The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 12, Issue 1, Article 8, pp. 1-7, 2015

ON EIGENVALUES AND BOUNDARY CURVATURE OF THE C*-ALGEBRA NUMERICAL RANG

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Received 10 November, 2014; accepted 16 April, 2015; published 26 May, 2015.

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ABSTRACT. Let $\mathcal{A}$ be a C*-algebra with unit 1 and $a \in \mathcal{A}$ be a nilpotent. By Donoghue's Theorem, all corner points of its numerical range $V(a)$ belong to the spectrum $\sigma(a)$. It is therefore natural to expect that, more generally, the distance from a point p on the boundary $\partial V(a)$ of $V(a)$ to $\sigma(a)$ should be in some sense bounded by the radius of curvature of $\partial V(a)$ at p.

Key words and phrases: C*-algebra numerical range, Eigenvalues, Curvature.

2000 *Mathematics Subject Classification.* Primary 47A12. Secondary 46K10, 15A42, 14H50.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND PRELIMINARIES

For a bounded linear operator A on a Hilbert space $\mathcal{H}$, the numerical range $W(A)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \rightarrow< Ax, x >$ associated with the operator. More precisely,

$$W(A) = \{< Ax, x >: x \in \mathcal{H}, \|x\| = 1\}.$$

Thus the numerical range of an operator, like the spectrum, is a subset of the complex plane whose geometrical properties should say something about the operator.

One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toeplitz-Hausdorff Theorem. Other important property of $W(A)$ is that its closure contains the spectrum of the operator, $W(A)$ is a connected set with a piecewise analytic boundary $\partial W(A)$ (see [4]). Hence, for all but finitely many points $p \in \partial W(A)$, the radius of curvature $R_p(A)$ of $\partial W(A)$ at p is well defined. By convention, $R_p(A) = 0$ if p is a corner point of $W(A)$, and $R_p(A) = \infty$ if p lies inside a flat portion of $\partial W(A)$.

Let $D_p(A)$ denote the distance from p to $\sigma(A)$, we define $M(A)$ as the smallest constant such that

$$D_p(A) \leq M(A)R_p(A) \tag{1.1}$$

for all $p \in \partial W(A)$ with finite non zero curvature.

By Donoghue's Theorem $D_p(A) = 0$ whenever $R_p(A) = 0$. Therefore, $M(A) = 0$ for all convexoid element A. Recall that a convexoid element is one that its numerical range coincides with the convex hull of its spectrum. For the non-convexoid element A,

$$M(A) = \sup \frac{D_p(A)}{R_p(A)} \tag{1.2}$$

where the supremum in the right-hand side is taken along all points $p \in \partial W(A)$ with finite non-zero curvature.

The computation of $M(A)$ for arbitrary $n \times n$ matrix A is an interesting open problem. For $n > 3$, we have not any exact value of

$$M_n = \sup\{M(A) : A \in \mathbb{C}^{n\times n}\}.$$

Hence, it is natural for Mathias to pose this question that whether there exists a universal constant M such that $M = \sup_n M_n$. In this way, Caston, et al [2] obtained the following approximations for M_n:

$$\frac{n}{2}\sin(\frac{\pi}{n}) \leq M_n \leq \frac{n}{2}. \tag{1.3}$$

Mirman found a sequence of $n \times n$ Toeplitz nilpotent matrices A_n with $M(A_n)$ ingrowing asymptotically as $\log n$ (see [2]). Hence, the answer to Mathia's question is negative. However, the lower bound in (1.3) is still of some interest, at least for small values of n. The question of the exact rate of growth of M_n (is it $\log n$, n or something in between) remains open.

For the study of numerical range of finite matrices, the matrix-theoretic properties can be exploited to yield special tools which are not available for general operators. For an n-by-n matrix A, let

$$p_A(x, y, z) = det(xRe(A) + yIm(A) + zI_n)$$

and let $C(A)$ denote the dual curve of $p_A(x, y, z) = 0$. Since p_A is a real homogeneous polynomial of degree n, the curve $C(A)$ is given by a real polynomial of degree at most $n(n-1)$, is of class n, and has n real foci $[a_j, b_j, 1]$, $j = 1, ..., n$, which correspond exactly to the n eigenvalues $a_j + ib_j$ of A. The connection of $C(A)$ with the numerical range $W(A)$ is provided by a result of Kippenhahn [3]: $W(A)$ is the convex hull of the real points of the curve $C(A)$, namely, $W(A)$ is the convex hull of the set

$$\left\{a + ib \in \mathbb{C} : a, b \in \mathbb{R}, ax + by + z = 0 \text{ is tangent to } p_A(x, y, z) = 0\right\}.$$

Kippenhahn's result can be easily verified by considering that

$$x = \max_{\theta} \left(\sigma(Re(e^{-i\theta}A))\right)$$

is a supporting line of $W(Re(e^{-i\theta}A))$ for any real θ. Since it can be shown that $\partial W(A)$ contains only finitely many line segments, the above result implies that $\partial W(A)$ is piecewise algebraic, that is, it is the union of finitely many algebraic curves. Hence one important way to yield $\partial W(A)$, is the Kippenhahn's result stated that the numerical range of A coincides with the convex hull of the real points of $C(A)$ [3]. On the other hand, a parametric representation of the boundary of $W(A)$ can be also obtained from the largest eigenvalue of $Re(e^{-i\theta}A)$ yielding useful information on $W(A)$.

For any $n \times n$ matrix A, let $\lambda(\theta)$ denote the maximum eigenvalue of $Re(e^{-i\theta}A)$. It is well known that $\lambda(\theta)$ is an analytic function of θ (possibly except for some isolated points), and a unit vector in $\mathbb{C}^n$ is such that $< Ax, x >$ belong to $\partial W(A) \cap L_\theta$ if and only if $Re(e^{-i\theta}A)x = \lambda(\theta)x$ [3]. Also $\partial W(A)$ admits a parametric representation:

$$x(\theta) = \lambda(\theta)cos(\theta) - \lambda^{'}(\theta)sin(\theta);$$

$$y(\theta) = \lambda(\theta)sin(\theta) + \lambda^{'}(\theta)cos(\theta).$$

(again, with possible exception of finitely many points). The curvature and radius of curvature of $\partial W(A)$ at $p = (x(\theta), y(\theta))$ are equals to

$$K(\theta) = \frac{1}{\lambda(\theta) + \lambda^{''}(\theta)}$$

and

$$R(\theta) = \lambda(\theta) + \lambda^{''}(\theta),$$

respectively.

2. MAIN RESULTS

2.1. **C*-algebras type.** Let $\mathcal{A}$ be a C*-algebra with unit 1 and let $\mathcal{S}$ be the state space of $\mathcal{A}$, i.e., $\mathcal{S} = \{\rho \in \mathcal{A}^* : \rho \geq 0, \rho(1) = 1\}$. For each $a \in \mathcal{A}$, the C*-algebra numerical range $V(a)$ and numerical radius $v(a)$ are defined, respectively, by

$$V(a) := \{\rho(a) : \rho \in \mathcal{S}\}, \text{ and } v(a) := \{|z| : z \in V(a)\}.$$

Remark 2.1. Let ρ be a state of $\mathcal{A}$. Then there exists a cyclic representation π_ρ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\rho$ and a unit cyclic vector x_ρ for $\mathcal{H}_\rho$ such that

$$\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho\rangle, \ a \in \mathcal{A}.$$

By Gelfand-Naimark Theorem the direct sum $\pi : a \mapsto \sum_{\rho\in\mathcal{S}} \oplus\pi_\rho(a)$ is a faithful representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H} = \sum_{\rho\in\mathcal{S}} \oplus\mathcal{H}_\rho$ (see [7]). Therefore, for each $\rho \in \mathcal{S}$, $\rho(a) \in W(\pi_\rho(a))(\subseteq W(\pi(a)))$ and hence $V(a)$ is contained in $W(\pi(a))$. On the other hand if x is a unit vector of $\mathcal{H}$, then the formula $\rho(b) = \langle \pi(b)x, x\rangle, b \in \mathcal{A}$ defines a state on $\mathcal{A}$ and hence $\rho(a) = \langle \pi(a)x, x\rangle \in V(a)$ and it follows that

$$W(A) = V(a), \tag{2.1}$$

where $A = \pi(a)$ (see also Theorem 3 of [1]).

Hence, for all but finitely many points $p \in \partial V(a)$, the radius of curvature $R_p(a)$ of $\partial V(a)$ at p is well defined. By conventionally, $R_p(a) = 0$ if p is a corner point of $V(a)$, and $R_p(a) = \infty$ if p lies inside a flat portion of $\partial V(a)$.

If $D_p(a)$ denotes the distance from p to $\sigma(a)$, we can define $M(a)$, just like that of operators, to be the smallest constant such that

$$D_p(a) \leq M(a)R_p(a) \tag{2.2}$$

for all $p \in \partial V(a)$ with finite non zero curvature.

Relation (2.1) implies that the Donoghue's Theorem is also true for the C*-algebra numerical range and therefore, $D_p(a) = 0$ whenever $R_p(a) = 0$. As a consequence, $M(a) = 0$ for all convexoid element a. For the non-convexoid element a,

$$M(a) = \sup \frac{D_p(a)}{R_p(a)} \tag{2.3}$$

where the supremum in the right-hand side is taken along all points $p \in \partial V(a)$ with finite non-zero curvature.

In this paper, we find upper and lower bounds of $M(a)$ when a is a nilpotent element of a C^*-algebra. In particular, the exact value of $M(a)$ when a is nontrivial self-inverse will be obtained.

2.2. **Nilpotency.** Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. We denote the numerical radius of A and the distance from the origin to the boundary of its numerical range by $w(A)$ and $w_0(A)$, respectively.

It has been shown that if A is a nonzero nilpotent operator with the power of nilpotency n such that $w(A) \leq (n-1)w_0(A)$, and A attains its numerical radius then the following conditions are equivalent(see [3]):

(1) $w(A) = (n-1)w_0(A)$;
(2) A is unitarily equivalent to an operator of the form $\eta A_n \oplus A'$, where η is a scalar satisfying $|\eta| = 2w_0(A)$, and A' is another operator;
(3) $W(A) = \lambda W(A_n)$, where λ is a scalar, and A_n is the $n \times n$ matrix

$$A_n := \begin{bmatrix} 0 & 1 & \dots & 1 \\ & 0 & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

Given $\mathcal{A}$ that is a C*-algebra with unit 1 and $a \in \mathcal{A}$ is a nilpotent element with the power of nilpotency $n \geq 1$, i.e., $a^n = 0$, $v(a) = (n-1)v_0(a)$, where $v_0(a)$ denotes the distance from the origin to the boundary of its numerical range and $A = \pi(a)$, where π is the faithful representation of $\mathcal{A}$ that follows from the Gelfand-Naimark constraction, then $W(A) = V(a)$ and so $w(A) = (n-1)w_0(A)$. Also $a^n = 0$ implies that $A^n = 0$, A is a nilpotent operator with nilpotency n. Then $V(a) = \lambda W(A_n)$, where λ is a scalar, which leads to the following:

Theorem 2.1. *Let $\mathcal{A}$ be a C*-algebra with unit 1 and $a \in \mathcal{A}$ be a nilpotent element with the power of nilpotency $n \geq 1$, i.e., $a^n = 0$. If $v(a) = (n-1)v_0(a)$, where $v_0(a)$ denotes the distance from the origin to the boundary of its numerical range, then*

$$\frac{n}{2}\sin(\frac{\pi}{n}) \leq M(a) \leq \frac{n}{2}.$$

Proof. Let $\partial W(A_n) = x(\theta) + iy(\theta)$ and $\partial V(a) = u(\theta) + iv(\theta)$. By a direct computation we obtain the following relations:

- $u(\theta) = Re(\lambda)x(\theta) - Im(\lambda)y(\theta)$
- $v(\theta) = Im(\lambda)x(\theta) + Re(\lambda)y(\theta)$
- $D_p(a) = |\lambda| D_p(A_n)$
- $R_p(a) = |\lambda| R_p(A_n)$.

Therefore, $M(a) = M(A_n)$. Put $M_n = Re(e^{-i\theta}A_n) - \lambda I_n$ and $P_n(\lambda) = det(M_n)$. Then

$$P_n(\lambda) = det\begin{bmatrix} -\lambda & \frac{e^{-i\theta}}{2} & \frac{e^{-i\theta}}{2} & \dots & \frac{e^{-i\theta}}{2} \\ \frac{e^{i\theta}}{2} & -\lambda & \frac{e^{-i\theta}}{2} & \dots & \frac{e^{-i\theta}}{2} \\ \frac{e^{i\theta}}{2} & \frac{e^{i\theta}}{2} & -\lambda & \dots & \frac{e^{-i\theta}}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{e^{i\theta}}{2} & \frac{e^{i\theta}}{2} & \frac{e^{i\theta}}{2} & \dots & -\lambda \end{bmatrix}.$$

Adding the -1 multiple of any row to before one, we get

$$P_n(\lambda) = det\begin{bmatrix} -\lambda - \frac{e^{i\theta}}{2} & \lambda + \frac{e^{-i\theta}}{2} & 0 & 0 & \dots & 0 \\ 0 & -\lambda - \frac{e^{i\theta}}{2} & \lambda + \frac{e^{-i\theta}}{2} & 0 & \dots & 0 \\ 0 & 0 & -\lambda - \frac{e^{i\theta}}{2} & \lambda + \frac{e^{-i\theta}}{2} & \dots & 0 \\ 0 & 0 & 0 & -\lambda - \frac{e^{i\theta}}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{e^{i\theta}}{2} & \frac{e^{i\theta}}{2} & \frac{e^{i\theta}}{2} & \frac{e^{i\theta}}{2} & \dots & -\lambda \end{bmatrix}.$$

Expanding this determinant about the first column, we get the following recursive formula:

$$P_n(\lambda) = (-\lambda - \frac{e^{i\theta}}{2})P_{n-1}(\lambda) + (-1)^{n-1}\frac{e^{i\theta}}{2}(\lambda + \frac{e^{-i\theta}}{2})^{n-1}, \quad (n \geq 1)$$

with the initial condition $P_0(\lambda) = 1$ (see also problem 392 of [8]). Solving this recursive formula, gives

$$P_n(\lambda) = \frac{(-1)^{n+1}}{e^{i\theta} - e^{-i\theta}}\left[e^{-i\theta}(\lambda + \frac{e^{i\theta}}{2})^n - e^{i\theta}(\lambda + \frac{e^{-i\theta}}{2})^n\right]. \tag{2.4}$$

Considering the equation $P_n(\lambda) = 0$, we get the equation

$$\left(\frac{\lambda + \frac{e^{i\theta}}{2}}{\lambda + \frac{e^{-i\theta}}{2}}\right)^n = e^{2i\theta}$$

in terms of λ. Solving the above equation in terms of λ, we have the following roots

$$\lambda_k = \frac{1}{2}\left[\sin\theta\cot(\frac{k\pi + \theta}{n}) - \cos\theta\right],$$

where $k = 0, 1, ..., n-1$. By a straightforward computation we see that $\lambda(\theta)$, the maximum eigenvalue of $Re(e^{-i\theta}A_n)$, is equal to λ_0 and $\lambda_0 \geq \lambda_k$, for each k. So

$$\lambda(\theta) = \frac{1}{2}\left[\sin(\theta)\cot(\frac{\theta}{n}) - \cos(\theta)\right] \quad (-\pi \leq \theta \leq \pi). \tag{2.5}$$

The equation (2.5) and the fact that $\sin k\theta \leq k\sin\theta$ imply that $w(A_n) = \frac{n-1}{2}$ and $w_0(A_n) = \frac{1}{2}$.

The above observations will be useful to verify the following results. Pick $-\pi \leq \theta \leq \pi$ and let

$$\gamma_1(\theta) := \big(\lambda(\theta) + i\lambda^{'}(\theta)\big)e^{i\theta}.$$

and

$$\gamma_2(t) = -\frac{1}{2} + i(\frac{-3}{2}\cot(\frac{\pi}{n})t + \frac{1}{2}\cot(\frac{\pi}{n}))$$

for $0 \leq t \leq 1$ where $\lambda^{'}(\theta) = \frac{d\lambda(\theta)}{d\theta}$. Then $\partial W(A_n) = \gamma_1 + \gamma_2$ and so $\partial W(A_n)$ is differentiable.

A direct calculation from (2.5) implies that

$$\lambda^{'}(\theta) = \frac{1}{2}\left[\cos(\theta)\cot(\frac{\theta}{n}) - \frac{1}{n}\sin(\theta)\csc^2(\frac{\theta}{n}) + \sin(\theta)\right] \tag{2.6}$$

for $-\pi \leq \theta \leq \pi$, $\theta \neq 0$; and $\lambda^{'}(0) = 0$.

$$\lambda^{''}(\theta) = \frac{1}{2}\Bigg[-\sin(\theta)\cot(\frac{\theta}{n}) - \frac{2}{n}\cos(\theta)\csc^2(\frac{\theta}{n}) + \frac{1}{n^2}\sin(\theta)\cos(\frac{\theta}{n})\csc^3(\frac{\theta}{n}) + \cos(\theta)\Bigg] \tag{2.7}$$

for $-\pi \leq \theta \leq \pi$, $\theta \neq 0$; and $\lambda^{''}(0) = 0$. Hence, the radius curvature function of the boundary of numerical range of A_n, i.e, $\partial W(A_n)$ is

$$R(\theta) = \frac{1}{n^2}\csc^3(\frac{\theta}{n})\left[\sin(\theta)\cos(\frac{\theta}{n}) - n\cos(\theta)\sin(\frac{\theta}{n})\right].$$

Note that $\frac{D(\pi)}{R(\pi)} = \frac{n}{2}\sin(\frac{\pi}{n})$ and since $\sigma(A_n) = \{0\}$,

$$M(A_n) = \sup_{\theta\neq 0}\frac{\sqrt{(\lambda(\theta))^2 + (\lambda^{'}(\theta))^2}}{\lambda(\theta) + \lambda^{''}(\theta)}. \tag{2.8}$$

Hence

$$M(A_n) \geq \frac{n}{2}\sin(\frac{\pi}{n}).$$

On the other hand, (1.3) implies that

$$M(A_n) \leq \frac{n}{2},$$

and therefore,

$$\frac{n}{2}\sin(\frac{\pi}{n}) \leq M(A_n) \leq \frac{n}{2}.$$

Since $M(a) = M(A_n)$, the proof is completed. In particular, if $n = 2$ then $M(a) = 1$. ∎

Remark 2.2. Let a be a self-inverse element, i.e., $a^2 = 1$ and $a \neq 1$, then $\partial V(a)$ is an ellipse with foci at ± 1 and major/minor axis $\|a\| \pm \frac{1}{\|a\|}$ [6]. If $\partial V(a) = t\cos(\theta) + is\sin(\theta)$ with $t^2 = s^2 + 1$, then

$$\begin{aligned} M(a) = \sup_{0\le\theta\le 2\pi} \frac{D(\theta)}{R(\theta)} &= \sup_{0\le\theta\le\frac{\pi}{2}} \frac{D(\theta)}{R(\theta)} \\ &= \sup_{0\le\theta\le\frac{\pi}{2}} \frac{ts\sqrt{(t\cos(\theta)-1)^2 + s^2\sin^2(\theta)}}{(t^2\sin^2(\theta)+s^2\cos^2(\theta))^{\frac{3}{2}}} \\ &= \sup_{0\le\theta\le\frac{\pi}{2}} \frac{ts(t-\cos(\theta))}{(\sin^2(\theta)+s^2)^{\frac{3}{2}}} \\ &= \max\left\{\frac{\sqrt{t^2-1}}{t}\,,\ \frac{t}{t+1}\right\}. \end{aligned}$$

Therefore, we have the following corollary:

Corollary 2.2. *Let a be a non trivial self inverse element of the C*-algebra $\mathcal{A}$. Then*

$$M(a) = \max\left\{\frac{\|a\|^2-1}{\|a\|^2+1}, \frac{\|a\|^2+1}{(\|a\|+1)^2}\right\}.$$

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