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PARTIAL SEMIGROUP ALGEBRAS ASSOCIATED TO PARTIAL ACTIONS

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ABSTRACT. For a given inverse semigroup S, we introduce the notion of algebraic crossed product by using a given partial action of S, and we will prove that under some condition it is associative. Also we will introduce the concept of partial semigroup algebra $K_{Par}(S)$, and we show that the suitable quotient of $K_{Par}(S)$ is a kind of crossed product.

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1. INTRODUCTION

The notion of a partial crossed product of a C^*-algebra by the group of integers was defined by R. Exel in [2]. Roughly, the automorphism used in the definition of a crossed product of a C^*-algebra $\mathcal{A}$ by the group of integers was replaced by an isomorphism between two ideals of $\mathcal{A}$, namely, partial automorphisms. K. McClanahan in [5] defined a partial crossed product of a C^*-algebra by a discrete group. The ideas involved were mostly straightforward generalization of the definitions given in [2]. K. McClanahan used partial actions of groups to define a partial crossed product. This motivates us to define the algebraic crossed products by a partial action of an inverse semigroup S, and we will discuss the associativity of this construction. To define a partial semigroup algebra, we need some definitions and terminologies that A. Buss and R. Exel defined in [1],namely, universal inverse semigroup $Pr(S)$ and partial action of an inverse semigroup. We will use the universal inverse semigroup $Pr(S)$ to define a partial semigroup algebra $K_{Par}(S)$. Here, we will assume that the reader is familiar with the inverse semigroups and the concepts of partial actions of inverse semigroups, multiplier algebras, and partial representation. For a through treatment of these subjects the reader is referred to [1, 3, 4, 5, 7]. In Section 2, we will introduce the algebraic crossed product, and we will discuss the conditions under which this construction is associative. In Section 3, to prove our main result, the suitable quotient of the partial semigroup algebra is a sort of crossed product, we need to define some partial action. To do this, we will show that we can define a partial representation of an inverse semigroup S from a partial action of it and vice versa. At the end, we will prove that the suitable quotient of $K_{Par}(S)$ is a sort of crossed product.

2. ALGEBRAIC CROSSED PRODUCTS

Throughout this work, we will assume that S is a unital inverse semigroup, $\mathcal{A}$ is an associative algebra, and α is a partial action of S on $\mathcal{A}$. Let α be a partial action of S on an algebra $\mathcal{A}$. Let $L = \{\sum_{s\in S}^{\text{finite}} a_s\delta_s \ : \ a_s \in X_s\}$, the set of all formal finite sums, with the following multiplication:

$$(a_s\delta_s)\cdot(b_t\delta_t) = \alpha_s(\alpha_{s^*}(a_s)b_t)\delta_{st}.$$

Note that $\alpha_s(\alpha_{s^*}(a_s)b_t)$ is an element of X_{st} simply because

$$\alpha_s(\alpha_{s^*}(a_s)b_t) \in \alpha_s(X_{s^*}\bigcap X_t) = X_s\bigcap X_{st}.$$

Thus, the multiplication is well-defined, and L is an algebra with this multiplication. Let $\mathcal{I}$ be the ideal of L generated by the set $\{a\delta_r - a\delta_t \ : \ a \in X_r,\ r \leq t\}$. Note that by [1, Proposition 3.8] $X_r \subseteq X_t$ if $r \leq t$. Now, we are at a position to introduce the algebraic crossed product $\mathcal{A} \times_\alpha S$ associated to a partial action α as we proposed in the beginning. the algebraic crossed product $\mathcal{A} \times_\alpha S$ is the algebra $\frac{L}{N}$. Let $\Phi : L_1 \to \mathcal{A} \times_\alpha S$ be the canonical homomorphism from L onto $\mathcal{A} \times_\alpha S$. We will denote $\Phi(a_s\delta_s)$ by $\overline{a_s\delta_s}$.

With the aid of the following pivotal Theorem, we are able to answer the associativity question of algebraic crossed product. We will show that the mentioned multiplication is associative if for each $s \in S$ the ideals X_s associated to the partial action α are (L, R)-associative, that is, when $L \circ \acute{R} = \acute{R} \circ L$ for all multipliers $(L, R), (\acute{L}, \acute{R})$ of X_s.

Theorem 2.1. *If α is a partial action of a unital inverse semigroup S on an algebra $\mathcal{A}$ such that each X_s $(s \in S)$ is (L,R)-associative, then the algebraic crossed product $\mathcal{A} \rtimes_\alpha S$ is associative.*

Proof. The proof is a slight modification of [3, Theorem 3.4]. ∎

As a consequence of Theorem2.1, we have the following corollary.

Corollary 2.2. *Let α be a partial action of a unital inverse semigroup S on an algebra $\mathcal{A}$ such that each X_s is an idempotent or non-degenerate ideal of $\mathcal{A}$ for each $s \in S$. Then $\mathcal{A} \rtimes_\alpha S$ is associative.*

Proof. This follows from the fact that (L, R)-associativity is equivalent to the condition of the Corollary and Theorem 2.1. ∎

Definition 2.1. We shall say that an algebra $\mathcal{A}$ is strongly associative if for any unital inverse semigroup S and any partial action of S on $\mathcal{A}$, namely α, the algebraic crossed product $\mathcal{A} \rtimes_\alpha S$ is associative.

As a consequence of Corollary 2.2 we have:

Corollary 2.3. *A semiprime algebra $\mathcal{A}$ is strongly associative.*

3. PARTIAL REPRESENTATIONS

Throughout this section, by π we mean a partial representation of an inverse semigroup S on a unital algebra $\mathcal{B}$. For simplicity hereafter, we will denote an element $\overline{a_s\delta_s}$ of $\mathcal{A} \rtimes_\alpha S$ by $a_s\delta_s$.

Lemma 3.1. *Suppose that α is a partial action of S on an algebra $\mathcal{A}$, where X_s (the range of α_s) is a unital algebra with the unit element 1_s. Then the map $\pi_\alpha : S \ni s \mapsto 1_s\delta_s \in \mathcal{A} \rtimes_\alpha S$ is a partial representation.*

Proof. First of all, note that since each X_s is unital, it is idempotent, hence, each X_s is (L, R)-associative, thus, $\mathcal{A} \rtimes_\alpha S$ is associative by Theorem 2.1. To prove the Lemma, let s, t be elements of S. Then

$$\begin{aligned} \pi_\alpha(s^*)\pi_\alpha(s)\pi_\alpha(t) &= 1_{s^*}\delta_{s^*}.1_s\delta_s.1_t\delta_t \\ &= \alpha_{s^*}(1_s.1_s)\delta_{s^*s}.1_t\delta_t \\ &= (1_{s^*}1_{s^*s})\delta_{s^*s}.1_t\delta_t. \end{aligned} \tag{3.1}$$

The last equality is obtained from the fact that 1_s1_t is the unit element of the algebra $X_s \cap X_t$. Since α is a partial action, the equality $\alpha_s(X_{s^*} \cap X_t) = X_s \cap X_{st}$ implies that

$$\alpha_s(1_{s^*}1_t) = 1_s1_{st}. \tag{3.2}$$

By the definition of the multiplication, the right hand side of (3.1) is equal to

$$\alpha_{s^*s}(\alpha_{s^*s}(1_{s^*}1_{s^*s})1_t)\delta_{s^*st} = \alpha_{s^*s}(1_{s^*}1_{s^*s}1_t)\delta_{s^*st}. \tag{3.3}$$

Note that $X_{s^*} \subseteq X_{s^*s}$ because

$$X_{s^*} = \alpha_{s^*}(X_s \cap X_s) = X_{s^*} \cap X_{s^*s} \subseteq X_{s^*s}.$$

Thus, $1_{s^*}1_{s^*s} = 1_{s^*s}1_{s^*} = 1_{s^*}$. The foregoing computations and equation (3.2) gives that the right hand side of (3.3) is equal to

$$1_{s^*}1_{s^*st}\delta_{s^*st}. \tag{3.4}$$

On the other hand,

$$\begin{aligned} \pi_\alpha(s^*)\pi_\alpha(st) &= 1_{s^*}\delta_{s^*}.1_{st}\delta_{st} \\ &= \alpha_{s^*}(\alpha_s(1_{s^*})1_{st})\delta_{s^*st} \\ &= \alpha_{s^*}(1_s1_{st})\delta_{s^*st} \\ &= 1_{s^*}1_{s^*st}\delta_{s^*st}. \end{aligned} \tag{3.5}$$

Comparing (3.4) and (3.5), one has that $\pi_\alpha(s^*)\pi_\alpha(s)\pi_\alpha(t) = \pi_\alpha(s^*)\pi_\alpha(st)$. For $s,t \in S$, we have

$$\begin{aligned}\pi_\alpha(s)\pi_\alpha(t)\pi_\alpha(t^*) &= 1_s\delta_s.1_t\delta_{tt^*}\\ &= \alpha_s(1_{s^*}1_t)\delta_{stt^*}\\ &= 1_s1_{st}1_{stt^*}\delta_{stt^*},\end{aligned} \tag{3.6}$$

by using (3.2) and the fact that $1_s1_{st} \in X_{stt^*}$. On the one hand, we have that

$$\begin{aligned}\pi_\alpha(st)\pi_\alpha(t^*) &= 1_{st}\delta_{st}.1_{t^*}\delta_{t^*}\\ &= \alpha_{st}(1_{(st)^*}1_{t^*})\delta_{stt^*}\\ &= 1_{st}1_{stt^*}\delta_{stt^*}.\end{aligned}$$

The last equality is obtained by using (3.2). Since $stt^* \leq s$, we have that $X_{stt^*} \subseteq X_s$ by [1, Proposition 3.8], therefore, $1_{st}1_{stt^*}\delta_{stt^*} = 1_s1_{st}1_{stt^*}\delta_{stt^*}$. Thus, the proof is complete if we compare this equality with (3.6). ∎

Definition 3.1. Two partial representations $\pi \ : S \mapsto \mathcal{B}$ and $\pi' \ : S \mapsto \mathcal{B}'$ are isomorphic if there exists an isomorphism of algebras $\phi \ : \mathcal{B} \mapsto \mathcal{B}'$ such that

$$\pi(s) = \phi(\pi'(s))$$

for all $s \in S$.

Remark 3.1. It is easy to see that if α and α' are two isomorphic partial actions for S then π_α and $\pi_{\alpha'}$ are isomorphic partial representations of S.

Our next goal is to define a partial action from a given partial representation of an inverse semigroup S. To do this, let $\pi : S \mapsto \mathcal{B}$ be a partial representation of S into a $\mathcal{K}$-algebra $\mathcal{B}$. Considering the elements $\varepsilon_s = \pi(s)\pi(s^*)$ for $s \in S$, we have the following Lemma.

Lemma 3.2. *Let s,t are in S. Then*

(*i*) *For given s,t in S, ε_s and ε_t are commuting and idempotent elements,*
(*ii*) *and $\pi(s)\varepsilon_t = \varepsilon_{st}\pi(s)$.*

Proof. Taking $s \in S$, one has that

$$\varepsilon_s\varepsilon_s = \pi(s)\pi(s^*)\pi(s)\pi(s^*) = \pi(s)\pi(s^*) = \varepsilon_s,$$

thus ε_s is idempotent. On the one hand, for $s,t \in S$ we have

$$\begin{aligned}\varepsilon_s\varepsilon_t &= \pi(s)\pi(s^*)\pi(t)\pi(t^*)\\ &= \pi(s)\pi(s^*t)\pi(t^*)\\ &= \pi(s)\pi(s^*t)\pi(t^*s)\pi(s^*t)\pi(t^*)\\ &= \pi(ss^*t)\pi(t^*s)\pi(s^*tt^*).\end{aligned} \tag{3.7}$$

Interchanging the role of s and t, from the foregoing argument we have that

$$\begin{aligned}\varepsilon_t\varepsilon_s &= \pi(tt^*s)\pi(s^*t)\pi(t^*ss^*)\\ &= \pi(tt^*s)\pi(s^*t)\pi(t^*s)\pi(s^*t)\pi(t^*ss^*)\\ &= \pi(tt^*ss^*t)\pi(t^*s)\pi(s^*tt^*ss^*)\\ &= \pi(ss^*tt^*t)\pi(t^*s)\pi(ss^*tt^*)\\ &= \pi(ss^*t)\pi(t^*s)\pi(s^*tt^*) = \varepsilon_s\varepsilon_t.\end{aligned}$$

For part (*ii*), let $s, t \in S$, then

$$\begin{aligned}\pi(s)\varepsilon_t &= \pi(s)\pi(t)\pi(t^*)\\ &= \pi(s)\pi(s^*)\pi(s)\pi(t)\pi(t^*)\\ &= \pi(s)\pi(t)\pi(t^*)\pi(s)\pi(s^*)\\ &= \pi(st)\pi(t^*s^*)\pi(s) = \varepsilon_{st}\pi(s). \end{aligned} \tag{3.8}$$

∎

Let $\mathcal{A}$ be the commutative subalgebra of $\mathcal{B}$ generated by all ε_s ($s \in S$), and for a fixed $s \in S$ set $X_s = \varepsilon_s\mathcal{A}$. Now, we can provide a partial action from π as follows:

Lemma 3.3. *The maps $\alpha_s^\pi : X_{s^*} \mapsto X_s$ ($s \in S$) defined by $\alpha_s^\pi(a) = \pi(s)a\pi(s^*)$ ($a \in X_{s^*}$) are isomorphisms of $\mathcal{K}$-algebras which determine a partial action α^π of S on the algebra $\mathcal{A}$.*

Proof. For simplicity, let $\alpha^\pi = \alpha$. We show that $\mathcal{A}$ is invariant under the map $a \mapsto \pi(s)a\pi(s^*)$ for each $s \in S$. Since $\mathcal{A}$ is spanned by elements of the form $\varepsilon_{t_1}...\varepsilon_{t_n}$ with $t_1...t_n \in S$. For such elements, one has that

$$\begin{aligned}\pi(s)\varepsilon_{t_1}...\varepsilon_{t_n}\pi(s^*) &= \varepsilon_{st_1}...\varepsilon_{st_n}\pi(s)\pi(s^*)\\ &= \varepsilon_{st_1}...\varepsilon_{st_n}\varepsilon_s \in \mathcal{A}\end{aligned}$$

Thus, $\pi(s)\mathcal{A}\pi(s^*) \subseteq \mathcal{A}$. Now, for $s \in S$,

$$\begin{aligned}\alpha_s(\varepsilon_{s^*}) &= \pi(s)\varepsilon_{s^*}\pi(s^*)\\ &= \pi(s)\pi(s^*)\pi(s)\pi(s^*)\\ &= \varepsilon_s\varepsilon_s = \varepsilon_s.\end{aligned}$$

Consequently, for the elements of the form $\varepsilon_{s^*}\varepsilon_{t_1}...\varepsilon_{t_n}$, one has that

$$\begin{aligned}\pi(s)\varepsilon_{s^*}\varepsilon_{t_1}...\varepsilon_{t_n}\pi(s^*) &= \varepsilon_{ss^*}\varepsilon_{st_1}...\varepsilon_{st_n}\pi(s)\pi(s^*)\\ &= \varepsilon_s\varepsilon_{ss^*}\varepsilon_{st_1}...\varepsilon_{st_n} \in X_s.\end{aligned}$$

Since X_{s^*} is spanned by the elements of the form $\varepsilon_{s^*}\varepsilon_{t_1}...\varepsilon_{t_n}$ with $t_1...t_n \in S$, we have that $\alpha_s : X_{s^*} \mapsto X_s$. Moreover, it is a homomorphism of algebras. Indeed, taking elements $a = \varepsilon_{s^*}\varepsilon_{t_1}...\varepsilon_{t_n}$ and $b = \varepsilon_{s^*}\varepsilon_{r_1}...\varepsilon_{r_n}$ in X_{s^*}, we see that

$$\begin{aligned}\alpha_s(a)\alpha_s(b) &= \pi(s)\varepsilon_{s^*}\varepsilon_{t_1}...\varepsilon_{t_n}\pi(s^*)\pi(s)\varepsilon_{s^*}\varepsilon_{r_1}...\varepsilon_{r_n}\pi(s^*)\\ &= \pi(s)\varepsilon_{s^*}\varepsilon_{t_1}...\varepsilon_{t_n}\varepsilon_{s^*}\varepsilon_{s^*}\varepsilon_{r_1}...\varepsilon_{r_n}\pi(s^*)\\ &= \pi(s)ab\pi(s^*) = \alpha_s(ab).\end{aligned}$$

Obviously, for $s \in S$, ε_s is the identity of X_s and $\alpha_s(\varepsilon_{s^*}) = \varepsilon_s$, thus, α_s is a homomorphism of algebras. Observe that for $s \in S$ we have $\alpha_s(\alpha_{s^*}(\varepsilon_s)) = \varepsilon_s$. By induction, we shall prove that $\alpha_s \circ \alpha_{s^*}(\varepsilon_s\varepsilon_{t_1}...\varepsilon_{t_n}) = \varepsilon_s\varepsilon_{t_1}...\varepsilon_{t_n}$. For $n = 1$,

$$\begin{aligned}\alpha_s \circ \alpha_{s^*}(\varepsilon_s\varepsilon_{t_1}) &= \alpha_s(\pi(s^*)\varepsilon_s\varepsilon_{t_1}\pi(s))\\ &= \alpha_s(\varepsilon_{s^*s}\varepsilon_{s^*t_1}\varepsilon_{s^*})\\ &= \pi(s)\varepsilon_{s^*}\varepsilon_{s^*s}\varepsilon_{s^*t_1}\pi(s^*)\\ &= \pi(s)\pi(s^*)\pi(s)\pi(s^*s)\pi(s^*s)\pi(s^*t_1)\pi(t_1^*s)\pi(s^*)\\ &= \pi(s)\pi(s^*)\pi(ss^*s)\pi(s^*s)\pi(s^*t_1)\pi(t_1^*s)\pi(s^*)\\ &= \pi(s)\pi(s^*)\pi(s)\pi(s^*)\pi(t_1)\pi(t_1^*)\pi(s)\pi(s^*)\\ &= \varepsilon_s\varepsilon_{t_1}.\end{aligned}$$

Now, suppose that for $n < k$ the statement is true. Let $\varepsilon_s \varepsilon_{t_1} ... \varepsilon_{t_k}$ be any element of X_s. Then

$$\begin{aligned}\alpha_s \circ \alpha_{s^*}(\varepsilon_s \varepsilon_{t_1} ... \varepsilon_{t_k}) &= \alpha_s \circ \alpha_{s^*}(\varepsilon_s \varepsilon_{t_1} ... \varepsilon_{t_{k-1}} \varepsilon_s \varepsilon_{t_k}) \\ &= \alpha_s \circ \alpha_{s^*}(\varepsilon_s \varepsilon_{t_1} ... \varepsilon_{t_{k-1}}) \alpha_s \circ \alpha_{s^*}(\varepsilon_s \varepsilon_{t_k}),\end{aligned}$$

so, the induction hypothesis implies that

$$\alpha_s \circ \alpha_{s^*}(\varepsilon_s \varepsilon_{t_1} ... \varepsilon_{t_k}) = \varepsilon_s \varepsilon_{t_1} ... \varepsilon_{t_k}.$$

Since X_s is spanned by such elements, we deduce that $\alpha_s \circ \alpha_{s^*}$ is the identity map on X_s. Interchanging the role of s and s^*, we see that $\alpha_{s^*} \circ \alpha_s$ is the identity map on X_{s^*}, hence, $\alpha_s^{-1} = \alpha_{s^*}$, thus, α_s is an isomorphism. Let $s, t \in S$ and write an element $a \in X_{s^*} \cap X_t$ as $a = \varepsilon_{s^*} \varepsilon_t b$ with $b \in \mathcal{A}$. So,

$$\begin{aligned}\alpha_s(a) &= \pi(s) \varepsilon_{s^*} \varepsilon_t b \pi(s^*) \\ &= \varepsilon_{ss^*} \varepsilon_{st} \pi(s) b \pi(s^*) \\ &= \varepsilon_{st} \varepsilon_{ss^*} \varepsilon_{st} \pi(s) b \pi(s^*) \in X_{st},\end{aligned}$$

where in the last step we have used the fact that $\pi(s)\mathcal{A}\pi(s^*) \subseteq \mathcal{A}$. This gives $\alpha_s(X_{s^*} \cap X_t) \subseteq X_s \cap X_{st}$. Taking $x \in X_s \cap X_{st}$, write x as $x = \varepsilon_s \varepsilon_{st} b$ with $b \in \mathcal{A}$. Moreover, we can assume that $b = \varepsilon_{t_1} ... \varepsilon_{t_n}$ with $t_1 ... t_n \in S$. Note that

$$\begin{aligned}\varepsilon_{st} \varepsilon_s &= \pi(st) \pi(t^* s^*) \pi(s) \pi(s^*) \\ &= \pi(st) \pi(t^*) \pi(s^*) \pi(s) \pi(s^*) \\ &= \pi(s) \pi(t) \pi(t^*) \pi(s^*) \pi(s) \pi(s^*) \\ &= \alpha_s(\varepsilon_t \varepsilon_{s^*}).\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}x &= \varepsilon_s \varepsilon_{st} \varepsilon_{t_1} ... \varepsilon_{t_n} \\ &= \varepsilon_s \varepsilon_{st} \pi(s) \pi(s^*) \varepsilon_s \varepsilon_{t_1} ... \varepsilon_{t_n} \\ &= \varepsilon_s \varepsilon_{st} \pi(s) \varepsilon_{s^* s} \varepsilon_{s^* t_1} ... \varepsilon_{s^* t_n} \pi(s^*) \\ &= \alpha_s(\varepsilon_t \varepsilon_{s^*}) \alpha_s(\varepsilon_{s^* s} \varepsilon_{s^* t_1} ... \varepsilon_{s^* t_n}) \\ &= \alpha_s(\varepsilon_t \varepsilon_{s^*} \varepsilon_{s^* s} \varepsilon_{s^* t_1} ... \varepsilon_{s^* t_n}).\end{aligned}$$

From the fact that $\varepsilon_t \varepsilon_{s^*} \varepsilon_{s^* s} \varepsilon_{s^* t_1} ... \varepsilon_{s^* t_n} \in X_{s^*} \cap X_t$, we have $x \in \alpha_s(X_{s^*} \cap X_t)$, and we deduce that $\alpha_s(X_{s^*} \cap X_t) = X_s \cap X_{st}$. Let $a \in \alpha_t^{-1}(X_t \cap X_{s^*}) \subseteq X_{t^*} \cap X_{t^* s^*}$. Then $\varepsilon_{t^*} a = a \varepsilon_{t^*} = a$, so,

$$\begin{aligned}\alpha_s \circ \alpha_t(a) = \alpha_s(\pi(t) a \pi(t^*)) &= \pi(s) \pi(t) a \pi(t^*) \pi(s^*) \\ &= \pi(s) \pi(t) a \varepsilon_{t^*} \pi(t^*) \pi(s^*) \\ &= \pi(s) \pi(t) a \varepsilon_{t^*} \pi(t^* s^*) \\ &= \pi(s) \pi(t) \varepsilon_{t^*} a \pi(t^* s^*) \\ &= \pi(st) \varepsilon_{t^*} a \pi(t^* s^*) \\ &= \alpha_{st}(a)\end{aligned}$$

Therefore, α is a partial action of S on $\mathcal{A}$. ∎

Remark 3.2. As in the previous case, it is easy to see that α^π and $\alpha^{\pi'}$ are isomorphic partial actions if π and π' are isomorphic representations.

Given an inverse semigroup S, let $\pi : S \to \mathcal{B}$ be a partial representation of S into $\mathcal{B}$, and let J be the ideal of $\mathcal{B}$ generated by all elements of the form $\pi(s) - \pi(t)$ where $s, t \in S$ and $s \leq t$. Define the map $\tilde{\pi} : S \to \frac{\mathcal{B}}{J}$ by $\tilde{\pi}(s) = \Phi(\pi(s))$, where $\Phi : \mathcal{B} \to \frac{\mathcal{B}}{J}$ is the quotient map. Obviously, $\tilde{\pi}$ is a partial representation of S on $\frac{\mathcal{B}}{J}$.

Proposition 3.4. *Let $\pi : S \to \mathcal{B}$ be a partial representation of S into $\mathcal{B}$ and suppose that $\tilde{\pi} : S \to \frac{\mathcal{B}}{J}$ is the partial representation mentioned above. Consider the subalgebra $\mathcal{A} \subseteq \frac{\mathcal{B}}{J}$ and partial action $\alpha^{\tilde{\pi}}$ as in Lemma 3.3. Then the map $\phi_{\tilde{\pi}} : \mathcal{A} \rtimes_{\alpha^{\tilde{\pi}}} S \ni \sum_{s\in S} a_s\delta_s \mapsto \sum_{s\in S} a_s\tilde{\pi}(s) \in \frac{\mathcal{B}}{J}$ is a homomorphism of $\mathcal{K}$-algebras such that $\phi_{\tilde{\pi}} \circ \pi_{\alpha^{\tilde{\pi}}} = \tilde{\pi}$.*

Proof. One can easily prove that the map $\phi_{\tilde{\pi}}$ is well-defined. Since $a_s\delta_s$ ($s \in S$) span L, it is enough to show that

$$\phi_{\tilde{\pi}}(a_s\delta_s \cdot b_t\delta_t) = \phi_{\tilde{\pi}}(a_s\delta_s) \cdot \phi_{\tilde{\pi}}(b_t\delta_t).$$

By the definition of $\phi_{\tilde{\pi}}$, we have

$$\phi_{\tilde{\pi}}(a_s\delta_s \cdot b_t\delta_t) = \phi_{\tilde{\pi}}(\alpha_s^{\tilde{\pi}}(\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t)\delta_{st}) = \alpha_s^{\tilde{\pi}}(\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t)\tilde{\pi}(st).$$

Since $\alpha_s^{\tilde{\pi}}(\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t) \in X_s$ and ε_s is the identity element of X_s,

$$\begin{aligned}
\alpha_s^{\tilde{\pi}}(\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t)\tilde{\pi}(st) &= \alpha_s^{\tilde{\pi}}(\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t)\varepsilon_s\tilde{\pi}(st) \\
&= \alpha_s^{\tilde{\pi}}(\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t)\varepsilon_s\tilde{\pi}(s)\tilde{\pi}(t) \\
&= \tilde{\pi}(s)(\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t)\tilde{\pi}(s^*)\tilde{\pi}(s)\tilde{\pi}(t) \\
&= \tilde{\pi}(s)(\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t)\varepsilon_{s^*}\tilde{\pi}(t) \\
&= \tilde{\pi}(s)\alpha_{s^*}^{\tilde{\pi}}(a_s)b_t\tilde{\pi}(t) \\
&= \tilde{\pi}(s)\tilde{\pi}(s^*)a_s\tilde{\pi}(s)b_t\tilde{\pi}(t) \\
&= a_s\tilde{\pi}(s)b_t\tilde{\pi}(t) = \phi_{\tilde{\pi}}(a_s\delta_s)\phi_{\tilde{\pi}}(b_t\delta_t).
\end{aligned}$$

∎

For an inverse semigroup S, let Pr(S) be the universal semigroup associated to S. We will refer the reader to [1] for more details about Pr(S).

Definition 3.2. Let S be an inverse semigroup. We will denote the semigroup algebra $KPr(S)$ by $K_{par}(S)$, and we will call it partial semigroup algebra.

Obviously, the canonical map $\iota_S : S \ni s \mapsto [s] \in K_{par}(S)$ is a partial homomorphism of S.

Remark 3.3. Suppose that $\pi : S \to \mathcal{B}$ is a partial homomorphism of S in $\mathcal{B}$. Then by [1, Proposition 2.20], there exists a unique semigroup homomorphism $\pi^* : Pr(S) \to \mathcal{B}$ such that $\pi^* \circ \iota_S = \pi$. Now define $\psi : K_{par}(S) \ni \Sigma_{s\in Pr(S)} a_s s \mapsto \Sigma_{s\in Pr(S)} a_s\pi^*(s) \in \mathcal{B}$. Obviously, ψ is a homomorphism and $\psi \circ \iota_S = \pi$. On the other hand, if $\psi : K_{par}(S) \to \mathcal{B}$ is an algebra homomorphism, then $\psi \circ \iota_S$ is a partial representation of S in $\mathcal{B}$.

Theorem 3.5. *Let ι_S be the partial representation as mentioned above. Considering the partial representation $\tilde{\iota_S}$ and the commutative subalgebra $\mathcal{A}$ of $\frac{K_{par}(S)}{J}$ generated by all elements $\tilde{\varepsilon_s} = \tilde{\iota_S}(s)\tilde{\iota_S}(s^*)$ defined in Proposition 3.4, we will prove that $\mathcal{A} \times_{\alpha^{\tilde{\iota_S}}} S$ is isomorphic to $\frac{K_{par}(S)}{J}$.*

Proof. By Lemma 3.1, $\pi_{\alpha^{\tilde{\iota_S}}} : S \ni s \mapsto \tilde{\varepsilon_s}\delta_s \in \mathcal{A} \rtimes_{\alpha^{\tilde{\iota_S}}} S$ is a partial representation of S on $\mathcal{A} \rtimes_{\alpha^{\tilde{\iota_S}}} S$. Now, by Remark 3.3, there exists $\psi : K_{par}(S) \to \mathcal{A} \rtimes_{\alpha^{\tilde{\iota_S}}} S$ such that $\psi \circ \tilde{\iota_S} = \pi_{\alpha^{\tilde{\iota_S}}}$, i.e $\psi([s]) = \tilde{\varepsilon_s}\delta_s$. For $s \in S$, we have

$$\phi_{\tilde{\iota_S}} \circ \psi([s]) = \phi_{\tilde{\iota_S}}(\tilde{\varepsilon_s}\delta_s) = \tilde{\varepsilon_s}\tilde{\iota_S}(s) = [s][s^*][s] = [s].$$

Consequently, $\phi_{\tilde{\iota_S}} \circ \psi = id_{K_{par}(S)}$. By definition of J, an element of J has the form $[s] - [t]$ where $s \leq t$. So, $\psi([s] - [t]) = \tilde{\varepsilon_s}\delta_s - \tilde{\varepsilon_t}\delta_t$ where $s \leq t$, but this is an element of the ideal generated by the elements of the form $a\delta_r - a\delta_k$ where $r \leq k$ and $a \in X_r$. So, ψ factors through the quotient $\frac{K_{Par}(S)}{J}$, so, $\phi_{\tilde{\iota_S}}$ is a surjective homomorphism. On the other hand, by the definition X_s, for each $s \in S$, is spanned by the element of the form $\tilde{\varepsilon_s}\tilde{\varepsilon_{t_1}}.\ldots.\tilde{\varepsilon_{t_n}}$. For $s \in S$, we have

$$\begin{aligned}
\tilde{\varepsilon_s}\delta_s\tilde{\varepsilon_{s^*}}\delta_{s^*} &= \alpha_s^{\tilde{\iota_S}}(\alpha_{s^*}^{\tilde{\iota_S}}(\tilde{\varepsilon_s})\tilde{\varepsilon_{s^*}})\delta_{ss^*} \\
&= \alpha_s^{\tilde{\iota_S}}(\tilde{\iota_S}(s^*)\tilde{\varepsilon_s}\tilde{\iota_S}(s)\tilde{\varepsilon_{s^*}})\delta_{ss^*} \\
&= \alpha_s^{\tilde{\iota_S}}([s^*][s][s^*][s]\tilde{\varepsilon_{s^*}}) \\
&= \alpha_s^{\tilde{\iota_S}}(\tilde{\varepsilon_{s^*}})\delta_{ss^*} \\
&= [s]\tilde{\varepsilon_{s^*}}[s^*] = \tilde{\varepsilon_s}\delta_{ss^*} = \tilde{\varepsilon_s}\delta_1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi \circ \phi_{\tilde{\iota_S}}((\tilde{\varepsilon_s}\tilde{\varepsilon_{t_1}}.\ldots.\tilde{\varepsilon_{t_n}})\delta_s) &= \psi(\tilde{\varepsilon_s}\tilde{\varepsilon_{t_1}}.\ldots.\tilde{\varepsilon_{t_n}}[s]) \\
&= \psi([s][s^*][t_1][t_1^*]...[t_n][t_n^*][s]) \\
&= \tilde{\varepsilon_s}\delta_s\tilde{\varepsilon_{s^*}}\delta_{s^*}...\tilde{\varepsilon_{t_n}}\delta_{t_n}\tilde{\varepsilon_{t_n}}\delta_{t_n^*}\tilde{\varepsilon_s}\delta_s \\
&= \tilde{\varepsilon_s}\delta_1\tilde{\varepsilon_{t_1}}\delta_1...\tilde{\varepsilon_{t_n}}\delta_1\tilde{\varepsilon_s}\delta_s \\
&= \tilde{\varepsilon_s}\tilde{\varepsilon_{t_1}}...\tilde{\varepsilon_{t_n}}\delta_s.
\end{aligned}$$

Thus, $\psi \circ \phi_{\tilde{\iota_S}}$ is also the identity map, by consequence, $\phi_{\tilde{\iota_S}}$ is an isomorphism. ∎

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