f \) is concave. Since \( f(0) = f(1) = 0 \) this implies that \( f(x) \geq 0 \) for \( x \in [0, 1] \), which proves the claim.

(i): The value at \( x = 0 \) of the given function equals 1, so it suffices to show that \( \cos(\frac{1}{2}\pi x) \leq 1 - x^2 \) for all \( x \in [0, 1] \). This follows from the double-angle formula for cosine and (3.1):
\[ \cos(\frac{1}{2}\pi x) = 1 - 2\sin^2(\frac{1}{4}\pi x) \leq 1 - x^2. \]

(ii): The given function has value 1 at \( x = 0 \), hence it suffices to show that for all \( x \in [0, 1] \),
\[ (x^2 + 1)\cos^2(\frac{1}{2}\pi x) \geq (1 - x^2)^2. \]

On the interval \([\frac{1}{2}, 1]\) we substitute \( x = 1 - y \). We then must prove that for \( y \in [0, \frac{1}{2}] \),
\[ (2 - 2y + y^2)\sin^2(\frac{1}{2}\pi y) \geq (2y - y^2)^2. \]

Since \( 2y \in [0, 1] \), we can use (3.1) to obtain \( \sqrt{2} \sin(\frac{1}{4}\pi \cdot 2y) \geq 2y \), and hence \( \sin^2(\frac{1}{2}\pi y) \geq 2y^2 \). This implies that
\[ (2 - 2y + y^2)\sin^2(\frac{1}{2}\pi y) \geq (2 - 2y + y^2)(2y^2) = (y^2 + (2 - y)^2)y^2 \geq (2 - y)^2y^2 = (2y - y^2)^2, \]
which concludes the proof on the interval \([\frac{1}{2}, 1]\).
For $x \in [0, \frac{1}{2}]$ we have
\[
(x^2 + 1) \cos^2\left(\frac{1}{2}\pi x\right) \geq (x^2 + 1)\left(1 - \frac{\pi^2}{8}x^2\right)^2
\]
\[
= (x^2 + 1)(1 - \frac{\pi^2}{4}x^2 + \frac{\pi^4}{64}x^4)
\]
\[
\geq 1 + (1 - \frac{\pi^2}{4})x^2 + (\frac{\pi^4}{64} - \frac{\pi^2}{4})x^4
\]
\[
= 1 + (1 - \frac{\pi^2}{4})x^2 + (\frac{\pi^4}{64} - \frac{\pi^2}{4} - 1)x^4 + x^4
\]
\[
\geq 1 + [(1 - \frac{\pi^2}{4}) + \frac{1}{4}(\frac{\pi^4}{64} - \frac{\pi^2}{4} - 1)]x^2 + x^4
\]
\[
\geq 1 - 2x^2 + x^4
\]
\[
= (1 - x^2)^2,
\]
noting that $\frac{\pi^2}{64} - \frac{\pi^2}{4} - 1 < 0$ and $(1 - \frac{\pi^2}{4}) + \frac{1}{4}(\frac{\pi^4}{64} - \frac{\pi^2}{4} - 1) \approx -1.9537471 \ldots > -2$.

**Proof of Lemma 2.3**

(i): We have
\[
h(x) + h(x - 1) = \frac{\sin^2(\pi x)}{\pi^2 x^2} + \frac{\sin^2(\pi x)}{\pi^2 (x - 1)^2} = \frac{(2x^2 - 2x + 1) \sin^2(\pi x)}{\pi^2 x^2(x - 1)^2} =: g(x).
\]
We must show that
\[
f(x) := g(x + \frac{1}{2}) = \frac{8}{\pi^2} \frac{4x^2 + 1}{(4x^2 - 1)^2} \cos^2(\pi x)
\]
has a global minimum in $x = 0$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$. But this follows from Lemma 3.1 and the fact that $f$ is even.

(ii): For $m = 1, 2, 3, \ldots$ we have
\[
h(x + m) + h(x - (m + 1)) = \frac{[2x^2 - 2x + (m + 1)^2 + m^2] \sin^2(\pi x)}{\pi^2 [(x + m)^2(x - (m + 1))^2]} =: g_m(x).
\]
We must show that
\[
f_m(x) := g_m(x + \frac{1}{2}) = \frac{8}{\pi^2} \frac{4x^2 + 4m^2 + 4m + 1}{[(4x^2 - (2m + 1)^2)^2]} \cos^2(\pi x)
\]
has a global maximum in $x = 0$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$. For this, it suffices to check that the functions
\[
\frac{4x^2 + 1}{(4x^2 - M^2)^2} \cos^2(\pi x) \quad \text{and} \quad \frac{1}{(4x^2 - M^2)^2} \cos^2(\pi x)
\]
are decreasing on $[0, \frac{1}{2}]$ for each $M \geq 3$, or equivalently, that
\[
\frac{\sqrt{x^2 + 1}}{M^2 - x^2} \cos\left(\frac{1}{2}\pi x\right) \quad \text{and} \quad \frac{1}{M^2 - x^2} \cos\left(\frac{1}{2}\pi x\right)
\]
are decreasing on $[0, 1]$ for each $M \geq 3$. It suffices to prove this for the first function, since this will immediately imply the result for the second function.

Straightforward algebra shows that the derivative of the function
\[
\psi_M(x) := \frac{\sqrt{x^2 + 1}}{M^2 - x^2} \cos\left(\frac{1}{2}\pi x\right)
\]
has a zero at $x$ if and only if
\[
2(x^2 + 2 + M^2) \cos\left(\frac{1}{2}\pi x\right) = \pi(M^2 - x^4 + (M^2 - 1)x^2) \sin\left(\frac{1}{2}\pi x\right).
\]
But,
\[ 2x(M^2 + 2 + x^2) \cos(\frac{1}{2} \pi x) \leq 2x(M^2 + 2 + x^2) \]
and, since \( 0 \leq x \leq 1 \),
\[ \pi(M^2 - x^4 + (M^2 - 1)x^2)x \leq \pi(M^2 - x^4 + (M^2 - 1)x^2) \sin(\frac{1}{2} \pi x), \]
while also, using that \( M \geq 3 \) and \( 0 \leq x \leq 1 \),
\[ 2(M^2 + 2 + x^2) \leq 2(M^2 + 2 + (M^2 - 1)x^2) \leq \pi(M^2 - x^4 + (M^2 - 1)x^2) \]
since \( 2(M^2 + 2) < \pi(M^2 - 1) \) for \( M \geq 3 \). It follows that the derivative of \( \psi_M \) has no zeros on \((0, 1]\), and then from
\[ \psi_M(0) = \frac{1}{M^2} > 0 = \psi_M(1) \]
it follows that \( \psi_M \) is decreasing on \([0, 1]\).

(iii): Proceeding as in (i), we have
\[ h(x) + h(x - 1) - ((h(x))^r + (h(x - 1))^r)^{1/r} \]
\[ = \frac{1}{\pi^2} \left[ \frac{1}{x^2} + \frac{1}{(1 - x)^2} - \left( \frac{1}{x^{2r}} + \frac{1}{(1 - x)^{2r}} \right)^{1/r} \right] \sin^2(\pi x) =: g(x). \]
We must show that
\[ f(x) := g(\frac{1}{2} + x) = \frac{1}{\pi^2} \left[ \left( \frac{1}{2} - x \right)^2 + \left( \frac{1}{2} + x \right)^2 - \left( \left( \frac{1}{2} - x \right)^{2r} + \left( \frac{1}{2} + x \right)^{2r} \right)^{1/r} \right] \cos^2(\pi x) \]
has a global maximum in \( x = 0 \) on the interval \([-\frac{1}{2}, \frac{1}{2}]\). The function \( f \) is even, and by Lemma 3.1, \( \cos^2(\pi x)/(\frac{1}{4} - x^2)^2 \) takes its maximum at \( x = 0 \). It thus remains to show that on the interval \([0, \frac{1}{2}]\) the function
\[ \phi_r(x) := \left( \frac{1}{2} - x \right)^2 + \left( \frac{1}{2} + x \right)^2 - \left( \left( \frac{1}{2} - x \right)^{2r} + \left( \frac{1}{2} + x \right)^{2r} \right)^{1/r} \]
is decreasing on \([0, \frac{1}{2}]\). The derivative of this function equals
\[ \phi'_r(x) = 4x - 2\left( \left( \frac{1}{2} - x \right)^{2r} + \left( \frac{1}{2} + x \right)^{2r} \right)^{1/r-1} \left( \left( \frac{1}{2} + x \right)^{2r-1} - \left( \frac{1}{2} - x \right)^{2r-1} \right). \]
To show that \( \phi'_r(x) \leq 0 \) we must show that
\[ \left( \left( \frac{1}{2} + x \right)^{2r} + \left( \frac{1}{2} - x \right)^{2r} \right)^{1/r-1} \left( \left( \frac{1}{2} + x \right)^{2r-1} - \left( \frac{1}{2} - x \right)^{2r-1} \right) \geq 2x \]
for \( x \in [0, \frac{1}{2}] \), or, after substituting \( a = \frac{1}{2} + x \) and \( b = \frac{1}{2} - x \), that
\[ a^{2r-1} - b^{2r-1} \geq (a - b)(a^{2r} + b^{2r})^{1-1/r} \]
for all \( a \in [\frac{1}{2}, 1] \). In view of
\[ (a^{2r} + b^{2r})^{1-1/r} = \left( (a^{2r} + b^{2r})^{1/(2r)} \right)^{2r-2} \]
\[ \leq \left( (a^{2r-1} + b^{2r-1})^{1/(2r-1)} \right)^{2r-2} = (a^{2r-1} + b^{2r-1})^{1-1/(2r-1)}, \]
with \( p := 2r - 1 \) it suffices to show that
\[ a^p - b^p \geq (a - b)(a^p + b^p)^{1-1/p} \]
for all \( a \geq b \geq 0 \). We can further simplify this upon dividing both sides by \( b^p \). In the new variable \( x = a/b \) we then have to prove that
\[ x^p - 1 \geq (x - 1)(x^p + 1)^{1-1/p} \]
for all \( x \geq 1 \).
Using that \((1 + y)^\alpha \leq 1 + \alpha y\) for \(y \geq 0\) and \(0 \leq \alpha \leq 1\), we have
\[
(x - 1)(x^p + 1)^{1-1/p} = (x^p - x^{p-1})(1 + x^{p-1})^{1-1/p} \leq (x^p - x^{p-1})[1 + (1 - \frac{1}{p})x^{-p}].
\]
Therefore it remains to prove that for \(x \geq 1\) and \(p \geq 1\) we have
\[
x^p - 1 \geq (x^p - x^{p-1})[1 + (1 - \frac{1}{p})x^{-p}],
\]
or, multiplying both sides with \(x\), that
\[
x^{p+1} - x \geq x^{p+1} - x^p + (1 - \frac{1}{p})(x - 1).
\]
that is, we must show that
\[
f_p(x) := x^p \geq x + (1 - \frac{1}{p})(x - 1) =: g_p(x).
\]
Now
\[
f'_p(x) = px^{p-1}, \quad g'_p(x) = 2 - \frac{1}{p}.
\]
It follows that \(f'_p(x) \geq g'_p(x) \geq 0\) for \(x \geq 1\), since \(p \geq 2 - \frac{1}{p}\) (multiply both sides by \(p\)). Together with \(f_p(1) = g_p(1)\) it follows that \(f_p(x) \geq g_p(x)\) for \(x \geq 1\) and \(p \geq 1\). This concludes the proof of (iii).

(iv): Fix \(m \geq 1\). For \(x \in [-\frac{1}{2}, \frac{1}{2}]\) we have
\[
\begin{align*}
(h(x + \frac{1}{2} + m) + h(x + \frac{1}{2} - (m + 1)))^r - h(x + \frac{1}{2} + m)^r - h(x + \frac{1}{2} - (m + 1))^r \\
= \left[\left(\frac{1}{(x + (m + 1))^2} + \frac{1}{(x - (m + \frac{1}{2}))^2}\right)^r - \left(\frac{1}{(x + (m + \frac{1}{2}))^2}\right)^r - \left(\frac{1}{(x - (m + \frac{1}{2}))^2}\right)^r\right] \\
\times \pi^{-2r} (\cos^2(\pi x))^r.
\end{align*}
\]
We must show that this function has a global maximum on \([-\frac{1}{2}, \frac{1}{2}]\) at \(x = 0\). Since by Lemma 3.1 \(\cos(\pi x)/(1 - 4x^2)\) has a global maximum at \(x = 0\), it suffices to prove that
\[
[((a + x)^{-2} + (a - x)^{-2})^r - (a + x)^{-2r} - (a - x)^{-2r}] \cdot (1 - 4x^2)^{2r}
\]
has a global maximum at \(x = 0\), where we have written \(a := m + \frac{1}{2} \geq \frac{3}{2}\). Since the function \(x \mapsto x^r\) is convex, we have \(\frac{1}{2}(a + x)^{-2r} + \frac{1}{2}(a - x)^{-2r} \geq \left(\frac{1}{2}(a + x)^{-2} + \frac{1}{2}(a - x)^{-2}\right)^r\) and hence
\[
2^{1-r}((a + x)^{-2} + (a - x)^{-2})^r - (a + x)^{-2r} - (a - x)^{-2r} \leq 0
\]
with equality for \(x = 0\). Therefore, it suffices to show that
\[
(1 - 2^{1-r})((a + x)^{-2} + (a - x)^{-2})^r(1 - 4x^2)^{2r}
\]
has a global maximum at \(x = 0\). It is enough to show that the function \(g(x) := ((a + x)^{-2} + (a - x)^{-2})(1 - 4x^2)^2\) is decreasing on \([0, \frac{1}{2}]\).
Computing the derivative of \(g\) we find
\[
g'(x) = -16x(1 - 4x^2)((a + x)^{-2} + (a - x)^{-2}) + (1 - 4x^2)^2(-2(a + x)^{-3} + 2(a - x)^{-3})
\]
\[
= (1 - 4x^2)(a + x)^{-3}(a - x)^{-3}k(x),
\]
In terms of the digamma function $\psi$, we have

\[ k(x) = -16x(a^2 - x^2)((a + x)^2 + (a - x)^2) + (1 - 4x^2)(2(a + x)^3 - 2(a - x)^3) \]
\[ = -16x \cdot 2(a^4 - x^4) + (1 - 4x^2) \cdot 4x \cdot (3a^2 + x^2) \]
\[ = 4x[-8(a^4 - x^4) + (1 - 4x^2)(3a^2 + x^2)] \]
\[ = 4x[4x^4 + (1 - 12a^2)x^2 + (3a^2 - 8a^4)] . \]

Since $a > \sqrt{3}$, the function $p(y) := 4y^2 + (1 - 12a^2)y + (3a^2 - 8a^4)$ has a positive and a negative root. The sum of the two roots equals $\frac{12a^2 - 1}{4}$ and therefore the positive root is larger than $3a^2 - \frac{1}{4} \geq \frac{20}{4}$. It follows that $p$ is negative on $[0, \frac{1}{4}]$ and hence that $g'(x) = (1 - 4x^2)(a + x)^{-3}(a - x)^{-3} \cdot 4x \cdot p(x^2) \leq 0$ on $[0, \frac{1}{4}]$, which finishes the proof. 

**Added in proof.** After this paper had been accepted for publication, Tom Koornwinder sent us the following interesting proof for the case that the parameter $r$ in Proposition 1.2 is integral. With his kind permission we reproduce it here.

We consider $f_r(x)$ on $(0, 1)$. In terms of the Hurwitz zeta-function $\zeta(s, q)$ (see [6, Eq. 25.11.1]) we have

\[ f_r(x) = \pi^{-2r} \sin^{2r}(\pi x) (\zeta(2r, x) + \zeta(2r, 1 - x)), \quad r = 1, 2, \ldots \]

In terms of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ (see [6, Eq. 25.11.12]) this can be rewritten as

\[ f_r(x) = \frac{\pi^{-2r} \sin^{2r}(\pi x)}{(2r - 1)!} \left( \frac{d}{dx} \right)^{2r-1} (\psi(x) - \psi(1 - x)) . \]

Applying the reflection formula $\psi(1 - z) - \psi(z) = \pi \cot(\pi z)$ (see [6, Eq. 5.5.4]) we obtain

\[ f_r(x) = -\frac{\pi^{1-2r} \sin^{2r}(\pi x)}{(2r - 1)!} \left( \frac{d}{dx} \right)^{2r-1} \cot(\pi x) . \]

Substitution of $t = \pi x$ simplifies this expression to

\[ \frac{(2r - 1)!}{\sin^{2r} t} \cdot f_r(t/\pi) = - \left( \frac{d}{dt} \right)^{2r-1} \cot t . \]

Since $(d/dt) \cot t = -1/\sin^2 t$, we have $f_1(t/\pi) = 1$. Also, we obtain the following recursion relation:

\[ (3.2) \quad \frac{(2r + 1)!}{\sin^{2r+2} t} \cdot f_{r+1}(t/\pi) = - \left( \frac{d}{dt} \right)^{2r+1} \cot t = (2r - 1)! \left( \frac{d}{dt} \right)^2 \frac{f_r(t/\pi)}{\sin^{2r} t} . \]

A small computation shows that

\[ \left( \frac{d}{dt} \right)^2 \frac{f_r(t/\pi)}{\sin^{2r} t} = \left( \frac{d}{dt} \right) \left[ \sin^{-2r} t \left( \frac{d}{dt} \right) f_r(t/\pi) - 2r \cos t (\sin^{2r-1} t) f_r(t/\pi) \right] \]
\[ = (\sin^{-2r} t) \left( \frac{d}{dt} \right)^2 f_r(t/\pi) - 4r \cos t (\sin^{2r-1} t) \left( \frac{d}{dt} \right) f_r(t/\pi) \]
\[ + (2r(2r + 1) \cos^2 t \sin^{-2r-2} t + 2r \sin^{-2r} t) \cdot f_r(t/\pi) . \]
Hence, (3.2) implies that
\[
\frac{(2r+1)!}{(2r-1)!} f_{r+1}(t/\pi) = (\sin^2 t) \left( \frac{d}{dt} \right)^2 f_r(t/\pi) - 4r(\cos t)(\sin t) \left( \frac{d}{dt} \right) f_r(t/\pi) + (2r(2r+1) \cos^2 t + 2r \sin^2 t) \cdot f_r(t/\pi). 
\]
(3.3)

Set \( y := \cos^2 t \) and \( D := d/dy \). So \( d/dt = -2(\sin t \cos t)D \) and
\[
\left( \frac{d}{dt} \right)^2 = \frac{d}{dt}(-2 \sin t \cos t)D \\
= -2(\cos^2 t - \sin^2 t)D - (2 \sin t \cos t) \left( \frac{d}{dt} \right) D \\
= -2(\cos^2 t - \sin^2 t)D + (4 \sin^2 t \cos^2 t)D^2 \\
= (-4y + 2)D + 4y(1 - y)D^2. 
\]

Equation (3.3) can therefore be rewritten as
\[
\frac{(2r+1)!}{(2r-1)!} f_{r+1}(t/\pi) = \left[ 4y(1 - y)^2D^2 + ((8r - 4)y + 2)(1 - y)D + 2r(2ry + 1) \right] f_r(t/\pi) \\
= \left[ 4y(r - yD)^2 + 8(r - yD)yD + 2yD + 2r + 4yD^2 + 2D \right] f_r(t/\pi). 
\]
(3.4)

Observe that \( (r - yD)y^k = (r - k)y^{k+1} \). Hence, if \( p = p(y) \) is a polynomial of degree \( n < r \) with nonnegative coefficients, then the same holds for \( (r - yD)p \). The recursion (3.4) and the fact that \( f_1(t/\pi) = 1 \) now imply that \( f_r(t/\pi) \) is a polynomial in \( y \) of degree \( r - 1 \) with nonnegative coefficients. The first few are given explicitly by
\[
f_1(t/\pi) = 1 \\
f_2(t/\pi) = \frac{1}{3} + \frac{2}{3} \cos^2 t \\
f_3(t/\pi) = \frac{2}{15} + \frac{11}{15} \cos^2 t + \frac{2}{15} \cos^4 t \\
f_4(t/\pi) = \frac{17}{315} + \frac{4}{7} \cos^2 t + \frac{38}{105} \cos^4 t + \frac{4}{315} \cos^6 t \\
f_5(t/\pi) = \frac{62}{2835} + \frac{1072}{2835} \cos^2 t + \frac{484}{945} \cos^4 t + \frac{247}{2835} \cos^6 t + \frac{2}{2835} \cos^8 t 
\]

For integers \( r \), Proposition 1.2 is an immediate consequence.

For half-integers \( r = n + \frac{1}{2} \) one could observe that the identity
\[
\psi^{(2n)}(x) = -(2n)! \sum_{m=0}^{\infty} \frac{1}{(x+m)^{2n+1}} 
\]
allows one to express the inequality of Proposition 1.2 in terms of the polygamma functions \( \psi^{(2n)} \). We have not been able, however, to use this fact to give a simpler proof in that case.

REFERENCES


