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STABILITY OF AN ALMOST SURJECTIVE EPSILON-ISOMETRY IN THE DUAL OF REAL BANACH SPACES

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ABSTRACT. In this paper, we study the stability of epsilon-isometry in the dual of real Banach spaces. We prove that the almost surjective epsilon-isometry mapping is stable in dual of each spaces. The proof uses Gâteaux differentiability space (GDS), weak-star exposed points, norm-attaining operator, and some studies about epsilon-isometry that have been done before.

Key words and phrases: Stability; Epsilon-isometry; Almost surjectivity; Dual of real Banach space.

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1. INTRODUCTION

Let X and Y are real Banach spaces. A mapping $f : X \to Y$ is said ε-isometry if there exist $\varepsilon > 0$ such that

$$|\|f(x) - f(y)\|_Y - \|x - y\|_X| \leq \varepsilon, \textit{ for every } x, y \in X.$$

We call a 0-isometry as isometry. If the vector space where the norm works is clear and does not lead to an ambiguous meaning, then it is simply written by $\|\cdot\|$ without writing the vector space.

Let a mapping ε-isometry $f : X \to Y$ is given with $\varepsilon > 0$, and $\rho > 0$ is a constant that depends on X and Y, is there exist a surjective isometry mapping $U : X \to Y$ such that

$$\|f(x) - U(x)\| \leq \rho\varepsilon, \tag{1.1}$$

for all $x \in X$?

The question above firstly appeared in the paper of D. H. Hyers and S. M. Ulam, and known as *Hyers-Ulam problem*. An ε-isometry mapping f is stable if there is an isometry mapping U and a constant $\rho < \infty$ that satisfies (1.1), see [5].

In this paper, we propose two questions:

(1) How is the development of *Hyers-Ulam problem* ?
(2) How is the stability of an almost surjective ε-isometry in the dual of real Banach spaces?

The paper is organized as follows. In the second section, we will describe briefly the development of *Hyers-Ulam problem* by stating theorems without writing the proofs, include Vestfrid's theorem (2015) and Lixin Cheng's lemma (2013). In the third section we will modify the Vestfrid's theorem by using Lixin Cheng's lemma. In addition, the third section also contains the results of several studies that have been done before, and they will be used to prove the main theorem of this paper.

2. THE DEVELOPMENT OF HYERS-ULAM PROBLEM

The studies of ε-isometry mapping (*Hyers-Ulam problem*) are divided into two cases, namely:

(1) f is a surjective mapping, and
(2) f is non-surjective mapping (almost surjective).

Based on the division above, this section is divided into two parts. The first part presents the related studies in surjective ε-isometry, and the second part provides the studies about ε-isometry mapping that is almost surjective.

The studies of ε-isometry mapping come from Mazur-Ulam study which stating that the surjective isometry mapping is affine, see [1]. In other word, a surjective isometry mapping in the normed space can be translated, such as for ε with $\varepsilon > 0$. The Mazur-Ulam's study brings up the question: if given any ε-isometry mapping, then is there always an isometry mapping?

In 1945, Hyers-Ulam [9] answered the question above by conducting a study of an ε-isometry mapping in Euclidean spaces. They found that for given any surjective ε-isometry, there is always a surjective isometry mapping that satisfies (1.1) with a constant $\rho = 10$. One year later, D. G. Bourgin [3] studied a surjective ε-isometry in the Lebesgue spaces and found that a surjective ε-isometry is stable with a constant $\rho = 12$. Many years later, Gevirtz [8] studied that a surjective ε-isometry is stable for any Banach spaces and found the constant ρ is 5. This constant was shaped to 2 by Omladič and Šemrl which is valid for any Banach spaces [14]. The Omladič and Šemrl's theorem is as follows.

Theorem 2.1. *Let X and Y are Banach spaces and $f : X \to Y$ is a surjective ε-isometry mapping with $f(0) = 0$, there exist a linear surjective isometry mapping $U : X \to Y$ such that*

$$\|f(x) - U(x)\| \leq 2\varepsilon, \text{ for every } x \in X.$$

The studies that mentioned above give the condition that the ε-isometry mapping should surjective. The surjective condition is weakened by Jussi Väisälä (2002) that the ε-isometry mapping is enough to be almost surjective if the target space of ε-isometry mapping f is uniformly convex, see [13] and [15].

Definition 2.1. Let $f : X \to Y$ is a mapping, Y_1 is a closed subset of Y, and $\delta \geq 0$. A mapping f is said as almost surjective onto Y if for every $y \in Y_1$, there exist $x \in X$ with $\|f(x) - y\| \leq \delta$ and for every $u \in X$, there exist $v \in Y_1$ with $\|f(u) - v\| \leq \delta$ [13].

Theorem 2.2. *Let E and F are Hilbert spaces and $f : E \to F$ is ε-isometry mapping with $f(0) = 0$. If f is almost surjective, i.e satisfies*

$$\sup_{\|y\|=1} \lim_{|t|\to\infty} \inf \left\| ty - \frac{f(E)}{t} \right\| < 1, \tag{2.1}$$

then there exist a linear surjective isometry $U : X \to Y$ such that

$$\|f(x) - U(x)\| \leq 2\varepsilon, \text{ for all } x \in X.$$

In 2015, Vestfrid [17] studied Väisälä's result for any Banach spaces and decrease the value 1 to $\frac{1}{2}$ in (2.1). Vestfrid's theorem is as follows.

Theorem 2.3. *Let X and Y are real Banach spaces and $f : X \to Y$ is an almost surjective ε-isometry mapping with $f(0) = 0$. If mapping f satisfies*

$$\sup_{y\in S_Y} \lim_{|t|\to\infty} \inf \left\| ty - \frac{f(X)}{t} \right\| < \frac{1}{2}, \tag{2.2}$$

then there exist a linear surjective isometry $U : X \to Y$ such that

$$\|f(x) - U(x)\| \leq 2\varepsilon, \text{ for all } x \in X. \tag{2.3}$$

On the other hand, Figiel (1968) showed that there exist a linear operator for an isometry mapping such that the inner product of both is an identity [7]. Figiel's theorem is as follows.

Theorem 2.4. *Let $U : X \to Y$ is an isometry mapping with $f(0) = 0$. There exist a linear operator $\phi : \overline{span}\, U(x) \to X$ with $\|\phi\| = 1$ such that $\phi \circ U = I$, the identity on X.*

The condition of an operator ϕ which is linear in Figiel's theorem above encourage the study about stability of an ε-isometry mapping f which is seen from the dual of the target space of a mapping f. Qian [12] gives the following result.

Lemma 2.5. *If $f : \mathbb{R} \to Y$ is surjective ε-isometry with $f(0) = 0$, there exist $\phi \in Y^*$ with $\|\phi\| = 1$ such that*

$$|\langle \phi, f(t) \rangle - t| \leq 5\varepsilon, \text{ for any } t \in \mathbb{R}.$$

Lixin Cheng et. al. (2013) reduce Qian's result from 5ε to 3ε [4].

Lemma 2.6. *Let Y be Banach space and $f : \mathbb{R} \to Y$ is surjective ε-isometry with $f(0) = 0$, there exist a linear functional $\phi \in Y^*$, $\|\phi\| = 1$, such that*

$$|\langle \phi, f(t) \rangle - t| \leq 3\varepsilon, \text{ for any } t \in \mathbb{R}.$$

3. STABILITY OF ALMOST SURJECTIVE ε-ISOMETRY IN THE DUAL OF REAL BANACH SPACES

This section contains a theorem which become the result of modifiying Theorem 2.3 using Lemma 2.6. The inequality (2.2) in Theorem 2.3 will be modified by Lemma 2.6 which will give the stability of an almost surjective ε-isometry mapping, $f : X \to Y$, in the dual X* and Y*. The theorem to be proved is as follows.

Theorem 3.1. *Let X and Y are real Banach spaces, $f : X \to Y$ is an ε-isometry with $f(0) = 0$, and $x_o \in S_x$. If the mapping f satisfies almost surjective condition, i.e*

$$\sup_{y \in S_Y} \lim_{|t| \to \infty} \inf \left\| ty - \frac{f(X)}{t} \right\| < \frac{1}{2}, \tag{3.1}$$

then for every $x^* \in X$, there exist a linear functional $\phi \in Y$* with $\|\phi\| = \|x^*\| = r$ such that*

$$|\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \le 4\varepsilon r, \tag{3.2}$$

for all $x \in X$.

Since the proof of Theorem 3.1 requires some results of previous studies, the proof of Theorem 3.1 will be granted at the end of this chapter.

Definition 3.1. Let X be a normed space and X* is the dual of X. Weak topology on X, denoted by w, is the weakest topology of X which make every element of X*, $x \mapsto x^*(x)$, continuous for every $x^* \in X$*. Weak star topology on X*, denoted by w^*, is the weakest topology of X* which make every element of

$$\begin{aligned} \iota(x) :& X \to X^{**} \\ & x^* \mapsto x^*(x) \end{aligned}$$

continuous for every $x \in X$.

Definition 3.2. Suppose $\|\cdot\|$ is a real function which is defined over X, $x_o \in S_X$, $x \in X$ and $t \in \mathbb{R}$. Left and right Gâteaux (directional) derivative for $\|\cdot\|$ at x_o in the direction x is defined as

$$G_-(x_o, x) = \lim_{t \to 0^-} \frac{\|x_o + tx\| - \|x_o\|}{t}$$

and

$$G_+(x_o, x) = \lim_{t \to 0^+} \frac{\|x_o + tx\| - \|x_o\|}{t}.$$

Banach space X is said Gâteaux differentiability space (GDS) if every convex functional on X is Gâteaux differentiable, $G_-(x_o, x) = G_+(x_o, x)$.

Corollary 3.2 (Fabian et. al. [6], Corollary 7.23)**.** *Let X be Banach space. If the dual norm of X* is strictly convex, then the norm of X is Gâteaux differentiable.*

Definition 3.3. A point x^* in a w^*-closed convex set $C \subset X$* is said to be w^*-exposed of C provided there exists a point $x \in X$ such that

$$\langle x^*, x \rangle > \langle y^*, x \rangle$$

for all $y^* \in C$ with $x^* \neq y^*$. The point x is called a w^*-exposing functional of C and exposing C at x^*. The set of all x^* which w^*-exposed of C is denoted by $w^* - expC$.

Definition 3.4. Let X and Y are Banach spaces and $T \in B(X,Y)$. For a bounded set C in X, we put

$$\|T\|_C = \sup\{\|T(x)\| : x \in C\}.$$

If there is $c \in C$ such that

$$\|T(c)\|_{\Re(T)} = \|T\|_C\,,$$

then T is attain its supremum over C.

Proposition 3.3 (Cheng et. al. [4], Proposition 2.2)**.** *Suppose that X is Banach space and that $C \subset X$* is a non-empty w^*-compact convex set. Then $x^* \in C$ is a w^*-expC and w^*-exposed by $x \in X$ if and only if $\sigma_C = \sup_C$ is Gâteaux differentiable at x and with Gâteaux derivative $d\sigma_C(x) = x^*$.*

Theorem 3.4 (Robert R. Phelps [11], Theorem 6.2)**.** *A Banach space X is Gâteaux differentiability space (GDS) if and only if every w^*-compact convex non-empty subset of X^* is w^*-closed convex hull of its w^*-exposed points.*

Lemma 3.5 (Bishop and R. Phelps' Lemma [2])**.** *Suppose X is normed space and $\varepsilon > 0$. If $f, g \in X$*, $\|f\| = 1 = \|g\|$ such that $|g(x)| \leq \varepsilon/2$ for $f(x) = 0$ and $\|x\| \leq 1$, then $\|f - g\| \leq \varepsilon$ or $\|f + g\| \leq \varepsilon$.*

Lemma 3.6 (Hausdorf's maximality principle)**.** *Every chain in $\mathcal{M}$ is contained in some maximal chain in $\mathcal{M}$.*

Theorem 3.7 (James Munkres [10], Theorem 26.9)**.** *Let Y be a normed space. Y is compact if and only if for every collection Φ of closed sets in Y has finite intersection property.*

Proof of Theorem 3.1. Proof will be divided into two parts. The first part is a finite-dimensional real Banach space X (i.e $X \cong X$*) and will shows the existence of $x^* \in X$* and $\phi \in Y$* that satisfies Theorem 3.1. The second part shows that Theorem 3.1 is valid for any real Banach spaces. In the proof, we use the ball theorem, so the analogy $\|\phi\| = \|x^*\| = r = 1$ does not eliminate the generalization.

Considering the definition of strictly convex space and functional $f : X \to Y$ is an ε-isometry, a simple proof will show that the dual space X* that satisfies inequality (3.1) is strictly convex. This result, by Corollary 3.2 and Definition 3.2, shows that the Banach space X that satisfies inequality (3.1) is GDS. The GDS condition of X ensures the existence of w^*-compact convex of its dual become w^*-closed convex hull of w^*-exposed points (by Theorem 3.4). Consequently, by Proposition 3.3, there exist $x_o \in S_x$ that Gâteaux differentiable at x_o such that

$$d\,\|x_o\| = x^*, \tag{3.3}$$

where $\sigma_C(x_o) = \|x_o\|$.

Since f is an almost surjective mapping, for a $y \in S_Y$, there exist an $x \in B_x$ such that $y = f(x)$. Hence, inequality (3.1) can be written in the form

$$\left\| f(x) - \frac{f(tx_o)}{t} \right\|. \tag{3.4}$$

By applying Lemma 2.6 to (3.4), we have

$$\left| \left\langle \phi, \frac{f(tx_o)}{t} \right\rangle - \langle \phi, f(x) \rangle - t \right| \leq 3\varepsilon. \tag{3.5}$$

Thus, for $t > 0$ we get

$$\begin{aligned} t - 3\varepsilon - \langle \phi, f(x) \rangle &\leq \left\langle \phi, \frac{f(tx_o)}{t} \right\rangle - \langle \phi, f(x) \rangle \\ &= \frac{1}{t} \langle \phi, f(tx_o) \rangle - \langle \phi, f(x) \rangle \\ &\leq \langle \phi, f(tx_o) \rangle - \langle \phi, f(x) \rangle \\ &\leq \|\phi\| \, \|f(tx_o)\| - \|\phi\| \, \|f(x)\| \\ &= \|f(tx_o)\| - \|f(x)\| \\ &\leq \|f(tx_o) - f(x)\| \leq \|tx_o - x\| + \varepsilon, \end{aligned}$$

or

$$t - \|tx_o - x\| - \langle \phi, f(x) \rangle \leq 4\varepsilon. \tag{3.6}$$

Since X is GDS and recalling that $x_o \in S_x$ or $\|x_o\| = 1$, from Definition 3.2 we get

$$d\, \|x_o\| = \lim_{t \to \infty} (t - \|tx_o - x\|) = x^*(x) = \langle x^*, x \rangle. \tag{3.7}$$

Substitute (3.7) into (3.6), for $t \to \infty$ we have

$$\langle x^*, x \rangle - \langle \phi, f(x) \rangle \leq 4\varepsilon. \tag{3.8}$$

For $t < 0$, (3.5) becomes

$$\begin{aligned} t - 3\varepsilon - \langle \phi, f(x) \rangle &\leq \left\langle \phi, \frac{f(-tx_o)}{-t} \right\rangle - \langle \phi, f(x) \rangle \\ &= \frac{1}{-t} \langle \phi, f(-tx_o) \rangle - \langle \phi, f(x) \rangle \\ &< \langle \phi, f(-tx_o) \rangle - \langle \phi, f(x) \rangle \\ &\leq \|\phi\| \, \|f(-tx_o)\| - \|\phi\| \, \|f(x)\| \\ &= \|f(-tx_o)\| - \|f(x)\| \\ &\leq \|f(-tx_o) - f(x)\| \\ &\leq \|-tx_o - x\| + \varepsilon, \end{aligned}$$

or

$$\begin{aligned} -4\varepsilon &\leq \|-tx_o - x\| - t + \langle \phi, f(x) \rangle \\ &= |-1| \, \|tx_o + x\| - t + \langle \phi, f(x) \rangle \\ &= \langle \phi, f(x) \rangle - (t - \|tx_o + x\|)\,. \end{aligned} \tag{3.9}$$

By applying (3.7) into (3.9) as before, for $t \to \infty$ we have

$$-4\varepsilon \leq \langle \phi, f(x) \rangle - \langle x^*, x \rangle. \tag{3.10}$$

From (3.8) and (3.10), we conclude

$$|\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r,$$

which is nothing but inequality (3.2).

We show that for every $x^* \in X*$, there exist $\phi \in Y*$ that satisfies inequality (3.2). Let $x^* \in S_{X^*}$. Recalling that X is GDS, by Theorem 3.4, there is a set w^*-compact convex being

w^*-closed convex hull of $w^* - expB_{X^*}$ points, or $co(w^* - expB_{X^*})$ exist. Since $w^* - expB_{X^*}$ is a set of all limit points of B_{X^*} and $co(w^* - expB_{X^*})$ is nothing but the cover of $w^* - expB_{X^*}$, so $\overline{co}(w^* - expB_{X^*}) = B_{X^*}$, i. e $co(w^* - expB_{X^*})$ is dense in B_{X^*} by definition. The dense property of $co(w^* - expB_{X^*})$ ensure the existence of limit points of $co(w^* - expB_{X^*})$. As a result, there is a sequence $(x_n^*) \subset co(w^* - expB_{X^*})$ which converges toward x^* in B_{X^*}. Since $co(w^* - expB_{X^*})$ is w^*-compact convex, there are m points of $w^* - expB_{X^*}(x_{n_1}^*, x_{n_2}^*, \dots, x_{n_m}^*)$ and m non-negative numbers $(\lambda_{n_1}, \lambda_{n_2}, \dots, \lambda_{n_m})$ with

$$\sum_{j=1}^{m} \lambda_{n_j} = 1$$

for all $m \in \mathbb{N}$ such that

$$x_n^* = \sum_{j=1}^{m} \lambda_{n_j} x_{n_j}^*.$$

Since

$$\left| \langle \phi_{n_j}, f(x) \rangle - \langle x_{n_j}^*, x \rangle \right| \leq 4\varepsilon r,$$

then Y* is w^*-compact convex also. Hence, $\phi_{n_j} \in S_{Y^*}$ exist for all $x \in X$, with $1 \leq j \leq m$. Let $\varphi_n \in S_{Y^*}$. ϕ_n can be expressed in the form of linear combination of ϕ_{n_j},

$$\varphi_n = \sum_{j=1}^{m} \lambda_{n_j} \phi_{n_j}.$$

Consequently,

$$\begin{aligned} \|\varphi_n\| &= \left\| \sum_{j=1}^{m} \lambda_{n_j} \phi_{n_j} \right\| \\ &\leq \sum_{j=1}^{m} \left\| \lambda_{n_j} \phi_{n_j} \right\|. \end{aligned}$$

Since $\sum_{j=1}^{m} \lambda_{n_j} = 1$ and $\phi_{n_j} \in S_{Y^*}$, we get $\|\varphi_n\| \leq 1$. Thus,

$$|\langle \varphi_n, f(x) \rangle - \langle x_n^*, x \rangle| \leq 4\varepsilon r$$

is hold for every $x \in X$. We know that $\varphi_n \in S_{Y^*}$ and φ_n is w^*-compact, there is a subnet from a sequence (φ_n) which w^*-converges to some $\phi \in B_{Y^*}$. Since $x_n^* \to x^*$ and $\varphi_n \to \phi$, the inequality

$$|\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r$$

is true for all $x \in X$. Besides,

$$\|\phi\| = \sup_{y \in S_Y} \|\phi(y)\| = 1$$

and

$$\|x^*\| = \sup_{x \in S_X} \|x^*(x)\| = 1.$$

Therefore, Theorem 3.1 is true for dim$X < \infty$. Now, we will prove that Theorem 3.1 is true for arbitrary real Banach space X. We know that real Banach space is a vector space X together with the norm which is defined on it, then the expansion of Theorem 3.1 can be done by broadening the bases of X which is finite-dimensional to infinite-dimensional space. This expansion can be performed by using *Hausdorf's maximality principle* (Lemma 3.6).

Assuming that there is a collection of linearly independent set of vector in X that satisfies Theorem 3.1. From *Hausdorf's maximality principle*, there is a maximum chain $\mathcal{M}$ of nested set of linearly independent A_α. We will show that

$$E := \cup_\alpha A_\alpha$$

is linearly independent and span X.

If $\sum_{i=1}^{\infty} \lambda_i a_i = 0$ for $a_i \in X$, then every vector a_i, where $i = 1, 2, \ldots$, is element of some A_{α_i}. Since A_α is nested, then every vector a_i is in some A_α. Hence,

$$\sum_{i=1}^{\infty} \lambda_i a_i = \bigcup_\alpha A_\alpha = E.$$

From the definition of E above, where A_α is nested linearly independent set, then $\lambda_i = 0$ for $i = 1, 2, \ldots$, which show that E is linearly independent.

Now, suppose that E does not span X, then there is a vector $v \notin span\{\cup_\alpha A_\alpha\}$, so $E \cup v$ is linearly independent. Since $E \cup v$ contains E and every A_α, lead to contradiction with the assumption that E is a maximal chain $\mathcal{M}$. Thus, E must span X. Since $E := \cup_\alpha A_\alpha$ is maximal chain $\mathcal{M}$ which is linearly independent and spans X, then Theorem 3.1 is true for arbitrary real Banach space X.

By using Lemma 3.5, we will show the existence of $x^* \in X$* and $\phi \in Y$.*. The explanation is as follows. Lemma 3.5 (Bishop and Phelps' lemma) states that norm-attaining functionals are always dense in the dual X^* of X. Assume $x^* \in X^*$ is a functional of X which satisfies Definition 3.3 (norm-attaining functional) with $\|x^*\| = 1$, so there is a $\|\phi\| = 1$ which satisfies Theorem 3.1. For every $x^* \in X$* with $\|x^*\| = 1$, Lemma 3.5 leads to the existence of norm-attaining functional sequence $(x_n^*) \subset X$* with $\|x_n^*\| = 1$ such that $x_n^* \to x^*$ for every $n \in N$. Let $\phi_n \in Y$* with $\|\phi_n\| = 1$ is a functional that is related to x_n^* such that

$$|\langle \phi_n, f(x) \rangle - \langle x_n^*, x \rangle| \leq 4\varepsilon r$$

for every $x \in X$. w^*-compact property of (ϕ_n) lead to the existence of a $\phi \in Y$* which being w^*-limit point of (ϕ_n). Therefore, Theorem 3.1 is true for functionals x^* and ϕ.

Given norm-attaining functional $x^* \in X$*. Let $x_o \in S_X$ such that $\langle x^*, x_o \rangle = 1$ and $\mathcal{F}$ is a collection of all subspaces of X which contain x_o. Since each $F \in \mathcal{F}$ is GDS, by using Theorem 3.1 which has been proved for any real Banach space, there is a $\phi_F \in S_{Y^*}$ such that

$$|\langle \phi_F, f(x) \rangle - \langle x_n^*, x \rangle| \leq 4\varepsilon r \tag{3.11}$$

for all $x \in X$. Let

$$\Phi_F = \{\phi_F \in Y^* \text{ satisfies (3.11) with } \|\phi_F\| \leq 1\},$$

and

$$\Phi = \{\Phi_F : F \in \mathcal{F}\}.$$

From (3.11), obviously for each $F \in \mathcal{F}$, Φ_F is w^*-compact convex set of Y*. Let $F, G \in \mathcal{F}$ and $H = span\{F, G\}$. It is clear that $\Phi_H \subset \Phi_F \cap \Phi_G$, so $\bigcap_{F \in \mathcal{F}} \Phi_F$ has finite intersection property. Furthermore, by Theorem 3.7, finite intersection property of subcollection of Φ show that the collection Φ is compact. Since every Φ_F is w^*-compact convex set and $\Phi_F \cap \Phi_G$ is non-empty, every $\Phi_F \cap \Phi_G$ is functional of Y* which satisfies (3.2). ■

4. CONCLUSIONS

The study about ε-isometry mapping (*Hyers-Ulam problem*) can be divided into two kinds, that is surjective ε-isometry mapping and almost surjective ε-isometry mapping. Both ε-isometries are stable, that there always exists isometry mapping where the norm of the difference of ε-isometry and isometry is bounded.

In addition, this paper also indicates that any ε-isometry mapping remained stable viewed from each dual space of domain and codomain.

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