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CREDIBILITY BASED FUZZY ENTROPY MEASURE

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ABSTRACT. Fuzzy entropy is the entropy of a fuzzy variable, loosely representing the information of uncertainty. This paper, first examines both previous membership and credibility based entropy measures in fuzzy environment, and then suggests an extended credibility based measure which satisfies mostly in Du Luca and Termini axioms. Furthermore, using credibility and the proposed measure, the relative entropy is defined to measure uncertainty between fuzzy numbers. Finally we provide some properties of this Credibility based fuzzy entropy measure and to clarify, give some examples.

Key words and phrases: Fuzzy variable; Entropy measure; Credibility; Relative entropy.

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1. INTRODUCTION

Theory of fuzzy sets, proposed by Zadeh [16] is to measure the degree of uncertainty due to the vagueness and complexity, where entropy, proposed by Shannon in 1948, measures the degree of uncertainty which results from randomness. Thus, to measure the degree of uncertainty caused by personal judgment, Du Luca and Termini [2] defined the synthesis of fuzzy entropy. He replaced the membership degrees of elements with the variables in the Shannon function. Later, he and his succeed researchers such as Kosko [3], Al-Sharhan et al. [1], Zhang et al. [17] debated the probabilistic and not-probabilistic fuzzy entropies. Lee et al. [6] introduced some modifications to the original fuzzy entropy. The fuzzy entropy is employed in various sciences such as finance, management, computer sciences and biology. For example, Li et al. [9] measured the degree of ambiguity in image by the entropy of the fuzzy set and applied it in breast cancer detection. Meng and Chen [12] developed an approach for fuzzy multi-criteria decision making using entropy with incomplete weight information. Ning, et al. [13] employed it in portfolio management where the fuzzy valued shares were optimized by entropy in a genetic algorithm. Later, Yari et al. [15] developed another approach of multi criteria methods in Valuation of American Options. They used Black Scholes differential equation and EM algorithm to estimate parameters of probability distributions. In information theory, the Kullback Leibler divergence [4, 5] which is called the relative entropy is defined as a measure of difference between two probability distributions. Afterwards, by the mixture of fuzzy and entropy, fuzzy relative entropy is extended and employed is many sciences. In 2002, Liu and Liu [10] presented a new formula for expected values of fuzzy variables using the credibility functions. Afterwards, many concepts and properties of fuzzy credibility functions were defined. Li and Liu [8] denoted a sufficient and necessary condition for credibility measures. Recently, an entropy measure for discrete and continuous Fuzzy variables is defined by Li and Liu [7] based on the credibility distributions. The paper is organized as follows. After the introduction in Section 1, some concepts and knowledge about fuzzy sets, fuzzy entropy, relative entropy and credibility are introduced in Section 2. In the following Section, after a brief discussion of Li and Liu [7]'s entropy and Du Luca and Termini's axioms, novel entropy measure and its corresponding properties and examples are given. Relative entropy between fuzzy numbers based on credibility function is also introduced in this section. Finally, we conclude in Section **??**.

2. PRELIMINARIES

We first remind fuzzy sets and then we can have the definitions of credibility measure and entropy. A fuzzy set over X is defined as below:

$$\tilde{A} = \{x, \mu_{\tilde{A}}(X), 1 - \mu_{\tilde{A}}(X) | x \in X\} \qquad \mu_{\tilde{A}}(X) : X \longrightarrow [0.1] \tag{2.1}$$

where $\mu_{\tilde{A}}(X)$ defines the degree of membership of x for every $x \in X$

For credibility measure, let ξ and r be a fuzzy value and a real number, respectively. Then the credibility of $\{\xi \leq r\}$ by Liu and Liu [10] is defined as:

$$Cr\{\xi \in \beta\} = \frac{1}{2}(Pos\{\xi \in \beta\} + Nec\{\xi \in \beta\}) = \frac{1}{2}(Sup_{x\in\beta}\mu(x) + 1 - Sup_{x\in\beta^c}\mu(x)) \tag{2.2}$$

or generally (Mandal et.al [11])

$$\xi \in \beta\} = \rho Pos\{\xi \in \beta\} + (1-\rho)Nec\{\xi \in \beta\}, \qquad 0 \leq \rho \leq 1 \tag{2.3}$$

De Luca and Termini [2] first axiomitized the fuzzy entropy measure as follows:

FS1: H($\widetilde{A}$)=0 iff $\widetilde{A}$ is a crisp set,i.e., $\mu_{\widetilde{A}}(x_i)$=0 or 1 $\forall x_i \in X$
FS2: H($\widetilde{A}$)= the maximum value iff $\mu_{\widetilde{A}}(x_i)$=0.5 $\forall x_i \in X$
FS3: H($\widetilde{A}) \leq H(\widetilde{B})$ if $\widetilde{A}$ is less fuzzy than $\widetilde{B}$ $\forall x_i \in X$
FS4: H($\widetilde{A}) = H(\overline{\widetilde{A}})$, where $\overline{\widetilde{A}}$ is the component of $\widetilde{A}$.
where H($\widetilde{A}$) is the entropy measure for fuzzy variable, ξ.
Later, the entropy axioms for fuzzy variable was defined by Li and Liu [8] using the credibility distributions.

$$H(\xi) = \int_{-\infty}^{+\infty} S(Cr\{\xi = x\})dx, \tag{2.4}$$

for continuous and similar for discrete using sigma
where

$$S(t) = -tLnt - (1-t)Ln(1-t), \qquad 0 \leq t \leq 1 \tag{2.5}$$

To measure the entropy of a fuzzy number related to another, Ohya,and Naritsuka [14] defined the discrete fuzzy relative entropy which was an extension of fuzzy entropy for two probability measures.

$$S_{fuz}(\xi|\eta) \equiv \begin{cases} k\sum_{i=1}^{n} f_i^v \;\; log\frac{f_i}{g_i} & if \;\; f \ll g \\ \infty & otherwise \end{cases} \tag{2.6}$$

where $v = log_2 e, k > 0,\ f_i \ and \ g_i, i \in \{1, 2, 3, ..., n\}$ are the membership functions over $\pi = \{w_1, w_2, ..., w_n,\}$, and f $\ll$ g denotes that g(w)=0 gives f(w)=0, we take 0log0=0.

3. ENTROPY AND RELATIVE ENTROPY MEASURES

3.1. Entropy for fuzzy variables.

Entropy is a measure of disorder in a system which is measured and axiomatized for fuzzy variables by De Luca and Termini [2]. But the entropy measure in Li and Liu [8] which is based on credibility, doesn't completely match the mentioned axioms, because the second measure is not directly related to the fuzziness of a value, but directly to the credibility.

Example 3.1. *Let ξ and η be two simple discrete fuzzy variables taking values in $\{x_1, x_2, x_3\}$ with possibilities $\{1, 0.5, 0.75\}$ and {1,0.5,0.25} respectively. Obviously, the entropy measures are equal by FS4, and this means x_3 doesn't differ with 1-x_3 because of its fuzziness. But, using credibility, the entropy values for ξ and η are respectively 1.25 and 1.125 when $\rho = 0.5$. That is; the lower membership of x_3 reduced the entropy value in credibility functions . On the other hand, a problem in the membership based entropy is measuring the fuzzy numbers separately. That is; in this system, the discrete points wouldn't be considered due to others and consequently we are facing to some uncorrelated variable instead of a system.*

To solve these two problems, we define the following credibility based entropy correlated measure.

$$H_c(\xi) = \int_{-\infty}^{+\infty} S(ECr\{\xi = x\})dx, for \;\; x \subset \beta \tag{3.1}$$

where β is the smallest interval containing the positive possibilities according to the following lemma.

(3.2) $$ECr\{\xi \in \beta\} = \rho Sup_{x\in\beta}\{1-\mu(x)\vee\gamma(x)\} + (1-\rho)(1-Sup_{x\in\beta^c}\{\mu(x)\vee\gamma(x)\})$$

where $\mu(x)\vee\gamma(x) = maxCount(\mu(x),\gamma(x)) = max(\mu(x),\gamma(x))$
where ECr is the credibility of not-disordering in a fuzzy variable and integral reduces to sigma for continuous distributions.

It is obvious that on the contrary of entropy measure in Li and Liu,$H_c(\xi)$ satisfies FS2 to FS4 in Du Luca and Termini. That is; by $H_c(\xi)$, the definitions in maximum entropy, fuzziness value and the component of fuzzy variables got compatible with Du Luca and Termini. But yet, similar to Li and Liu, the measure does not match FS1 and of course this is the positive matter of Li and Liu's credibility based entropy measure, because the correlation between the elements (in discrete numbers) or intervals (in continuous numbers) is not ignored.

Example 3.2. *Let ξ be a triangular fuzzy number, then, entropy measure with the membership function, μ, is given as:*

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} & a \le x < b \\ \frac{c-x}{c-b} & b \le x \le c \\ 0 & otherwise \end{cases}$$

$$ECr(\xi = x) = \begin{cases} \rho\frac{x-a}{b-a} & a \le x < \frac{a+b}{2} \\ \rho\frac{b-x}{b-a} & \frac{a+b}{2} \le x < b \\ \rho\frac{x-b}{c-b} & b \le x < \frac{b+c}{2} \\ \rho\frac{c-x}{c-b} & \frac{b+c}{2} \le x < c \\ 0 & otherwise \end{cases}$$

$$\begin{aligned} H_c(\xi) &= H_{c\{x\in[a,\frac{a+b}{2}]\}}(\xi) + H_{c\{x\in[\frac{a+b}{2},b]\}}(\xi) + H_{c\{x\in[b,\frac{b+c}{2}]\}}(\xi) + H_{c\{x\in[\frac{b+c}{2},c]\}}(\xi) \\ &= \frac{9Ln3 - 8Ln4 + 4}{16}(c-a), \quad for \rho = \frac{1}{2} \end{aligned}$$

More generally:

$$H_c(\xi \le x) = \begin{cases} 0 & x < a \\ \frac{9Ln3-8Ln4+8}{16}(x-a) & a \le x < c \\ \frac{9Ln3-8Ln4+8}{16}(c-a) & x \ge c \end{cases}$$

Properties:

1) Upper and lower bounds: For a continuous fuzzy variable, ξ, with positive possibility in interval [a, b], we have $0 \le H_c(\xi) \le (b-a)Ln2$ when discrete environment with n membership degrees, we have $0 \le H_c(\xi) \le nLn2$

Because $-ECrLn(ECr) - (1-ECr)Ln(1-ECr)$ reaches its maximum value in Ln2, for$ECr = \frac{1}{2}$.

2) Rearrangement: Let ξ and η be two continuous fuzzy variables with membership functions $\mu(= \bigcup_{i=1}^{n} u_i)$ and $\lambda(= \bigcup_{i=1}^{n} v_i)$, respectively, where $\{u_1, u_2, ..., u_n\}$ is a rearrangement of the sequence $\{u_1, u_2, ..., u_n\}$, and u_i, $v_i = 1, 2, ..., n$ nare disjoint intervals. Then, we have $H_c(\xi) = H_c(\eta)$

This Property works for discrete variables when u_is and v_is are constant as the membership values. Proof: This lemma can be easily demonstrated from the definition of $H_c(\xi)$.

3) Non-Monotonicity: Let ξ and η be two continuous fuzzy variables with membership disjoint functions μ and η, respectively. Then, we have $H_c(\xi) > H_c(\xi \oplus \eta)$ when η reduces the fuzziness of ξ, and we have $H_c(\xi) < H_c(\xi \oplus \eta)$ when η grows the fuzziness of ξ.
where $\oplus$ is as operation between two fuzzy numbers and defined as: $\xi \oplus \eta = \{\langle x, \mu_\xi(x) + \mu_\eta(x) - \mu_\xi(x)\mu_\eta(x), 1 - (\mu_\xi(x) + \mu_\eta(x) - \mu_\xi(x)\mu_\eta(x))\rangle | x \in X\}$.

This Property also works for discrete variables.

As a simple example, let ξ and η be two discrete fuzzy variables with the similar membership degrees $\{1, \frac{1}{2}\}$. It is obvious that $\xi \oplus \eta$ (with the membership degrees $\{1, \frac{3}{4}\}$ reduces the fuzziness and consequently reduces the entropy. Therefore: $H_c(\xi) > H_c(\xi \oplus \eta)$

We have$H_c(\xi) = H_c(\eta) = 1.125$ where $H_c(\xi \oplus \eta) = 0.754$

But if ξ and η be two discrete fuzzy variables with the similar membership degrees $\{1, \frac{1}{4}\}$, we the membership degrees $\{1, \frac{7}{16}\}$ for $\xi \oplus \eta$ and $H_c(\xi) < H_c(\xi \oplus \eta)$, where $H_c(\xi) = H_c(\eta) = 0.754$ where $H_c(\xi \oplus \eta) = 1.37$.

3.2. **Relative entropy for fuzzy variabless.**

Definition 3.1. Let ξ and η be two discrete fuzzy numbers taking values in $\{x_1, x_2, ..., x_n\}$ with membership degrees $\{\mu_1, \mu_2, ..., \mu_n\}$ and $\{\lambda_1, \lambda_2, ..., \lambda_n\}$, respectively. The relative entropy between these two numbers is defined as:

$$ReECr(\xi\eta) \equiv \begin{cases} k\sum_{i=1}^{n} ECr^v(\xi = x_i)log\frac{ECr(\xi=x_i)}{ECr(\xi=x_i)}) & if \mu_i \ll \lambda_i \\ \infty & otherwise \end{cases} \tag{3.3}$$

where $v = log_2 e$ and k>0. We have

$$ReECr(\xi\eta) \equiv \begin{cases} k\int_{-\infty}^{+\infty} ECr^v(\xi = x)log\frac{ECr(\xi=x)}{ECr(\xi=x)}) & if \mu \ll \lambda \\ \infty & otherwise \end{cases} \tag{3.4}$$

For continuous fuzzy variables, ξ and η with membership functions, ξ and λ, taking values in R, and β is the smallest interval containing the simultaneously positive possibilities in ξ and η, and is employed by the following theorem. Thus, for triangular fuzzy numbers, if there is any $x \in U$ (universe of discourse), such that $b_1 \neq b_2$ or $\mu(x) > 0$ (while $\lambda(x) = 0$), c.

Theorem 3.1. *Let ξ be a continuous fuzzy variable that takes value in R. If there exist an interval, B such that $Cr\{\xi \in B\} = 1$, then for every interval, $\alpha(s.t.\alpha \cap \beta) = \phi$, we have $Cr\{\xi \in \alpha\} = 0$.*

Proof: The proof is similar to lemma 1 in Li and Liu [7] *for discrete fuzzy variables.*

Example 3.3. *Let ξ and η be two triangular fuzzy numbers with the membership functions, μ and λ, respectively, then, the relative entropy measure, $ReECr(\xi\eta)$, for k, Î¡=1 is given as:*

$$\mu(x) = \begin{cases} \frac{x-a_1}{b_1-a_1} & a_1 \leq x < b_1 \\ \frac{c_1-x}{c_1-b_1} & b_1 \leq x \leq c_1 \\ 0 & otherwise \end{cases}$$

$$\lambda(x) = \begin{cases} \frac{x-a_2}{b_2-a_2} & a_2 \leq x < b_2 \\ \frac{c_2-x}{c_2-b_2} & b_2 \leq x \leq c_2 \\ 0 & otherwise \end{cases}$$

$$\begin{aligned} ReECr(\xi\eta) &= \int_{a_2}^{c_2} Cr^v(\eta = x) log \frac{Cr(\eta = x)}{Cr(\eta = x)})dx \\ &= \frac{4\alpha^2 Ln(\tau+\alpha) - \alpha^2 Ln(\alpha) - \tau^2 Ln(\tau) - \alpha\tau}{4\tau} \\ &\quad + \frac{4\beta^2 Ln(\theta+\beta) - \beta^2 Ln(\beta) - \theta^2 Ln(\theta) - \beta\theta}{4\tau} \end{aligned}$$

where $\alpha = a_2 - a_1, \quad \beta = c_2 - c_1, \quad \tau = b - a_2 \quad and \;\; \theta = c_2 - b.$

4. CONCLUSION

Du Luca and Termini axioms for fuzzy entropy measures were first recalled in this paper. We then proposed credibility based fuzzy entropy measure which satisfied mostly with Du Luca and Termini. Furthermore, to illustrate the proposed measure, the properties and some examples were provided. Afterwards, based on this credibility measure, the relative entropy was defined as a measure between two fuzzy numbers. Therefore, hereinafter these measures can be used in the scientific works as the entropy measure in fuzzy environment.

REFERENCES

[1] S. AL-SHARHAN, S. KARRAY, F. GUEAIEB and O. BASIR, Fuzzy entropy: a brief survey. *P. 10th. IEEE. Int. Conf. Fuzzy. Syst.*, (2001), pp. 1135–1139.

[2] D. LUCA and S. TERMINI, A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, *Inform. Control.*, **20** (1972), pp. 301–312.

[3] A. KAUFFMAN, *Introduction to the Theory of Fuzzy Subsets*, Academic, New York, (1975).

[4] S. KULLBACK, Letter to the Editor: The KullbackâĂŞLeibler distance, *Am. Stat.*, **41** (1987), no. 4, pp. 340–341.

[5] S. KULLBACK and R.A. LEIBLER, On information and sufficiency, *Ann. Math. Stat.*, **22** (1951), no. 1, pp. 79–86.

[6] H. LEE, C. CHEN, J. CHEN and Y. JOU, An efficient fuzzy classifier with feature selection based on fuzzy entropy, *IEEE. T. Syst. Man. And. Cy. B.*, **31**(2001), pp. 426–432.

[7] P. LI and B. LIU, Entropy of Credibility Distributions for Fuzzy Variables, *IEEE. T. Fuzzy. Syst.*, **16**(2008), no. 1, pp. 123–129.

[8] X. LI and B. LIU, A sufficient and necessary condition for credibility measures, *Int. J. Unc. Fuzz. Knowl. Based. Syst.*, **14**(2006), pp. 527–535.

[9] X. LI, Z. ZHAO and H.D. CHENG, Fuzzy entropy threshold approach to breast cancer detection, *Inform. Sci. Appl.*, **4**(1995), no. 1, pp. 49–56.

[10] B. LIU and Y.K. LIU, Expected value of fuzzy variable and fuzzy expected value models, *IEEE. T. Fuzzy. Syst.*, **10**(2002), no. 4, pp. 445–450.

[11] S. MANDAL, K. MAITYA, S. MONDAL and M. MAITI, Optimal production inventory policy for defective items with fuzzy time period, *Appl. Math. Model.*, **34**(2010), no. 8, pp. 810–822.

[12] F. MENG and X. CHEN, Interval-valued intuitionistic fuzzy multi-criteria group decision making based on cross entropy and 2-additive measures, *Soft. Comput.*, **19**(2010), pp. 2071–2082.

[13] Y. NING, H. KE and Z. FU, Triangular entropy of uncertain variables with application to portfolio selection, *Soft. Comput.*, **19**(2014), pp. 2203–2209.

[14] M. OHYA and S. NARITSUKA, On fuzzy relative entropy, *Open. Syst. Inform. Dyn.*, **1**(1995), no. 3, pp. 397–407.

[15] G. YARI, M. RAHIMI and P. KUMAR, Multi-Period Multi-Criteria (MPMC) Valuation of American Options Based on Entropy Optimization Principles, *Iran. J. Sci. Technol. A*, (Articles in Press).

[16] L. A. ZADEH, Fuzzy sets, *Inform. Comput.*, **8**(1965), pp. 338–356.

[17] Q. ZHANG, H. XING, F. LIU, J. YE and P. TANG, Some new entropy measures for interval-valued intuitionistic fuzzy sets based on distances and their relationships with similarity and inclusion measures, *Inform. Sciences.*, **283**(2014), pp. 55–69.