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ON THE RAYLEIGH-LOVE ROD ACCRETING IN BOTH LENGTH AND CROSS-SECTIONAL AREA: FORCED AND DAMPED VIBRATIONS

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ABSTRACT. In this paper an elastic cylindrical rod that is subjected to forced and damped vibrations is considered. The rod is assumed to be isotropic. The applied external force of excitation is assumed to be harmonic, and the damping force is that of Kelvin-Voigt. The longitudinally vibrating rod is fixed at the left end and free at the other end. The rod is assumed to be accreting in length and cross-sectional area as it vibrates. The problem arising and the dynamics of the vibrating rod are described and investigated within the Rayleigh-Love theories of the rod. A partial differential equation describing the longitudinal displacement of the rod is formulated. The formulated partial differential equation, together with the corresponding boundary conditions as per the configuration of the rod, is solved numerically using the Galerkin-Kantorovich method. The frequency of vibration of the harmonic exciting force is kept constant in this investigation.

It is shown that in this periodically forced viscoelastic damped vibration, all the modes of vibration are subjected to the resonance behaviour within a proper time interval, depending on the length of the accreting rod.

Key words and phrases: Elastic rod; isotropic rod; Force of excitation; Kelvin-Voigt damping; Galerkin-Kantorovich method; Rayleigh-Love theory; Frequency of vibration; Amplitude of vibration.

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1. INTRODUCTION

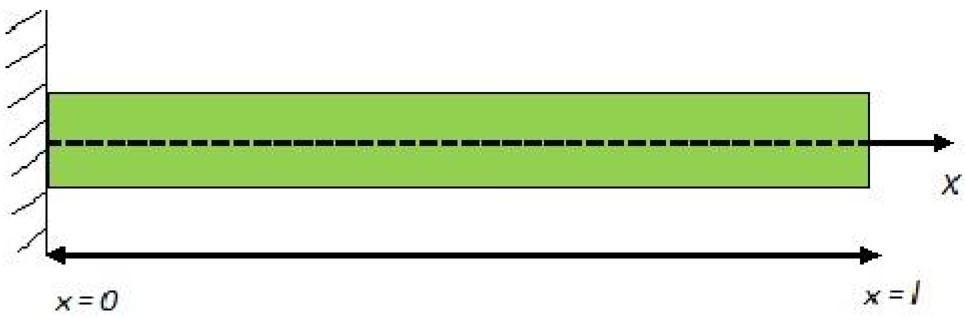
An extensive review of the fundamental models for the longitudinal vibration of rods has been a topic of interest to researchers, engineers, mathematicians and scientists alike. One can immediately think of the works by Shatalov et al in [[1], 2012], Fedotov et al in [[2],2011], and many more others in research papers and books. In most of these reviews the focus has always been on the behavioural patterns of the longitudinally vibrating rod while its dimensions were not accreting. While such a structural investigation may have well provided the researcher or the design engineer with an invaluable information as to how the stress and strains of the rod were or to be distributed, so that they can have a plan of circumventing and preventing the immnent catastrophic situation as a result of the unexpected amplitude of vibration growth, one critical assumption and possibility has always not featured in such discourse and investigations. The assumption and possibility is that the longitudinally vibrating rod may be accreting in one or several of its dimensions.

In this paper we investigate the behavioural patterns of a longitudinally vibrating rod that is assumed to be accreting in both length and cross-sectional area. It is further assumed that the rod is subjected to forced and damped vibrations. The rod is fixed at the left end and free at the other end. The dynamics and the problem arising is described by the Rayleigh-Love model. The longitudinal displacement in the Rayleigh-Love rod is described in accordance with the assumptions made in various vibration theories.

2. THE EQUATION OF MOTION

The Rayleigh-Love rod considered in this paper is assumed to be of unit length, elastic and isotropic. The rod is configured as in the figure below [[3], 2015]:

Figure 1: Fixed-Free Longitudinally Vibrating Rod



The equation of motion will therefore be given as derived by many authors, for example, Shatalov et al in [[4], 2010]. In this theory the partial differential equation describing the longitudinal displacement $u = u\,(t, x)$ for the longitudinally vibrating rod as subjected to forced and damped vibration is given by

$$\frac{\partial}{\partial t}\left[\rho\, A(t)\frac{\partial u}{\partial t} - \rho\,\nu^2 I_p(t)\,\frac{\partial^3 u}{\partial t \partial x^2}\right] - 2\delta\frac{\partial^3 u}{\partial t \partial x^2} - E\,A(t)\,\frac{\partial^2 u}{\partial x^2} = F_0 \sin\left(\omega\, t\right). \tag{2.1}$$

where $F_0 \sin\left(\omega\, t\right)$ is the external harmonically varying force and $2\delta\frac{\partial^3 u}{\partial t \partial x^2}$ is the damping mechanism known as the Kelvi-Voigt damping. The associated boundary conditions, as per the configuration of the rod, are defined as

$$\begin{aligned} x &= 0: \;\; u\,(t,x) = 0 \\ x &= 1: \;\; u'_x\,(t,x) = 0. \end{aligned} \tag{2.2}$$

The cross-sectional area of the rod is assumed to be a variable function of time t such that its growth is effected by the growth of its radius. It is further assumed that the cross-sectional area accretes proportionally with time.(2.2). The partial differential equation (2.1) can be simplified and re-written as

$$\rho\ A(t)\frac{\partial u}{\partial t}+\rho\frac{\partial u}{\partial t}\frac{\partial A}{\partial t}-\rho\nu^2 I_p(t)\frac{\partial^4 u}{\partial t^2\partial x^2}-\rho\nu^2\frac{\partial I_p}{\partial t}\frac{\partial^3 u}{\partial t\partial x^2}-2\delta\frac{\partial^3 u}{\partial t\partial x^2}-$$

$$E\ A(t)\ \frac{\partial^2 u}{\partial x^2}=F_0\sin\left(\omega\ t\right). \tag{2.3}$$

The radius of the cross-sectional area of the rod, which necessarily effects the growth of the rod is defined by

$$r\left(t\right)=r_0+\varepsilon\eta\sqrt{t}, \tag{2.4}$$

where ε and η are small parameters to effect the slow growth rate of the cross-sectional area. It is further assumed that the longitudinally vibrating rod is accreting linearly by the algebraic rule [[5], 2015]

$$f\left(t\right)=t, \tag{2.5}$$

and the linear growth of the rod in its longitudinal direction be

$$l(t)=1+\varepsilon.f\ \ (t)\,. \tag{2.6}$$

The change of variables $(t=\tau$, and $x=y\left(1+\varepsilon.f\left(t\right)\right))$ is introduced so that equation (2.3) together with the accompanying boundary conditions (2.2) can now be converted into a standard boundary-value problem as:

$$\frac{\partial^2\tilde{u}}{\partial\tau^2}+\frac{\varepsilon\eta}{\sqrt{\tau}\left(r_0+\varepsilon\eta\sqrt{\tau}\right)}\frac{\partial\tilde{u}}{\partial\tau}-\frac{\nu^2\left(r_0^2+2r_0\varepsilon\eta\sqrt{\tau}\right)}{2\left(1+\varepsilon\tau\right)^2}\frac{\partial^4\tilde{u}}{\partial\tau^2\partial y^2}-\frac{2y\varepsilon}{\left(1+\varepsilon\tau\right)}\frac{\partial^2\tilde{u}}{\partial\tau\partial y}-$$

$$\frac{2\delta}{\rho\pi\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2\left(1+\varepsilon\tau\right)^2}\frac{\partial^3\tilde{u}}{\partial\tau\partial y^2}+\frac{\varepsilon\eta\nu^2 r_0}{\sqrt{\tau}\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}\frac{\partial^3\tilde{u}}{\partial\tau\partial y^2}+\frac{2\varepsilon\nu^2 r_0^2}{\left(1+\varepsilon\tau\right)^3}\frac{\partial^3\tilde{u}}{\partial\tau\partial y^2}+$$

$$\frac{y\varepsilon\nu^2 r_0^2}{\left(1+\varepsilon\tau\right)^3}\frac{\partial^4\tilde{u}}{\partial\tau\partial y^3}-\frac{c^2}{\left(1+\varepsilon\tau\right)^2}\frac{\partial^2\tilde{u}}{\partial y^2}\approx\frac{F_0}{\rho\pi\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}\sin\left(\omega\ t\right). \tag{2.7}$$

The equation (2.7) was obtained after intentionally and deliberately neglecting the terms of order $O\left(\varepsilon^2\right)$ and/or $O\left(\epsilon\delta\right)$, since the two parameters have already been defined to be arbitrarily small. The associated boundary conditions of the partial differential equation (2.7) in new variables are given as

$$\begin{aligned} y &= 0: \ \tilde{u}\left(\tau,0\right)=0 \\ y &= 1: \ \tilde{u}_y{}'\left(\tau,1\right)=0. \end{aligned} \tag{2.8}$$

Thus the equations (2.7) and (2.8) form a boundary value problem which is solved numerically using the Galerkin-Kantorovich method.

3. NUMERICAL SOLUTION

The following linear combination of functions is chosen as an approximate solution of the boundary-value problem (2.7) and (2.8) [[6], 2002]:

$$u = u(\tau, y) = \sum_{m=1}^{\infty} C_m(\tau) \sin\left[\frac{(2m-1)\pi}{2} y\right]. \tag{3.1}$$

The unknown coefficients $C_m(\tau)$ are determined in the process of solving the boundary-value problem. In order to determine these coefficients, a sequence of linearly independent functions of the form

$$u(y) = 2\sin\left[\frac{(2n-1)\pi}{2} y\right], n = 1, 2, 3, ..., \tag{3.2}$$

is considered. Equation (3.1) is substituted into equation (2.7), such that the transformed partial differential equation is multiplied by equation (3.2). The result hereof is then integrated over the region $0 \le y \le 1$. The derived integrals are then each equated to zero, since for the exact solution, these integrals must equal zero. The following system of coupled ordinary differential equations is obtained using Mathematica® software:

$$\begin{gathered}
\left(1 + \frac{\pi^2 \varepsilon\eta\nu^2\sqrt{\tau} r_0^2}{4(1+\varepsilon\tau)^2}\right)\frac{d^2C_1}{d\tau^2} - \\
\frac{\varepsilon}{(1+\varepsilon\tau)} - \frac{\pi^2\varepsilon\eta\nu^2 r_0}{4\sqrt{\tau}(1+\varepsilon\tau)^2} + \frac{3\pi^2\varepsilon\nu^2 r_0^2}{8(1+\varepsilon\tau)^2} - \\
\frac{\pi\delta}{2\rho(1+\varepsilon\tau)^2(r_0+\varepsilon\eta\sqrt{\tau})^2} - \frac{\varepsilon\eta}{\sqrt{\tau}(\varepsilon\eta\sqrt{\tau}+r_0)})\frac{dC_1}{d\tau} + \\
\frac{c^2\pi^2}{4(1+\varepsilon\tau)^2}C_1 + \left(\frac{9\varepsilon}{2(1+\varepsilon\tau)} + \frac{81\varepsilon\pi^2\nu^2 r_0^2}{16(1+\varepsilon\tau)^3}\right)\frac{dC_2}{d\tau} \\
-\left(\frac{25\varepsilon}{6(1+\varepsilon\tau)} + \frac{625\varepsilon\pi^2\nu^2 r_0^2}{48(1+\varepsilon\tau)^3}\right)\frac{dC_3}{d\tau} + \\
\left(\frac{49\varepsilon}{12(1+\varepsilon\tau)} + \frac{2401\varepsilon\pi^2\nu^2 r_0^2}{96(1+\varepsilon\tau)^3}\right)\frac{dC_4}{d\tau} \\
-\left(\frac{81\varepsilon}{20(1+\varepsilon\tau)} + \frac{6561\varepsilon\pi^2\nu^2 r_0^2}{160(1+\varepsilon\tau)^3}\right)\frac{dC_5}{d\tau} = \frac{F_0}{\rho\pi(r_0+\varepsilon\eta\sqrt{\tau})^2}\sin(\omega\, t) \qquad (3.3) \\
\left(1 + \frac{9\pi^2 \varepsilon\eta\nu^2\sqrt{\tau} r_0^2}{4(1+\varepsilon\tau)^2}\right)\frac{d^2C_2}{d\tau^2} - \\
\frac{\varepsilon}{(1+\varepsilon\tau)} - \frac{9\pi^2\varepsilon\eta\nu^2 r_0}{4\sqrt{\tau}(1+\varepsilon\tau)^2} + \frac{27\pi^2\varepsilon\nu^2 r_0^2}{8(1+\varepsilon\tau)^2} - \\
\frac{9\pi\delta}{2\rho(1+\varepsilon\tau)^2(r_0+\varepsilon\eta\sqrt{\tau})^2} - \frac{\varepsilon\eta}{\sqrt{\tau}(\varepsilon\eta\sqrt{\tau}+r_0)})\frac{dC_2}{d\tau} + \\
\frac{9c^2\pi^2}{4(1+\varepsilon\tau)^2}C_2 - \left(\frac{\varepsilon}{2(1+\varepsilon\tau)} + \frac{\varepsilon\pi^2\nu^2 r_0^2}{16(1+\varepsilon\tau)^3}\right)\frac{dC_1}{d\tau} + \\
\left(\frac{25\varepsilon}{4(1+\varepsilon\tau)} + \frac{625\varepsilon\pi^2\nu^2 r_0^2}{32(1+\varepsilon\tau)^3}\right)\frac{dC_3}{d\tau}
\end{gathered}$$

$$-\left(\frac{49\varepsilon}{10\left(1+\varepsilon\tau\right)}+\frac{2401\varepsilon\pi^2\nu^2r_0^2}{80\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_4}{d\tau}$$

$$+\left(\frac{9\varepsilon}{2\left(1+\varepsilon\tau\right)}+\frac{729\varepsilon\pi^2\nu^2r_0^2}{16\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_5}{d\tau}=\frac{4F_0}{3\rho\pi\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}\sin\left(\omega\, t\right) \tag{3.4}$$

$$\left(1+\frac{25\pi^2\varepsilon\eta\nu^2\sqrt{\tau}r_0^2}{4\left(1+\varepsilon\tau\right)^2}\right)\frac{d^2C_3}{d\tau^2}-$$

$$\left(\frac{\varepsilon}{\left(1+\varepsilon\tau\right)}-\frac{25\pi^2\varepsilon\eta\nu^2r_0}{4\sqrt{\tau}\left(1+\varepsilon\tau\right)^2}+\frac{75\pi^2\varepsilon\nu^2r_0^2}{8\left(1+\varepsilon\tau\right)^3}-\right)$$

$$\frac{25\pi\delta}{2\rho\left(1+\varepsilon\tau\right)^2\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}-\frac{\varepsilon\eta}{\sqrt{\tau}\left(\varepsilon\eta\sqrt{\tau}+r_0\right)})\frac{dC_3}{d\tau}+$$

$$\frac{25c^2\pi^2}{4\left(1+\varepsilon\tau\right)^2}C_3+\left(\frac{\varepsilon}{6\left(1+\varepsilon\tau\right)}+\frac{\varepsilon\pi^2\nu^2r_0^2}{48\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_1}{d\tau}-$$

$$\left(\frac{9\varepsilon}{4\left(1+\varepsilon\tau\right)}+\frac{81\varepsilon\pi^2\nu^2r_0^2}{32\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_3}{d\tau}$$

$$+\left(\frac{49\varepsilon}{6\left(1+\varepsilon\tau\right)}+\frac{2401\varepsilon\pi^2\nu^2r_0^2}{48\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_4}{d\tau}$$

$$-\left(\frac{81\varepsilon}{14\left(1+\varepsilon\tau\right)}+\frac{6561\varepsilon\pi^2\nu^2r_0^2}{112\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_5}{d\tau}=\frac{4F_0}{5\rho\pi\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}\sin\left(\omega\, t\right) \tag{3.5}$$

$$\left(1+\frac{49\pi^2\varepsilon\eta\nu^2\sqrt{\tau}r_0^2}{4\left(1+\varepsilon\tau\right)^2}\right)\frac{d^2C_4}{d\tau^2}-$$

$$\frac{\varepsilon}{\left(1+\varepsilon\tau\right)}-\frac{49\pi^2\varepsilon\eta\nu^2r_0}{4\sqrt{\tau}\left(1+\varepsilon\tau\right)^2}+\frac{147\pi^2\varepsilon\nu^2r_0^2}{8\left(1+\varepsilon\tau\right)^3}-$$

$$\frac{49\pi\delta}{2\rho\left(1+\varepsilon\tau\right)^2\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}-\frac{\varepsilon\eta}{\sqrt{\tau}\left(\varepsilon\eta\sqrt{\tau}+r_0\right)})\frac{dC_4}{d\tau}+$$

$$\frac{49c^2\pi^2}{4\left(1+\varepsilon\tau\right)^2}C_4-\left(\frac{\varepsilon}{12\left(1+\varepsilon\tau\right)}+\frac{\varepsilon\pi^2\nu^2r_0^2}{96\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_1}{d\tau}+$$

$$\left(\frac{9\varepsilon}{10\left(1+\varepsilon\tau\right)}+\frac{81\varepsilon\pi^2\nu^2r_0^2}{80\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_2}{d\tau}$$

$$-\left(\frac{25\varepsilon}{6\left(1+\varepsilon\tau\right)}+\frac{625\varepsilon\pi^2\nu^2r_0^2}{48\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_3}{d\tau}$$

$$+\left(\frac{81\varepsilon}{8\left(1+\varepsilon\tau\right)}+\frac{6561\varepsilon\pi^2\nu^2r_0^2}{64\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_5}{d\tau}=\frac{4F_0}{7\rho\pi\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}\sin\left(\omega\, t\right) \tag{3.6}$$

$$\left(1+\frac{81\pi^2\varepsilon\eta\nu^2\sqrt{\tau}r_0^2}{4\left(1+\varepsilon\tau\right)^2}\right)\frac{d^2C_5}{d\tau^2}-$$

$$\left(\frac{\varepsilon}{\left(1+\varepsilon\tau\right)}-\frac{81\pi^2\varepsilon\eta\nu^2r_0}{4\sqrt{\tau}\left(1+\varepsilon\tau\right)^2}+\frac{243\pi^2\varepsilon\nu^2r_0^2}{8\left(1+\varepsilon\tau\right)^3}-\right)$$

$$\frac{81\pi\delta}{2\rho\left(1+\varepsilon\tau\right)^2\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}-\frac{\varepsilon\eta}{\sqrt{\tau}\left(\varepsilon\eta\sqrt{\tau}+r_0\right)})\frac{dC_5}{d\tau}+$$

$$\frac{81c^2\pi^2}{4\left(1+\varepsilon\tau\right)^2}C_4+\left(\frac{\varepsilon}{20\left(1+\varepsilon\tau\right)}+\frac{\varepsilon\pi^2\nu^2r_0^2}{160\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_1}{d\tau}-$$
$$\left(\frac{\varepsilon}{2\left(1+\varepsilon\tau\right)}+\frac{9\varepsilon\pi^2\nu^2r_0^2}{16\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_2}{d\tau}$$
$$+\left(\frac{25\varepsilon}{14\left(1+\varepsilon\tau\right)}+\frac{625\varepsilon\pi^2\nu^2r_0^2}{112\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_3}{d\tau}$$
$$-\left(\frac{49\varepsilon}{8\left(1+\varepsilon\tau\right)}+\frac{2401\varepsilon\pi^2\nu^2r_0^2}{64\left(1+\varepsilon\tau\right)^3}\right)\frac{dC_4}{d\tau}=\frac{4F_0}{9\rho\pi\left(r_0+\varepsilon\eta\sqrt{\tau}\right)^2}\sin\left(\omega\, t\right). \tag{3.7}$$

The system of equations (3.3) through to (3.7) is solved numerically by the use of the computer software Mathematica®. The solutions, as given graphically in the figures below, are obtained at the following assumed values of the parameters: $c = 1; \varepsilon = 0.05; r_0 = 0.1; \eta = 0.05; \delta = 0.05; \omega = 0.4\pi; F_0 = 1; \rho = 0.33$.

Figure 2: Rayleigh-Love Vibration: First Mode

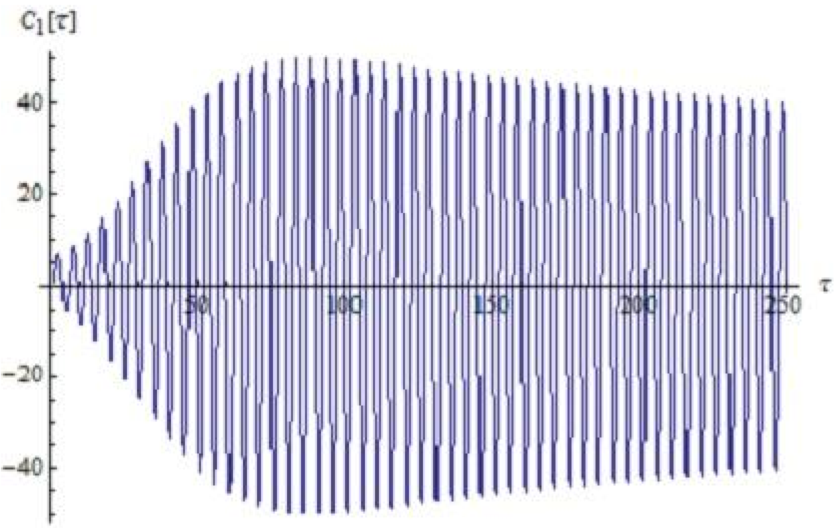


Figure 3: Rayleigh-Love Vibration: Second Mode

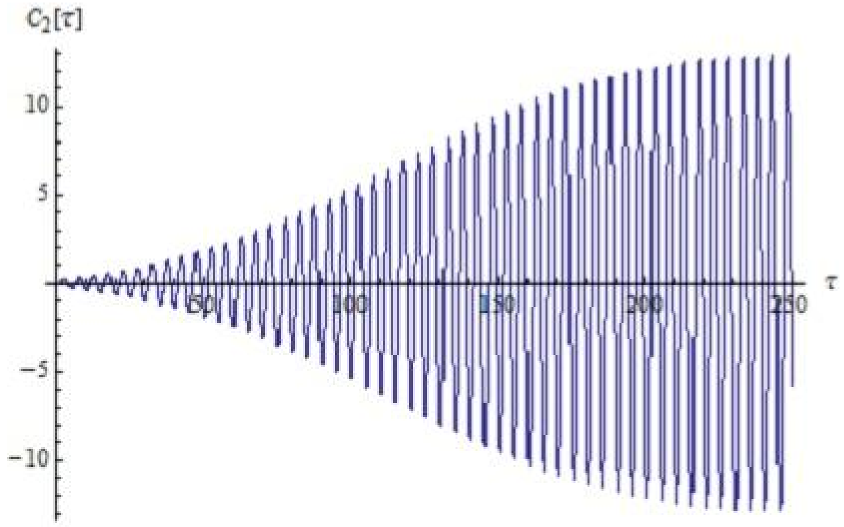


Figure 4: Rayleigh-Love Vibration: Third Mode

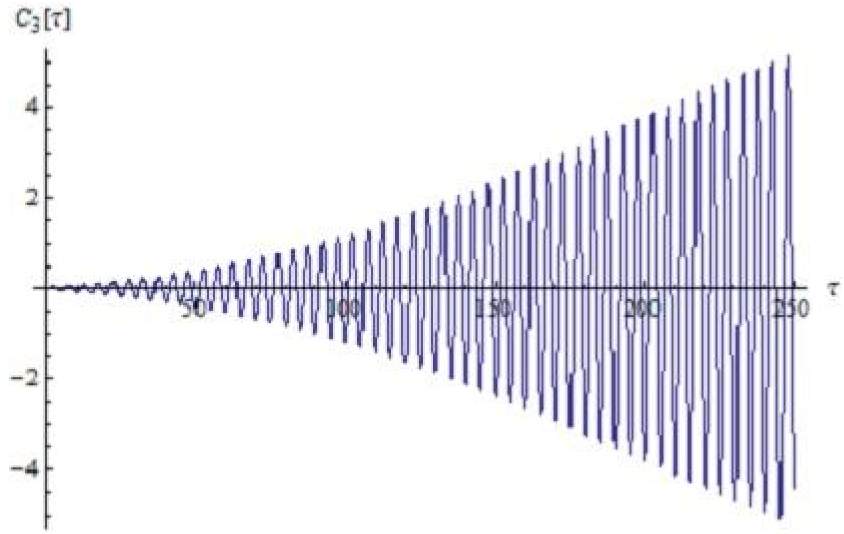


Figure 5: Rayleigh-Love Vibration: Fourth Mode

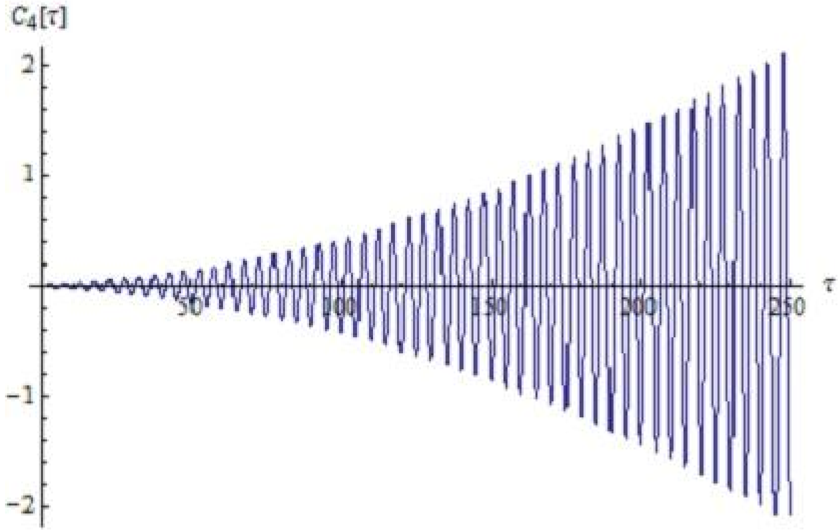
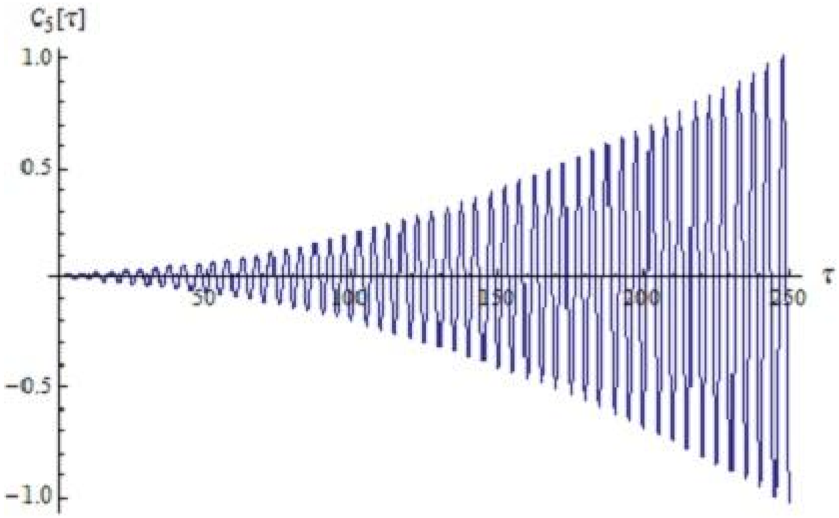


Figure 6: Rayleigh-Love Vibration: Fifth Mode



4. DISCUSSION AND CONCLUSION

The graphical solutions in this model exhibit a tendency that in the periodically forced viscoelastically damped case, all the modes of vibration are subjected to the resonance behaviour within a proper time interval depending on the length of the growing rod. At the first mode of vibration the amplitude is higher and decreases with time. This behaviour can be explained by the effect of the damping force. The same phenomenon can be seen at the second mode of vibration, albeit at a lower amplitude of vibration than in the first mode of vibration. As from the third mode of vibration, the exciting force dominates the damping force, hence the occurence of resonance phenomenon throughout the vibration. The decrease in amplitude of vibration from one mode to the other can still be attested to the slight influence of the damping force as the rod increase in both length and cross-sectional area.

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