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SOME DOUBLE $\lambda-$CONVERGENT SEQUENCE SPACES OVER n-NORMED SPACES

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ABSTRACT. In this paper we introduce some double generalized $\lambda-$convergent sequence spaces over $n-$normed spaces defined by Musielak-Orlicz function $\mathcal{M} = (M_{k,l})$. We also made an attempt to study some topological and algebraic properties of these sequence spaces.

Key words and phrases: Orlicz function, Musielak-Orlicz function, Paranorm space, Sequence space, Double sequence, p-convergent, $n-$normed space.

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1. INTRODUCTION

The initial work on double sequences was founded in Bromwich [4]. Later on, it was studied by Hardy [10], Móricz [15], Móricz and Rhoades [16], Tripathy ([26], [27]), Başarır and Sonalcan [2] and many others. Hardy [10] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [30] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [18] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [17] and Mursaleen and Edely [19] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Başar and Sonalcan [2] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{jk})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{jk} - L| < \epsilon$ whenever $j, k > n$. We shall write more briefly as P-convergent.

The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7],[8]) and Gunawan and Mashadi [9]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$, where $\mathbb{R}$ is field of real numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{R}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [12] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$ and for $L > 1$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [13],[21]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty, \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty, \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

By w^2 we denote the space of all real or complex valued double sequences. If $x \in w^2$, then we simply write $x = x_{k,l}$ instead of $x = (x_{k,l})_{k,l=1}^{\infty}$. Also, we will use the conventions that $e = (1, 1, ...)$. Any vector subspace of w^2 is called a double sequence space. We will write l_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences respectively. Further, by $l_p(1 \leq p < \infty)$, we denote the sequence space of all $p-$absolutely convergent series, that is $l_p = \{x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p \leq \infty$. Throughout the article,

$w^2(X), l_\infty(X)$ and $l_p(X)$ denote respectively, the spaces of all, bounded and $p-$absolutely summable sequences with the elements in X, where (X,q) is a seminormed space. By $\theta = (0,0,...)$, we denote the zero element in X. Also $P_{s,t}$ denote the class of subsets $\sigma = \sigma_1 \times \sigma_2$ in $\mathbb{N} \times \mathbb{N}$ such that the number of elements of σ_1 and σ_2 are atmost s and t respectively. Besides $\{\varphi_{k,l}\}$ is taken as a non-decreasing double sequence of positive real numbers such that $k\varphi_{k+1,l} \leq (k+1)\varphi_{k,l}$ and $l\varphi_{k,l+1} \leq (l+1)\varphi_{k,l}$. The class of all the sequences $(\varphi_{k,l})$ satisfying this property is denoted by φ. In [20], the notion of $\lambda-$ convergent and bounded sequences is introduced as follows:
let $\lambda = (\lambda_{k,l})$ be a strictly increasing double sequence of positive reals tending to infinity, that is

$$\lambda_{k+1,l} \leq \lambda_{k,l} + 1, \quad \lambda_{k,l+1} \leq \lambda_{k,l} + 1,$$
$$\lambda_{k,l} - \lambda_{k+1,l} \leq \lambda_{k,l} + 1 - \lambda_{k+1,l+1}, \lambda_{1,1} = 1,$$

We say that a sequence $x = (x_{k,l}) \in w^2$ is a double $\lambda-$ convergent to the number $L \in \mathbb{C}$, called as the double $\lambda-$ limit of x, if $\Lambda_{m,n}(x) \to L$ as $m,n \to \infty$, where

$$\Lambda_{m,n}(x) = \frac{1}{\lambda_{m,n}} \sum_{k,l=0}^{m,n} |\lambda_{k,l} - \lambda_{k+1,l}| x_{k,l}, \quad m,n \in \mathbb{N}.$$

In particular, we say that x is a double $\lambda-$ null sequence if $\Lambda_{m,n}(x) \to 0 \quad as \quad m,n \to \infty$. Further, we say that x is double $\lambda-$ bounded if sup $|\Lambda_{m,n}(x)| < \infty$. If $\lim_{m,n} x_{m,n} = a$ in the ordinary sense of convergence, then

$$\lim_{m,n} \Big(\frac{1}{\lambda_{m,n}} \sum_{k,l=0}^{m,n} |(\lambda_{k,l} - \lambda_{k+1,l})(x_{k,l} - a)| \Big) = 0.$$

This implies that

$$\lim_{m,n} |\Lambda_{m,n}(x) - a| = \lim_{m,n} \Big| \frac{1}{\lambda_{m,n}} \sum_{k,l=0}^{m,n} (\lambda_{k,l} - \lambda_{k+1,l})(x_{k,l} - a) \Big| = 0,$$

which yields that $\lim_{m,n} \Lambda_{m,n}(x) = a$ and hence x is double $\lambda-$ convergent to a. We therefore deduce that the ordinary convergence implies the double $\lambda-$ convergence to the same limit.
The notion of difference sequence spaces was introduced by Kizmaz[11], who studied the difference sequence spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$. The notion was further generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\triangle^n)$, $c(\triangle^n)$ and $c_0(\triangle^n)$. Let m,n be non-negative integers, then for Z a given sequence space, we have

$$Z(\triangle_m^n) = \{x = (x_k) \in w : (\triangle_m^n x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\triangle_m^n x = (\triangle_m^n x_k) = (\triangle_m^{n-1} x_k - \triangle_m^{n-1} x_{k+m})$ and $\triangle_m^0 = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

Taking $m = 1$, we get the spaces $l_\infty(\triangle^n), c(\triangle^n)$ and $c_0(\triangle^n)$ studied by Et and Çolak [5]. Taking $m = n = 1$, we get the spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ introduced and studied by Kizmaz [11]. Similarly, we can define difference operators on double sequence spaces as:

$$\begin{aligned} \Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}, \end{aligned}$$

$$\Delta^n x_{k,l} = \Delta^{n-1}x_{k,l} - \Delta^{n-1}x_{k,l+1} - \Delta^{n-1}x_{k+1,l} + \Delta^{n-1}x_{k+1,l+1}$$

and

$$\Delta^n_m x_{k,l} = \Delta^{n-1}_m x_{k,l} - \Delta^{n-1}_m x_{k,l+1} - \Delta^{n-1}_m x_{k+1,l} + \Delta^{n-1}_m x_{k+1,l+1}.$$

For more details about sequence spaces (see [3], [22], [23], [24]) and refrences therein.
The space $m(\phi)$ introduced and studied by Sargent [25] is defined as follows:

$$m(\phi) = \{x = (x_k) \in s : \|x\|_{m(\phi)} = \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma}|x_k| < \infty\}.$$

Let M be an Orlicz function, then Tripathy and Mahanta [28] defined and studied the following sequence space:

$$m(M,\varphi) = \{x = (x_k) \in s : \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\ , \quad \text{for some} \quad \rho > 0\}.$$

Recently, Altun and Bilgin [1] defined and studied the following sequence spaces:

$$m(M,A,\varphi,p) = \{x = (x_k) \in s : \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma} M\Big(\frac{|A_i x|}{\rho}\Big)^{p_k} < \infty\ , \quad \text{for some} \quad \rho > 0\}.$$

We define the following sequences spaces which we shall discuss in the second section of the present paper:
$m(\mathcal{M}\ ,\varphi,q,\Lambda,\triangle^n_m,p,\|.,...,.\|) =$

$$\Big\{x = (x_{k,l}) \in w^2 : P - \lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle^n_m x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} = 0, \quad \text{for some} \quad \rho > 0\Big\}.$$

If we take$M_{k,l}(x) = x$, we get
$m(\varphi,q,\Lambda,\triangle^n_m,p,\|.,...,.\|) =$

$$\Big\{x = (x_{k,l}) \in w^2 : P - \lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} \Big(q\Big(\|\frac{\Lambda_{k,l}\triangle^n_m x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} = 0, \quad \text{for some} \quad \rho > 0\Big\}.$$

If we take $p = (p_{k,l}) = 1$, we get
$m(\mathcal{M}\ ,\varphi,q,\Lambda,\triangle^n_m,\|.,...,.\|) =$

$$\Big\{x = (x_{k,l}) \in w^2 : P - \lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle^n_m x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big) = 0, \quad \text{for some} \quad \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_{k,l} \leq \sup p_{k,l} = H, K = \max(1, 2^{H-1})$ then

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K\{|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\} \tag{1.1}$$

for all k, l and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.
In this paper we study some topological properties and inclusion relation between spaces $m(\mathcal{M}\ \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$ which we have defined above and the spaces $m(\mathcal{M}\ , A, \varphi, \triangle_m^n, p, \|., ..., .\|)$ which we shall discuss in the third section of the paper.

2. DOUBLE SEQUENCE SPACES DEFINED BY MUSIELAK-ORLICZ FUNCTION

In this section we study some topological properties and inclusion relation between the spaces $m(\mathcal{M}\ , \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$.

Theorem 2.1. *Let $\mathcal{M} = (M_{k,l})$ be Musielak-Orlicz function and $p = (p_{k,l})$ be a double sequence of strictly positive real numbers.Then the sequence space $m(\mathcal{M}\ , \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$ is a linear space over the complex field $\mathbb{C}$.*

Proof. Let $x = (x_{k,l})$ and $y = (y_{k,l}) \in m(\mathcal{M}\ , \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho_1}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} = 0$$

and

$$\lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n y_{k,l}}{\rho_2}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|., ..., .\|$ is a $n-$ norm on X and $M_{k,l}$ are non-decreasing and convex function so by using inequality (1.1), we have

$$\begin{aligned}
&\lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n(\alpha x_{k,l} + \beta y_{k,l})}{\rho_3}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&\leq \lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} \Big[M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n(\alpha x_{k,l})}{\rho_3}, z_1, ..., z_{n-1}\|\Big)\Big) \\
&+ \Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n(\beta y_{k,l})}{\rho_3}, z_1, ..., z_{n-1}\|\Big)\Big)\Big]^{p_{k,l}} \\
&\leq K \lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} \frac{1}{2^{p_{k,l}}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho_1}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&+ K \lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} \frac{1}{2^{p_{k,l}}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_k\triangle_m^n y_{k,l}}{\rho_2}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&= 0.
\end{aligned}$$

Thus we have $\alpha x+\beta y \in m(\mathcal{M}\ , \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$. Hence $m(\mathcal{M}\ , \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$ is a linear space. ∎

Theorem 2.2. *For any Musielak Orlicz function $\mathcal{M} = (M_{k,l})$ and a bounded sequence $p = (p_{k,l})$ of positive real numbers $m(\mathcal{M}\ , \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1,$$

$$r=1,2,3,...\Big\},$$

where $M=\max(1,\sup_{k,l} p_{k,l}<\infty)$.

Proof. Clearly $g(x)\geq 0$ for $x=(x_{k,l})\in m(\mathcal{M},\varphi,q,\Lambda,\triangle_m^n,p,\|.,...,.\|)$.. Since $M_{k,l}(0)=0$, we get $g(0)=0$. Again, if $g(x)=0$, then

$$g(x)=\inf\Big\{\rho^{\frac{p_r}{M}}:\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1\Big\}=0.$$

This implies that for a given $\epsilon>0$, there exist some $\rho_\epsilon(0<\rho_\epsilon<\epsilon)$ such that

$$\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho_\epsilon},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1.$$

Thus

$$\begin{aligned}\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}&\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\epsilon},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\\ &\leq \Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho_\epsilon},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\\ &\leq 1.\end{aligned}$$

Suppose that $x_{k,l}\neq 0$ for each $k,l\in\mathbb{N}$. This implies that $\Lambda_{k,l}\triangle_m^n x_{k,l}\neq 0$ for each $k,l\in\mathbb{N}$. Let $\epsilon\to 0$, then $\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\epsilon},z_1,...,z_{n-1}\|\to\infty$. It follows that

$$\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\epsilon},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\to\infty,$$

which is a contradiction. Therefore $\Lambda_{k,l}\triangle_m^n x_{k,l}=0$ for each k,l and thus $x_{k,l}=0$ for each $k,l\in\mathbb{N}$. Let $\rho_1>0$ and $\rho_2>0$ be such that

$$\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho_1},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1$$

and

$$\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n y_{k,l}}{\rho_2},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1.$$

Let $\rho=\rho_1+\rho_2$. Then by Minkowski inequality, we have

$$\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n(x_{k,l}+y_{k,l})}{\rho},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}$$

$$\begin{aligned}
&= \Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n(x_{k,l}+y_{k,l})}{\rho_1+\rho_2},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\\
&\leq \Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\frac{\rho_1}{\rho_1+\rho_2}\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho_1},z_1,...,z_{n-1}\|\\
&+ \frac{\rho_2}{\rho_1+\rho_2}\|\frac{\Lambda_{k,l}\triangle_m^n y_{k,l}}{\rho_2},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\\
&\leq \frac{\rho_1}{\rho_1+\rho_2}\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho_1},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\\
&+ \frac{\rho_2}{\rho_1+\rho_2}\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n y_{k,l}}{\rho_2},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\\
&\leq 1.
\end{aligned}$$

Since $\rho's$ are non-negative, so we have

$$g(x+y)=$$

$$\inf\Big\{\rho^{\frac{p_r}{M}}:\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n(x_{k,l}+y_{k,l})}{\rho},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1\Big\}$$

$$\begin{aligned}
&\leq \inf\Big\{\rho_1^{\frac{p_r}{M}}:\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho_1},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1\Big\}\\
&+ \inf\Big\{\rho_2^{\frac{p_r}{M}}:\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n y_{k,l}}{\rho_2},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1\Big\}.
\end{aligned}$$

Therefore, $g(x+y)\leq g(x)+g(y)$. Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x)=\inf\Big\{\rho^{\frac{p_r}{M}}:\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\mu\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1\Big\}.$$

Then

$$g(\mu x)=\inf\Big\{(|\mu|t)^{\frac{p_r}{M}}:\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{t},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\leq 1\Big\},$$

where $t=\frac{\rho}{|\mu|}$. Since $|\mu|^{p_r}\leq\max(1,|\mu|\sup p_r)$, we have

$$g(\mu x)=$$

$$\max(1,|\mu|\sup p_r)\inf\Big\{(t)^{\frac{p_r}{M}}:\Big(\lim_{k,l}\frac{1}{\varphi_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{t},z_1,...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}$$

$$\leq 1\Big\},$$

So, the fact scalar multiplication is continuous follows from the above inequality. ∎

Theorem 2.3. *If $\mathcal{M}=(M_{k,l})$ be any Musielak Orlicz function. Then*

$$m(\mathcal{M},\varphi^{*},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)\subset m(\mathcal{M},\varphi^{**},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)$$

if and only if $\sup_{s,t\geq 1}\frac{\varphi^{*}_{s,t}}{\varphi^{**}_{s,t}}<\infty$.

Proof. Let $x\in m(\mathcal{M},\varphi^{*},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)$ and $N=\sup_{s,t\geq 1}\frac{\varphi^{*}_{s,t}}{\varphi^{**}_{s,t}}<\infty$. Then we get

$$\begin{aligned}
&\frac{1}{\varphi^{**}_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_{m}^{n}x_{k,l}}{\rho},z_{1},...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\\
&\leq \sup_{s,t\geq 1}\frac{\varphi^{*}_{s,t}}{\varphi^{**}_{s,t}}\frac{1}{\varphi^{*}_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_{m}^{n}x_{k,l}}{\rho},z_{1},...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\\
&= N\frac{1}{\varphi^{*}_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_{m}^{n}x_{k,l}}{\rho},z_{1},...,z_{n-1}\|\Big)\Big)^{p_{k,l}}\\
&= 0.
\end{aligned}$$

Thus $x\in m(\mathcal{M},\varphi^{**},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)$.
Conversely, suppose that

$$m(\mathcal{M},\varphi^{*},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)\subset m(\mathcal{M},\varphi^{**},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)$$

and $x\in m(\mathcal{M},\varphi^{*},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)$. Then there exists $\rho>o$ such that

$$\frac{1}{\varphi^{*}_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_{m}^{n}x_{k,l}}{\rho},z_{1},...,z_{n-1}\|\Big)\Big)^{p_{k,l}}<\epsilon,$$

for every $\epsilon>0$. Suppose that $\sup_{s,t\geq 1}\frac{\varphi^{*}_{s,t}}{\varphi^{**}_{s,t}}=\infty$, then there exists a sequence of numbers (s_i,t_j) such that $\lim_{i,j\to\infty}\frac{\varphi^{*}_{s_i,t_j}}{\varphi^{**}_{s_i,t_j}}=\infty$. Hence, we have

$$\frac{1}{\varphi^{**}_{s,t}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_{m}^{n}x_{k,l}}{\rho},z_{1},...,z_{n-1}\|\Big)\Big)^{p_{k,l}}$$

$$\leq \sup_{i,j\geq 1}\frac{\varphi^{*}_{s_i,t_j}}{\varphi^{**}_{s,t}}\frac{1}{\varphi^{*}_{s_i,t_j}}\sum_{k,l\in\sigma,\sigma\in P_{s,t}}M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_{m}^{n}x_{k,l}}{\rho},z_{1},...,z_{n-1}\|\Big)\Big)^{p_{k,l}}=\infty.$$

Therefore, $x\notin m(\mathcal{M},\varphi^{**},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)$, which is a contradiction. This completes the proof of the theorem. ∎

Corollary 2.4. *If $\mathcal{M}=(M_{k,l})$ be any Musielak Orlicz function. Then*

$$m(\mathcal{M},\varphi^{*},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)=m(\mathcal{M},\varphi^{**},q,\Lambda,\triangle_{m}^{n},p,\|.,...,.\|)$$

if and only if $\sup_{s,t\geq 1}\frac{\varphi^{*}_{s,t}}{\varphi^{**}_{s,t}}<\infty,\quad \sup_{s,t\geq 1}\frac{\varphi^{**}_{s,t}}{\varphi^{*}_{s,t}}>\infty$.

The corollary is easy to prove so for brevity it is omitted.

Theorem 2.5. *For Musielak Orlicz functions $\mathcal{M} = (M_{k,l}), \mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ which satisfy Δ_2- condition and q, q_1, q_2 are seminorms. Then the following relation holds:*
(i) $m(\mathcal{M}', \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|) \subset m(\mathcal{M} \circ \mathcal{M}', \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$
(ii) $m(\mathcal{M}', \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|) \cap m(\mathcal{M}'', \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$

$$\subset \quad m(\mathcal{M}' + \mathcal{M}'', \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$$

(iii) $m(\mathcal{M}, \varphi, q_1, \Lambda, \triangle_m^n, p, \|., ..., .\|) \cap m(\mathcal{M}, \varphi, q_2, \Lambda, \triangle_m^n, p, \|., ..., .\|)$

$$\subset \quad m(\mathcal{M}, \varphi, q_1 + q_2, \Lambda, \triangle_m^n, p, \|., ..., .\|).$$

Proof. The proof of theorem along the same lines as the proof of the Theorem 2.5 of [1]. ∎

Corollary 2.6. *If $\mathcal{M} = (M_{k,l})$ be any Musielak Orlicz function which satisfy Δ_2- condition.Then $m(\mathcal{M}, \varphi^*, q, \Lambda, \triangle_m^n, p, \|., ..., .\|) \subset m(\mathcal{M}, \varphi^{**}, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$.*

Theorem 2.7. *If $\mathcal{M} = (M_{k,l})$ be any Musielak Orlicz function. Then the sequence space $m(\mathcal{M}, \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$ is solid.*

Proof. Let $x \in m(\mathcal{M}, \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$. Then there exists $\rho > 0$ such that

$$\Big(\lim_{k,l} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} < \epsilon,$$

for every $\epsilon > 0$. Let $(\alpha_{k,l})$ be a sequence of scalars with $|\alpha_{k,l}| \leq 1$ for all $n \in \mathbb{N}$.
Then we have

$$\begin{aligned}
&\tfrac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n (\alpha_{k,l} x_{k,l})}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&\leq \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} |\alpha_{k,l}| M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&\leq \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma,\sigma\in P_{s,t}} M_{k,l}\Big(q\Big(\|\frac{\Lambda_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}},
\end{aligned}$$

which proves that $m(\mathcal{M}, \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$ is solid space. ∎

Theorem 2.8. *If $\mathcal{M} = (M_{k,l})$ be any Musielak Orlicz function. The sequence space $m(\mathcal{M}, \varphi, q, \Lambda, \triangle_m^n, p, \|., ..., .\|)$ is monotone.*

The proof is obvious.

3. DOUBLE GENERALIZED SEQUENCE SPACES

Let $A = (\alpha_{i,j,k,l})$ be an infinite matrix of complex numbers. Then, we have

$$A(x) = (A_{i,j}(x)) \text{ if } A_{i,j}(x) = \sum_{k,l=1,1}^{\infty,\infty} \alpha_{i,j,k,l} x_{k,l}$$

converges for each i, j. For more details see [1].
In this section we introduce the following sequences spaces which are actually the generalizations of sequences spaces defined by Altun and Bilgin [1]. Thus we have the spaces:

$$m(\mathcal{M}\ , A, \varphi, \triangle_m^n, p, \|.,...,.\|) =$$
$$\Big\{x = x_{k,l} \in w^2 : \sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(q\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} < \infty,$$
$$\text{for some } \rho > 0\Big\}.$$

If $M_{k,l}(x) = x$, we have

$$m(A, \varphi, \triangle_m^n, p, \|.,...,.\|) =$$
$$\Big\{x = x_{k,l} \in w^2 : \sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} \Big(q\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} < \infty,$$
$$\text{for some } \rho > 0\Big\}.$$

In this section of the present paper we shall also study some topological properties and inclusion relations between the spaces $m(\mathcal{M}\ , A, \varphi, \triangle_m^n, p, \|.,...,.\|)$.

Theorem 3.1. *Let $\mathcal{M} = (M_{k,l})$ Musielak- Orlicz function. Then the sequence space $m(\mathcal{M}, A, \varphi, \triangle_m^n, p, \|.,...,.\|)$ is a linear space over the complex field $\mathbb{C}$.*

Proof. Let $x = (x_{k,l})$ and $y = (y_{k,l}) \in m(\mathcal{M}, A, \varphi, \triangle_m^n, p, \|.,...,.\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(q\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_1}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} < \infty$$

and

$$\sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(q\Big(\|\frac{A_{k,l}\triangle_m^n y_{k,l}}{\rho_2}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} < \infty$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|.,...,.\|$ is a $n-$ norm on X and $M_{k,l}$ are non-decreasing and convex function so by using inequality (1.1), we have

$$\begin{aligned}
&\sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k\in\sigma} M_{k,l}\Big(q\Big(\|\frac{A_{k,l}\triangle_m^n (\alpha x_{k,l} + \beta y_{k,l})}{\rho_3}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&\leq \sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(q\Big(\|\frac{A_{k,l}\triangle_m^n (\alpha x_{k,l})}{\rho_1}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&+ \sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(q\Big(\|\frac{A_{k,l}\triangle_m^n (\beta y_{k,l})}{\rho_2}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&\leq K \sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} \frac{1}{2^{p_{k,l}}} M_{k,l}\Big(q\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_1}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&+ K \sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} \frac{1}{2^{p_{k,l}}} M_{k,l}\Big(q\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_2}, z_1, ..., z_{n-1}\|\Big)\Big)^{p_{k,l}} \\
&< \infty.
\end{aligned}$$

Thus, we have $\alpha x+\beta y \in m(\mathcal{M}, A, \phi, \triangle_m^n, p, \|., ..., .\|)$. Hence $m(\mathcal{M}, A, \phi, \triangle_m^n, p, \|., ..., .\|)$ is a linear space. ∎

Theorem 3.2. *For any Musielak Orlicz function $\mathcal{M} = (M_{k,l})$ and a bounded sequence $p = (p_{k,l})$ of positive real numbers $m(\mathcal{M}, \varphi, A, \triangle_m^n, p, \|., ..., .\|)$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1,$$

$$r = 1, 2, 3, ...\Big\},$$

where $M = \max(1, \sup_{k,l} p_{k,l} < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_{k,l}) \in m(\mathcal{M}, \varphi, A, \triangle_m^n, p, \|., ..., .\|)$. Since $M_{k,l}(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x)$$
$$= \inf\Big\{\rho^{\frac{p_r}{M}} : \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exist some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_\epsilon}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1.$$

Thus

$$\begin{aligned}
&\Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\epsilon}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \\
&\leq \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_\epsilon}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \\
&\leq 1.
\end{aligned}$$

Suppose that $x_{k,l} \neq 0$ for each $k, l \in \mathbb{N}$. This implies that $A_{k,l}\triangle_m^n x_{k,l} \neq 0$ for each $k, l \in \mathbb{N}$. Let $\epsilon \to 0$, then $\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\epsilon}, z_1, ..., z_{n-1}\| \to \infty$. It follows that

$$\Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\epsilon}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \to \infty,$$

which is a contradiction. Therefore $A_{k,l}\triangle_m^n x_{k,l} = 0$ for each k, l and thus $x_{k,l} = 0$ for each $k, l \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_1}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1$$

and

$$\Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n y_{k,l}}{\rho_2}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by Minkowski inequality, we have

$$\begin{aligned}
&\Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n(x_{k,l}+y_{k,l})}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \\
&= \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n(x_{k,l}+y_{k,l})}{\rho_1+\rho_2}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \\
&\leq \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\frac{\rho_1}{\rho_1+\rho_2}\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_1}, z_1, ..., z_{n-1}\| \\
&+ \frac{\rho_2}{\rho_1+\rho_2}\|\frac{A_{k,l}\triangle_m^n y_{k,l}}{\rho_2}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \\
&\leq \frac{\rho_1}{\rho_1+\rho_2}\Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_1}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \\
&+ \frac{\rho_2}{\rho_1+\rho_2}\Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n y_{k,l}}{\rho_2}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \\
&\leq 1.
\end{aligned}$$

Since $\rho's$ are non-negative, so we have

$$\begin{aligned}
&g(x+y) = \\
&\inf\Big\{\rho^{\frac{pr}{M}} : \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n(x_{k,l}+y_{k,l})}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1\Big\} \\
&\leq \inf\Big\{\rho_1^{\frac{pr}{M}} : \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho_1}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1\Big\} \\
&+ \inf\Big\{\rho_2^{\frac{pr}{M}} : \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n y_{k,l}}{\rho_2}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1\Big\}.
\end{aligned}$$

Therefore, $g(x+y) \leq g(x) + g(y)$. Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x) = \inf\Big\{\rho^{\frac{pr}{M}} : \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{\mu A_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \leq 1\Big\}.$$

Then

$$g(\mu x) = \inf\Big\{(|\mu| t)^{\frac{pr}{M}} : \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n(\mu x_{k,l})}{,} z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}}\Big\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_r} \leq \max(1, |\mu| \sup p_r)$, we have

$$\begin{aligned}
&g(\mu x) = \\
&\max(1, |\mu| \sup p_r) \inf\Big\{(t)^{\frac{pr}{m}} : \Big(\sup_{s,t\geq 1,\sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{t}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}}\Big)^{\frac{1}{M}} \\
&\leq 1\Big\},
\end{aligned}$$

So, the fact that scalar multiplication is continuous follows from the above inequality. ∎

Theorem 3.3. *If $\mathcal{M} = (M_{k,l})$ be any Musielak Orlicz function. Then*

$$m(\mathcal{M}, \varphi^*, A, \triangle_m^n, p, \|., ..., .\|) \subset m(\mathcal{M}, \varphi^{**}, A, \triangle_m^n, p, \|., ..., .\|)$$

if and only if $\sup_{s,t\geq 1} \frac{\varphi^*_{s,t}}{\varphi^{**}_{s,t}} < \infty$.

The proof is trivial so we omit it.

Corollary 3.4. *If $\mathcal{M} = (M_{k,l})$ be any Musielak Orlicz function. Then*

$$m(\mathcal{M}, \varphi^*, A, \triangle_m^n, p, \|., ..., .\|) = m(\mathcal{M}, \varphi^{**}, A, \triangle_m^n, p, \|., ..., .\|)$$

if and only if $\sup_{s,t\geq 1} \frac{\varphi^*_{s,t}}{\varphi^{**}_{s,t}} < \infty, \sup_{s,t\geq 1} \frac{\varphi^{**}_{s,t}}{\varphi^{*}_{s,t}} > \infty$.

The corollary is easy to prove so it is omitted.

Theorem 3.5. *For Musielak Orlicz functions $\mathcal{M} = (M_{k,l}), \mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ which satisfy Δ_2- condition. Then the following relation holds:*
(i) $m(\mathcal{M}', \varphi, A, \triangle_m^n, p, \|., ..., .\|) \subset m(\mathcal{M} \circ \mathcal{M}', \varphi, A, \triangle_m^n, p, \|., ..., .\|)$
(ii) $m(\mathcal{M}', \varphi, A, \triangle_m^n, p, \|., ..., .\|) \cap m(\mathcal{M}'', \varphi, A, \triangle_m^n, p, \|., ..., .\|)$

$$\subset \quad m(\mathcal{M}' + \mathcal{M}'', \varphi, A, \triangle_m^n, p\|., ..., .\|)$$

Proof. The proof is along the same lines as the proof of the Theorem 2.5 of [1]. ∎

Corollary 3.6. *If $\mathcal{M} = (M_{k,l})$ be any Musielak Orlicz function which satisfy Δ_2- condition. Then $m(\mathcal{M}, \varphi^*, A, \triangle_m^n, p, \|., ..., .\|) \subset m(\mathcal{M}, \varphi^{**}, A, \triangle_m^n, p, \|., ..., .\|)$.*

Theorem 3.7. *If $\mathcal{M} = (M_{k,l})$ be any Musielak Orlicz function. Then the sequence space $m(\mathcal{M}, \varphi, A, \triangle_m^n, p, \|., ..., .\|)$ is solid.*

Proof. Let $x \in m(\mathcal{M}, \varphi, A, \triangle_m^n, p, \|., ..., .\|)$. Then there exists $\rho > 0$ such that

$$\sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}} < \epsilon,$$

for every $\epsilon > 0$. Let $(\alpha_{k,l})$ be a sequence of scalars with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n (\alpha_{k,l} x_{k,l})}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}} \\
&\leq \sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} |\alpha_{k,l}| M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}} \\
&\leq \sup_{s,t\geq 1, \sigma\in P_{s,t}} \frac{1}{\varphi_{s,t}} \sum_{k,l\in\sigma} M_{k,l}\Big(\|\frac{A_{k,l}\triangle_m^n x_{k,l}}{\rho}, z_1, ..., z_{n-1}\|\Big)^{p_{k,l}},
\end{aligned}$$

which proves that $m(\mathcal{M}, \varphi, A, \triangle_m^n, p, \|., ..., .\|)$ is solid space. ∎

Corollary 3.8. *If $\mathcal{M} = (M_{k,l})$ be any Musielak Orlicz function. Then the sequence space $m(\mathcal{M}, \varphi, A, \triangle_m^n, p, \|., ..., .\|)$ is monotone.*

The proof is obvious.

REFERENCES

[1] Y. ALTUN and T. BILGIN, On a new class of sequences related to the l_p space defined by Orlicz function, *Taiwanese J. Math.,* **13** (2003), pp. 1189-1196.

[2] M. BAŞARŀR and O. SONALCAN, On some double sequence spaces, *J. Indian Acad. Math.,* **21** (1999), pp. 193-200.

[3] T. BILGIN, Some new difference sequence spaces defined by an Orlicz function, *Filomat*, **17** (2003), pp. 1-8.

[4] T. J. BROMWICH, *An Introduction to the Theory of Infinite Series*, Macmillan and co. Ltd., New York 1965.

[5] M. ET and R. ÇOLAK, On some generalized sequence spaces, *Soochow. J. Math.,* **21** (1995), pp. 377-386.

[6] S. GÄHLER, Linear 2-normietre, *Rume, Math. Nachr.,* **28** (1965), pp. 1-43.

[7] H. GUNAWAN, On n-Inner Product, n-Norms, and the Cauchy-Schwartz Inequality, *Sci. Math. Jpn.,* **5** (2001), pp. 47-54.

[8] H. GUNAWAN, The space of p-summable sequence and its natural n-norm, *Bull. Aust. Math. Soc.,* **64** (2001), pp. 137-147.

[9] H. GUNAWAN and M. MASHADI, On n-normed spaces, *Int. J. Math. Sci.,* **27** (2001), pp. 631-639.

[10] G. H. HARDY, On the convergence of certain multiple series, *Proc. Camb. Phil., Soc.,* **19** (1917), pp. 86-95.

[11] H. KŀZMAZ, On certain sequence spaces, *Canad. Math-Bull.,* **24** (1981), pp. 169-176.

[12] J. LINDENSTRAUSS and L. TZAFRIRI, On Orlicz sequence spaces, *Israel J. Math.,* **10** (1971), pp. 379-390.

[13] L. MALIGRANDA, *Orlicz Spaces and Interpolation*, Seminars in Mathematics **5**, Polish Academy of Science, 1989.

[14] A. MISIAK, n-Inner product spaces, *Math. Nachr.,* **140** (1989), pp. 299-319.

[15] F. MÓRICZ, Extension of the spaces c and c_0 from single to double sequences, *Acta Math. Hungarica,* **57** (1991), pp. 129-136.

[16] F. MÓRICZ and B. E. RHOADES, Almost convergence of double sequences and strong reqularity of summability matrices, *Math. Proc. Camb. Phil. Soc.,* **104** (1988), pp. 283-294.

[17] M. MURSALEEN, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.,* **293** (2004), pp. 523-531.

[18] M. MURSALEEN and O. H. H. EDELY, Statistical convergence of double sequences, *J. Math. Anal. Appl.,* **288** (2003), pp. 223-231.

[19] M. MURSALEEN and O. H. H. EDELY, Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.,* **293** (2004), pp. 532-540.

[20] M. MURSALEEN and A. K. NOMAN, On the spaces of λ-convergent and bounded sequences, *Thai J. Math.,* **8** (2010), pp. 311-329.

[21] J. MUSIELAK, *Orlicz Spaces and Modular Spaces*, Lecture notes in Mathematics, **1034** 1983.

[22] K. RAJ and A. KILICMAN, On certain generalized paranormed spaces, *J. Inequal. Appl.,* (2015), 2015: 37.

[23] K. RAJ, S. JAMWAL and S.K. SHARMA, New classes of generalized sequence spaces defined by an Orlicz function, *J. Comput. Anal. Appl.,* **15** (2013), pp. 730-737.

[24] S.A.MOHIUDDINE, K. RAJ and A. ALOTAIBI, Some paranormed double difference sequence spaces for Orlicz functions and bounded-regular matrices, *Abstr. Appl. Anal.*, Volume 2014, Article ID 419064, 10 pages.

[25] W. L. C. SARGENT, Some sequence spaces related to the l_p spaces, *Journal of London Mathematical Society*, **35** (1960), pp. 161-171.

[26] B. C. TRIPATHY, Generalized difference paranormed statistically convergent sequences defined by Orlicz function in a locally convex spaces, *Soochow J. Math.*, **30** (2004), pp. 431-446.

[27] B. C. TRIPATHY, Statistically convergent double sequences, *Tamkang J. Math.*, **34** (2003), pp. 231-237.

[28] B. C. TRIPATHY and S. MAHANTA, Some sequence spaces related to the l_p spaces defined by Orlicz function, *Soochow J. Math.*, **29** (2003), pp. 379-391.

[29] A. WILANSKY, Summability through Functional Analysis, *North-Holland Math. Stud.*, **85** 1984.

[30] M. ZELTSER, Investigation of double sequence spaces by soft and hard analytical methods, *Diss. Math. Univ. Tartu.*, **25**, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu 2001.