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EXISTENCE AND REGULARITY OF MINIMA OF AN INTEGRAL FUNCTIONAL IN UNBOUNDED DOMAIN

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ABSTRACT. We prove the existence and the regularity of minima for a functional defined on a suitable Sobolev space.

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1. INTRODUCTION

This work is devoted to some results concerning the existence and the regularity of minima for the following functional defined on $W^{1,p}(\mathbb{R}^N)\ \ (1 < p < N,\ N \geq 2)$ with value on $\overline{\mathbb{R}}$, by

$$J(v) = \int_{\mathbb{R}^N} \frac{|\nabla v|^p + |v|^p}{(a(x) + |v|)^{\alpha p}}\, dx - \int_{\mathbb{R}^N} f.|v|^m\, dx, \qquad v \in\ W^{1,p}(\mathbb{R}^N), \tag{1.1}$$

for some real number α such that

$$0 < \alpha < \frac{p-m}{p}, \tag{1.2}$$

and where a is a measurable function on $\mathbb{R}^N$ that belongs to some Lebesgue space (we will specify that later).
As regards the datum f, we will assume that belongs to a suitably Lebesgue space which make sense to second term of functional J.
Recalling that, the previous problem has been studied by Boccardo and Orsina [3] in bounded domain, more precisely the authors they proved the existence and regularity of minima for the following functional:

$$\tilde{J}(v) = \int_\Omega a(x,v)|\nabla v|^p\, dx - \int_\Omega f.v\, dx,$$

where a is a carathéodory function satisfying the following condition:

$$\frac{\beta_0}{(1+|t|)^{\alpha p}} \leq a(x,t) \leq \beta_1$$

where β_0, β_1 are positive constants and α verified the condition (1.2). In the same way we cite the recent works [1, 2] of Boccardo and Orsina.
The main goal of this work is to extend the previous result to a general domain.
One of the main difficulties is to find a suitable Sobolev space in which the functional J is finite, since even the second term is for example finite in $W^{1,p}(\mathbb{R}^N)$ when $f \in\ L^r(\mathbb{R}^N)$, with $(q^*)' \leq r \leq q'$, the first term is not in general finite when $a(x)$ is small enough. An additional difficulty also arises : the generate coerciveness in $W^{1,p}(\mathbb{R}^N)$ of the principal part of the functional J. thus even J is lower semi continuous on $W_0^{1,p}(\Omega)$ as a consequence of the De Giorgi theorem, the lack of coerciveness implies that J may not attain its minimum on $W_0^{1,p}(\Omega)$ even in the case in which J is bounded from below (see example 3.2 of [3]), which will lead us to defined the functional on another space, $W_0^{1,q}(\Omega)$ for some $q < p$ depending on α. In this setting we prove several existence and regularity minima (depending on the summability of the datum f) for functional J, where α satisfies the condition (1.2) and a satisfies the following condition:

$$0 < a(x) \leq M \ \text{ a.e in } \quad \mathbb{R}^N, \tag{1.3}$$

with M a positive constant

$$a(x) \in\ L^{q^*}(\mathbb{R}^N), \tag{1.4}$$

with

$$q = \frac{Np(1-\alpha)}{N - \alpha p}. \tag{1.5}$$

We also assume that:

$$f \in\ L^{r_0}(\mathbb{R}^N) \ \text{ for some } \quad \left(\frac{q^*}{m}\right)' \leq r_0 \leq \left(\frac{q}{m}\right)'. \tag{1.6}$$

Notations :

In the sequel we will use several times the following functions of a real variable depending on a parameter $k > 0$

$$T_k(s) = \max(-k, \min(k, s)), \quad G_k(s) = T_k(s). \tag{1.7}$$

Furthermore, we will denote with c or $c_1, c_2, ...$, various constants which may depend on the data of the problem, whose value may vary from line to line.
If $1 < \sigma < N$, we denote by $\sigma^* = \frac{N\sigma}{N-\sigma}$ the Sobolev embedding exponent for the space $W_0^{1,\sigma}(\Omega)$.
If $u : I\!R^N \to I\!R$ is a Lebesgue measurable function, we define, for all $k \geq 0$

$$A_k = \left\{x \in I\!R^N \ : \ |u(x)| \geq k\right\}, \quad B_k = \left\{x \in I\!R^N \ : \ k \leq |u(x)| \leq k+1\right\}. \tag{1.8}$$

If E is a lebesgue measurable subset of $I\!R^N$, we denote by $|E|$ its N-dimensional Lebesgue measure.
We defined the following functional on $W^{1,q}(I\!R^N)$, with q as in (1.5)

$$I(v) = \begin{cases} J(v) & \text{if } \displaystyle\int_{I\!R^N} \frac{|\nabla v|^p + |v|^p}{(a(x)+|v|)^{\alpha p}}\, dx < +\infty \\ +\infty & \text{Otherwise} \end{cases}. \tag{1.9}$$

Our results are the following :

Theorem 1.1. *Under the hypotheses* $(1.2), (1.3), (1.4), (1.5)$ *and* (1.6), *there exists a minimum* u *of* I *on* $W^{1,q}(I\!R^N)$.

The second result considers the case where f has a high summability.

Theorem 1.2. *Let* $f \in L^r(I\!R^N)$ *with* $r > \frac{N}{p}$. *Suppose that* $(1.2), (1.3), (1.4), (1.5)$ *and* (1.6) *holds true, then any minimum* u *of* I *on* $W^{1,q}(I\!R^N)$ *belongs to* $W^{1,p}(I\!R^N) \cap L^\infty(I\!R^N)$*; thus* J *attains its minimum on* $W^{1,p}(I\!R^N)$.

Remark 1.1.

(1) Observe that $\dfrac{N}{p} > \left(\dfrac{q^*}{m}\right)'$.
(2) We do not need the hypotheses (1.6) in the case where

$$\frac{N}{p} < \left(\frac{q}{m}\right)' \text{ and } \frac{N}{p} < r \leq \left(\frac{q}{m}\right)'.$$

Remark 1.2. Observe that the condition on r does not depend on α, and the result also does not depend on α. The main tool of the proof will be an $L^\infty(I\!R^N)$ estimate, which then implies the $W^{1,p}(I\!R^N)$ estimate.

Theorem 1.3. *Assume that* $(1.2), (1.3), (1.4), (1.5), (1.6)$ *and* $f \in L^r(I\!R^N)$ *with*

$$\left(\frac{p^*}{m+\alpha p}\right)' \leq r < \frac{N}{p}.$$

Then any minimum u of I on $W^{1,p}(I\!R^N)$ belongs to $W^{1,p}(I\!R^N) \cap L^s(I\!R^N)$; thus J attains its minimum on $W^{1,p}(I\!R^N)$, where

$$s = \frac{Nr\,(p(1-\alpha)-m)}{N-rp}.$$

Remark 1.3.

(1) Since $0 < \alpha < \dfrac{p-m}{p}$ we have

$$\left(\frac{p^*}{m+\alpha p}\right)' < \frac{N}{p}.$$

(2) We do not need the hypotheses (1.6) in the case where

$$\left(\frac{p^*}{m+\alpha p}\right)' < \left(\frac{q}{m}\right)' \text{ and } \left(\frac{p^*}{m+\alpha p}\right)' \le r < \min\left(\left(\frac{q}{m}\right)', \frac{N}{p}\right).$$

Remark 1.4. Remark that if the minima are not bounded, we still have that they belong to $W^{1,p}(I\!R^N)$. The $W^{1,p}(I\!R^N)$ regularity result will be proved combining the information that u belongs to $L^s(I\!R^N)$ with the fact that u is minimum.

Remark 1.5. As a consequence of the previous theorem, if $r = \dfrac{N}{p}$ and $f \in L^1(I\!R^N)$, we have that any minimum u belongs to $W^{1,p}(I\!R^N)$ and to $L^s(I\!R^N)$, for every $s < +\infty$.

If we decrease the summability of f, we find minima of I which do not in general belong any more $W^{1,p}(I\!R^N)$.

Theorem 1.4. *Under the assumptions $(1.2) - (1.6)$, if $f \in L^r(I\!R^N)$ with*

$$\left(\frac{p^*(1-\alpha)}{m}\right)' \le r < \left(\frac{p^*}{m+\alpha p}\right)'.$$

Then any minimum u of I on $W^{1,p}(I\!R^N)$ belongs to $W^{1,\rho}(I\!R^N) \cap L^s(I\!R^N)$; thus J attains its minimum on $W^{1,\rho}(I\!R^N)$, where

$$\rho = \frac{Nr\,(p(1-\alpha)-m)}{N-r(m+\alpha p)}.$$

Remark 1.6. Note that the condition $0 < \alpha < \dfrac{p-m}{p}$ implies that

$$\left(\frac{p^*(1-\alpha)}{m}\right)' \le r < \left(\frac{p^*}{m+\alpha p}\right)'.$$

Remark 1.7. If α tends to $\dfrac{p-m}{p}$ both $\left(\dfrac{p^*}{m+\alpha p}\right)'$ and $\left(\frac{p^*(1-\alpha)}{m}\right)'$ converge to $\dfrac{N}{p}$, so that theorem 1.3 and 1.4 cannot applied if $\alpha = \dfrac{p-m}{p}$.

The paper is organized as follow : in the next section we prove the existence of a minimum for I, in the third section we give the proof of theorem 1.2 (proof of bounded minima), while the fourth section is devoted to the proof of Theorems 1.3 and 1.4.

2. EXISTENCE OF MINIMA

In order to prove that there exists a minimum of I on $W^{1,q}(I\!R^N)$, we are going to prove that I is both coercive and weakly lower semicontinuous on $W^{1,q}(I\!R^N)$.

Theorem 2.1. *Let m such that $1 < m < p(1-\alpha)$. Suppose that the hypotheses $(1.2), (1.4), (1.5), (1.6)$ and $a(x) > 0$ a.e in $I\!R^N$. Then I is coercive and weakly lower semicontinuous on $W^{1,q}(I\!R^N)$.*

Proof. It is clear that, by theorem du to De Giorgi (see [8]), the functional I is weak lower semicontinuous. For what concerns the coercivity, it is enough to consider v in $W^{1,q}(I\!R^N)$ such that $I(v)$ is finite.
We have

$$\int_{I\!R^N} |\nabla v|^q + |v|^q \, dx = \int_{I\!R^N} \frac{|\nabla v|^q + |v|^q}{(a(x)+|v|)^{\alpha q}} (a(x)+|v|)^{\alpha q} \, dx,$$

hence, by the Hölder inequality, since $q < p$, we have

$$\int_{I\!R^N} |\nabla v|^q + |v|^q \, dx \le c \left(\int_{I\!R^N} \frac{|\nabla v|^p + |v|^p}{(a(x)+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \times \left(\int_{I\!R^N} (a(x)+|v|)^{\frac{\alpha p q}{p-q}} \, dx \right)^{1-\frac{q}{p}}.$$

By easy calculation, we show that:

$$q^* = \frac{\alpha p q}{p-q}, \tag{2.1}$$

and that, from Sobolev embedding theorem we obtain:

$$\int_{I\!R^N} |\nabla v|^q + |v|^q \, dx \le c \left(\int_{I\!R^N} \frac{|\nabla v|^p + |v|^p}{(a(x)+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \times \left(1 + \left(\int_{I\!R^N} |\nabla v|^q + |v|^q \, dx \right)^{\frac{q^*}{q}} \right)^{1-\frac{q}{p}},$$

which implies that if $R = \|v\|_{W^{1,q}(I\!R^N)}$

$$R^p \le \left(\int_{I\!R^N} \frac{|\nabla v|^p + |v|^p}{(a(x)+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \left(1 + R^{q^*}\right)^{1-\frac{q}{p}}. \tag{2.2}$$

On the other hand, since $\left(\frac{q^*}{m}\right)' \le r_0 \le \left(\frac{q}{m}\right)'$, we have

$$\begin{aligned} \left| \int_{I\!R^N} f.|v|^m \, dx \right| &\le c \left(\int_{I\!R^N} |f|^{r_0} \, dx \right)^{\frac{1}{r_0}} \left(\int_{I\!R^N} |\nabla v|^q + |v|^q \, dx \right)^{\frac{m}{q}} \\ &\le cR. \end{aligned} \tag{2.3}$$

Hence, from (2.2) and (2.3) it follows that

$$I(v) \ge c \frac{R^p}{(1+R^{q^*})^{\frac{p}{q}-1}} - cR^m,$$

then recalling (2.1), it is easy to check that:

$$p - q^* \left(\frac{p}{q} - 1 \right) > 1.$$

Thus, we deduce that:

$$\lim_{R\to+\infty} I(v) = +\infty,$$

and then the coerciveness of I on $W^{1,q}(I\!R^N)$ is proven.

∎

It is will known that (see [5]), since I is coercive and weakly semi-continuous, there exists minimum of I on $W^{1,q}(I\!R^N)$. From which theorem 1.1 follows.

3. BOUNDED MINIMA

In this section, we will prove the boundedness of minimum of I. to do this we need the following lemma. We first recall that by theorem 1.2 there exists u in $W^{1,q}(I\!R^N)$ such that

$$I(u) = \min\left\{I(u), \quad v \in W^{1,q}(I\!R^N)\right\},$$

i.e

$$I(u) \le I(v) \quad \text{for all} \quad v \in W^{1,q}(I\!R^N). \tag{3.1}$$

Lemma 3.1. [4] *Let w be a function in $W^{1,\sigma}(\Omega)$ such that, for k is greater than some k_0*

$$\int_{A_k} |\nabla w|^\sigma \, dx \le ck^{\theta\sigma}|A_k|^{\frac{\sigma}{\sigma^*}+\varepsilon},$$

where $\varepsilon > 0, \quad 0 \le \theta < 1$.
Then the norm of w in $L^\infty(\Omega)$ is bounded by a constant which depends on $c, \theta, \sigma, N, \varepsilon, k_0$

The proof of this lemma can be found in the appendix of [4], its proof based on the lemma due of Stammpacchia [10].

Lemma 3.2. *Let u be a minimum of I on $W^{1,q}(I\!R^N)$, then there exists two positive constant c_1, c_2 such that:*

$$\int_{A_k} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \le c_1 k^m \int_{A_k} |f|\, dx + c_2 \int_{A_k} |f|.|G_k(u)|^m\, dx \ \ \forall\ k>0, \tag{3.2}$$

where A_k is as in (1.8) *and G_k is the function defined in* (1.7).

Proof. We have, $I(u) \le I(0) = 0$, then

$$\int_{I\!R^N} \frac{|\nabla u|^p + |u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \le \int_{I\!R^N} |f|.|u|^m\, dx < +\infty.$$

On the other hand, we have for all $k>0$,

$$\begin{aligned}\int_{I\!R^N} \frac{|\nabla T_k(u)|^p + |T_k(u)|^p}{(a(x)+|T_k(u)|)^{\alpha p}}\, dx &= \int_{\{|u|\le k\}} \frac{|\nabla u|^p + |u|^p}{(a(x)+|u|)^{\alpha p}}\, dx + \int_{\{|u|\ge k\}} \frac{k^p}{(a(x)+k)^{\alpha p}}\, dx \\ &\le \int_{I\!R^N} \frac{|\nabla u| + |u|^p}{(a(x)+|u|)^{\alpha p}}\, dx + k^{p(1-\alpha)}.\ \text{meas}\{|u|>k\} < +\infty,\end{aligned}$$

we take $v = T_k(u)$ as test function in (3.1) to obtain: $\forall\ k>0$.

$$\begin{aligned}\int_{A_k} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \ &\le \ \int_{A_k} |f|.(|u|^m - k^m)\, dx = \int_{A_k} |f|.\left((|u|-k+k)^m - k^m\right)\, dx, \\ &\le \ c_1 k^m \int_{A_k} |f|\, dx + c_2 \int_{A_k} |f|.|G_k(u)|^m\, dx.\end{aligned}$$

∎

Proof of Theorem 1.2. Let us consider σ a positive number such that $1 < \sigma < q < p$. Suppose that

$$1 < \sigma < q < p, \text{ with } \frac{1}{r} + \frac{m}{\sigma^*} < 1,$$

(see the remark below for the possibility of this choice).
By Lemma 3.2, Hölder and Sobolev inequalities, we have:
(3.3)

$$\begin{aligned}\int_{A_k} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\,dx\,dx &\leq c\|f\|_{L^r} k^m |A_k|^{1-\frac{1}{r}} + c\|f\|_{L^r}\left[\int_{A_k} |G_k(u)|^{\sigma^*}\,dx\right]^{\frac{m}{\sigma^*}} .|A_k|^{1-\frac{m}{\sigma^*}-\frac{1}{r}} \\ &\leq ck^m|A_k|^{1-\frac{1}{r}} + c\left[\int_{A_k} |\nabla u|^\sigma\,dx\right]^{\frac{m}{\sigma}} .|A_k|^{1-\frac{m}{\sigma^*}-\frac{1}{r}}.\end{aligned}$$

On the other hand, by Hölder inequality, we obtain:

$$\begin{aligned}\int_{A_k} |\nabla u|^\sigma\,dx &= \int_{A_k} \frac{|\nabla u|^\sigma}{(a(x)+|u|)^{\alpha\sigma}}(a(x)+|u|)^{\alpha\sigma}\,dx \\ &\leq \left[\int_{A_k} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\,dx\right]^{\frac{\sigma}{p}} \left[\int_{A_k} (a(x)+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{1-\frac{\sigma}{p}}.\end{aligned}$$

Moreover, from (3.3), we deduce, since $a \in L^\infty(I\!R^N)$,

$$\tag{3.4}\begin{aligned}\int_{A_k} |\nabla u|^\sigma\,dx \leq ck^{\frac{m\sigma}{p}} |A_k|^{(1-\frac{1}{r})\frac{\sigma}{p}} \left[\int_{A_k} (1+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{\frac{p-\sigma}{p}} \\ +c|A_k|^{\left(1-\frac{m}{\sigma^*}-\frac{1}{r}\right)\frac{\sigma}{p}} \left[\int_{A_k} |\nabla u|^\sigma\,dx\right]^{\frac{m}{p}} \left[\int_{A_k} (1+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{\frac{p-\sigma}{p}}.\end{aligned}$$

Applying Young's inequality with exponents $\frac{p}{m}$ and $\frac{p}{p-m}$, on the second terms of the right side, we get:

$$\tag{3.5}\begin{aligned}\int_{A_k} |\nabla u|^\sigma\,dx \leq ck^{\frac{m\sigma}{p}} |A_k|^{(1-\frac{1}{r})\frac{\sigma}{p}} \left[\int_{A_k} (1+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{\frac{p-\sigma}{p}} \\ +c|A_k|^{\left(1-\frac{m}{\sigma^*}-\frac{1}{r}\right)\frac{\sigma}{p-m}} \left[\int_{A_k} (1+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{\frac{p-\sigma}{p-m}}.\end{aligned}$$

Because it can be easily verified that if $k \geq 1$, one has that $1+|u| \leq 2(k+|G_k(u)|)$ on A_k, we can write:

$$\begin{aligned}\int_{A_k} |\nabla u|^\sigma\,dx \leq c\Big\{ & k^{(\frac{m}{p}+\alpha)\sigma}|A_k|^{\left(1-\frac{1}{r}\right)\frac{\sigma}{p}+1-\frac{\sigma}{p}} + k^{\frac{\alpha\sigma p}{p-m}}|A_k|^{\left(1-\frac{m}{\sigma^*}-\frac{1}{r}\right)\frac{\sigma}{p-m}+\frac{p-\sigma}{p-m}} \\ & +k^{\frac{m\sigma}{p}}|A_k|^{\left(1-\frac{1}{r}\right)\frac{\sigma}{p}} \left[\int_{A_k} |G_k(u)|^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{\frac{p-\sigma}{p}} \\ & + |A_k|^{\left(1-\frac{m}{\sigma^*}-\frac{1}{r}\right)\frac{\sigma}{p-m}} \left[\int_{A_k} |G_k(u)|^{\frac{\alpha\sigma p}{p-\sigma}}\,dx\right]^{\frac{p-\sigma}{p-m}}\Big\}.\end{aligned}$$

Now, observe that we can choose σ such that $\dfrac{\alpha\sigma p}{p-\sigma} < \sigma^*$ (see also remark below for the possibility of this choice). By this property we deduce, by using the Höder's and Sobolev's inequalities,

$$\begin{aligned}\int_{A_k} |\nabla u|^\sigma\, dx \;\leq\;& c\Big\{ k^{(\frac{m}{p}+\alpha)\sigma} |A_k|^{\left(1-\frac{\sigma}{rp}\right)} + k^{\frac{\alpha\sigma p}{p-m}} |A_k|^{\left(1-\frac{m}{\sigma^*}-\frac{1}{r}\right)\frac{\sigma}{p-m}+\frac{p-\sigma}{p-m}} \\ &+ k^{\frac{m\sigma}{p}} |A_k|^{\left(p-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p}} \left[\int_{A_k} |\nabla u|^\sigma\, dx\right]^\alpha \\ &+ |A_k|^{\left(p-m-\frac{\sigma}{r}+\frac{m\sigma}{N}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p-m}} \left[\int_{A_k} |\nabla u|^\sigma\, dx\right]^{\frac{\alpha p}{p-m}} \Big\}.\end{aligned}$$

Applying Young's inequality with exponents $\frac{1}{\alpha}$ and $\frac{1}{1-\alpha}$ (respectively $\frac{p-m}{\alpha p}$ and $\frac{p-m}{p-m-\alpha p}$), on the second (respectively third) terms of the right side, we get:

$$\begin{aligned}& k^{\frac{m\sigma}{p}} |A_k|^{\left(p-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p}} \left[\int_{A_k} |\nabla u|^\sigma\, dx\right]^\alpha \\ &\qquad \leq \frac{1}{4}\int_{A_k} |\nabla u|^\sigma\, dx + c k^{\frac{m\sigma}{p(1-\alpha)}} |A_k|^{\left(p-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p(1-\alpha)}}\end{aligned}$$

$$\begin{aligned}& |A_k|^{\left(p-m-\frac{\sigma}{r}+\frac{m\sigma}{N}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p-m}} \left[\int_{A_k} |\nabla u|^\sigma\, dx\right]^{\frac{\alpha p}{p-m}} \\ &\qquad \leq \frac{1}{4}\int_{A_k} |\nabla u|^\sigma\, dx + c|A_k|^{\left(p-m-\frac{\sigma}{r}+\frac{m\sigma}{N}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p-m-\alpha p}},\end{aligned}$$

so that we have:

(3.6)

$$\begin{aligned}\int_{A_k} |\nabla u|^\sigma\, dx \;\leq\;& c\Big\{ k^{(\frac{m}{p}+\alpha)\sigma} |A_k|^{\left(1-\frac{\sigma}{rp}\right)} + k^{\frac{\alpha\sigma p}{p-m}} |A_k|^{\left(1-\frac{m}{\sigma^*}-\frac{1}{r}\right)\frac{\sigma}{p-m}+\frac{p-\sigma}{p-m}} \\ &+ k^{\frac{m\sigma}{p(1-\alpha)}} |A_k|^{\left(p-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p(1-\alpha)}} + |A_k|^{\left(p-m-\frac{\sigma}{r}+\frac{m\sigma}{N}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p-m-\alpha p}} \Big\}.\end{aligned}$$

As can seen by means of straightforward calculations, the assumptions on r and α, imply that:

$$(p-m-\frac{\sigma}{r}+\frac{\sigma m}{N})\frac{1}{p-m} < \left(p-m-\frac{\sigma}{r}+\frac{\sigma m}{N}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p-m-\alpha p}.$$

We note that, from the fact that u belongs to $W^{1,q}(I\!R^N)$, it follows that the sequence $(|A_k|)_k$ tends to zero as k tends to infinity, thus there exists k_0 such that if $k \geq k_0$, we have

$$|A_k|^{\left(p-m-\frac{\sigma}{r}+\frac{\sigma m}{N}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p-m-\alpha p}} < |A_k|^{(p-m-\frac{\sigma}{r}+\frac{\sigma m}{N})\frac{1}{p-m}},$$

and so (3.6) implies that:

$$\begin{aligned}\int_{A_k} |\nabla u|^\sigma\, dx \;\leq\;& c\Big\{ k^{(\frac{m}{p}+\alpha)\sigma} |A_k|^{\left(1-\frac{\sigma}{rp}\right)} + k^{\frac{\alpha\sigma p}{p-m}} |A_k|^{\left(p-m-\frac{\sigma}{r}+\frac{\sigma m}{N}\right)\frac{1}{p-m}} \\ &+ k^{\frac{m\sigma}{p(1-\alpha)}} |A_k|^{\left(p-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p(1-\alpha)}} \Big\} \quad \forall\, k > k_0,\end{aligned}$$

which implies that:

$$\int_{A_k} |\nabla u|^\sigma\, dx \leq c k^{\varepsilon\sigma} |A_k|^\mu,$$

where

$$\varepsilon = \max\left(\frac{m}{p}+\alpha,\ \frac{\alpha p}{p-m},\ \frac{m}{p(1-\alpha)}\right),$$

and

$$\mu = \min\left(1-\frac{\sigma}{rp},\ \left(p-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{p(1-\alpha)},\ \left(p-m-\frac{\sigma}{r}+\frac{\sigma m}{N}\right)\frac{1}{p-m}\right).$$

We also observe that $\mu - \dfrac{\sigma}{\sigma^*} > 0$ since $r > \dfrac{N}{p}$ and ε belongs to $(0,1)$ since $0 < \alpha < \dfrac{p-m}{p}$.
Thus from Lemma 3.1 it follows that u belongs to $L^\infty(I\!R^N)$.
On the other hand, since

$$\int_{I\!R^N} \frac{|\nabla u|^p + |u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \le \int_{I\!R^N} |f|.|u|^m\, dx < +\infty,$$

the $L^\infty(I\!R^N)$ estimate implies that

$$\frac{1}{(\|a\|_{L^\infty(I\!R^N)} + \|u\|_{L^\infty(I\!R^N)})^{\alpha p}} \int_{I\!R^N} |\nabla u|^p + |u|^p\, dx \le \int_{I\!R^N} \frac{|\nabla u|^p + |u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \le c,$$

and so u belongs to $W^{1,p}(I\!R^N)$.
Finally, Theorem 1.2 is completely proved. ∎

Remark 3.1. Remark that

(1) $\dfrac{\alpha\sigma p}{p-\sigma} < \sigma^*$ is equivalent to $\sigma < q$.

(2) $\dfrac{1}{r} + \dfrac{m}{\sigma^*} < 1$ is equivalent to $\sigma > \dfrac{mNr}{Nr - N + mr}$,

(3) $\dfrac{mNr}{Nr-N+mr} < q$, for every $\alpha \in \left(1, \dfrac{p-m}{p}\right)$, $p \in (1,N)$ and $r > \dfrac{N}{p}$, which implies that we can choose σ satisfies $1)$ and $2)$.

4. SUMMABILITY OF UNBOUNDED MINIMA

In this section, we will prove Theorems 1.3 and 1.4. We first begin with the technical results, which will be used later

Lemma 4.1. *Let u be a minimum of I in $W^{1,q}(I\!R^N)$, then for all $k \in I\!N$, we have*

$$\int_{B_k} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \le \frac{c}{1+k} \int_{A_k} |f|.|u|^m\, dx \quad \forall\, k \in I\!N, \tag{4.1}$$

where A_k and B_k are as in (1.8) .
Moreover, if $a \in L^\infty(I\!R^N)$, we have

$$\int_{B_k} |\nabla u|^p\, dx \le c(1+k)^{\alpha p - 1} \int_{A_k} |f|.|u|^m\, dx \quad \forall\, k \in I\!N, \tag{4.2}$$

where c is a constant independent of k.

Proof. - Let us observe that, if $k = 0$, since u is a minimum of I, we get

$$\int_{I\!R^N} \frac{|\nabla u|^p + |u|^p}{a(x)+|u|)^{\alpha p}}\, dx \le \int_{I\!R^N} |f|.|u|^m\, dx,$$

this implies that:

$$\int_{\{0\le|u|<1\}} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \le \int_{I\!R^N} |f|.|u|^m\, dx = \int_{A_0} |f|.|u|^m\, dx.$$

Thus (4.1) (and then (4.2)) is proven.

- Assume now that $k > 0$ and let $v = u - T_1(u - T_k(u))$, observe that $v \in W^{1,q}(I\!R^N)$ and by a simple calculation, we get

$$\int_{I\!R^N} \frac{|\nabla v|^p + |v|^p}{(a(x) + |v|)^{\alpha p}}\, dx < +\infty.$$

We take v as test function in (3.1), we obtain:

$$\int_{I\!R^N} \frac{|\nabla u|^p + |u|^p}{(a(x) + |u|)^{\alpha p}}\, dx - \int_{I\!R^N} f.|u|^m\, dx \le \int_{I\!R^N} \frac{|\nabla v|^p + |v|^p}{(a(x) + |v|)^{\alpha p}}\, dx - \int_{I\!R^N} f.|v|^m\, dx,$$

then

$$\begin{aligned}\int_{I\!R^N} \frac{|\nabla u|^p + |u|^p}{(a(x) + |u|)^{\alpha p}}\, dx &- \int_{I\!R^N} \frac{|\nabla v|^p + |v|^p}{(a(x) + |v|)^{\alpha p}}\, dx \\ &\le \int_{A_k} |f|. \left(|u|^m - |v|^m\right)\, dx \le m \int_{A_k} |f|.|u|^{m-1}\, dx,\end{aligned}$$

which implies that:

$$\begin{aligned}\int_{I\!R^N} \frac{|\nabla u|^p + |u|^p}{(a(x) + |u|)^{\alpha p}}\, dx &- \int_{I\!R^N} \frac{|\nabla v|^p + |v|^p}{(a(x) + |v|)^{\alpha p}}\, dx \\ &\le \frac{m}{k} \int_{A_k} |f|.|u|^m\, dx \le \frac{2m}{k+1} \int_{A_k} |f|.|u|^m\, dx.\end{aligned}$$

By definition of v, the previous inequality implies that:

$$\begin{aligned}\int_{B_k} &\left\{ \frac{|\nabla u|^p + |u|^p}{(a(x) + |u|)^{\alpha p}} - \frac{k^p}{(a(x) + k)^{\alpha p}} \right\} dx \\ &+ \int_{A_{k+1}} \left\{ \frac{|\nabla u|^p + |u|^p}{(a(x) + |u|)^{\alpha p}} - \frac{|\nabla v|^p + |v|^p}{(a(x) + |v|)^{\alpha p}} \right\} dx \le \frac{2m}{k+1} \int_{A_k} |f|.|u|^m\, dx,\end{aligned}$$

since the function

$$f_x : s \longrightarrow \frac{s^p}{(a(x) + s)^{\alpha p}},$$

is increasing for all $x \in I\!R^N$, we have:

$$\begin{aligned}\int_{B_k} \frac{|\nabla u|^p}{(a(x) + |u|)^{\alpha p}}\, dx \;&\le\; \int_{A_{k+1}} |\nabla u|^p \left\{ \frac{1}{(a(x) + |v|)^{\alpha p}} - \frac{1}{(a(x) + |u|)^{\alpha p}} \right\} dx \\ &\quad + \frac{2m}{k+1} \int_{A_k} |f|.|u|^m\, dx \\ &\le\; \int_{A_{k+1}} |\nabla u|^p \frac{(a(x) + |u|)^{\alpha p} - (a(x) + |v|)^{\alpha p}}{(a(x) + |v|)^{\alpha p}(a(x) + |u|)^{\alpha p}}\, dx \\ &\quad + \frac{2}{k+1} \int_{A_k} |f|.|u|^m\, dx.\end{aligned} \tag{4.3}$$

Since $|v| = |u| - 1$ on A_{k+1}, we easily obtain that there exists a positive constant c (independent of x) such that

$$(a(x) + |u|)^{\alpha p} - (a(x) + |v|)^{\alpha p} \le c(a(x) + |v|)^{\alpha p - 1}.$$

Thus (4.3) becomes

$$\int_{B_k} \frac{|\nabla u|^p}{(a(x) + |u|)^{\alpha p}}\, dx \le c \int_{A_{k+1}} \frac{|\nabla u|^p}{(a(x) + |v|)(a(x) + |u|)^{\alpha p}}\, dx + \frac{2m}{k+1} \int_{A_k} |f|.|u|^m\, dx.$$

Since $|v| \geq k$ on A_{k+1}, we have:

$$\int_{B_k} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \leq \frac{c}{k} \int_{A_{k+1}} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\, dx + \frac{2m}{k+1} \int_{A_k} |f|.|u|^m\, dx.$$

Finally, we have:

$$\int_{B_k} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\, dx \leq \frac{c}{k+1} \int_{A_k} |f|.|u|^m\, dx. \tag{4.4}$$

Moreover, if $a \in L^\infty(I\!R^N)$, we get:

$$a(x) + |u(x)| \leq \|a\|_\infty + k + 1 \leq c(k+1), \quad \text{for all} \quad x \in B_k,$$

thus and (4.4), we deduce that:

$$\int_{B_k} \frac{|\nabla u|^p}{(1+k)^{\alpha p}}\, dx \leq \frac{c}{k+1} \int_{A_k} |f|.|u|^m\, dx,$$

which implies that:

$$\int_{B_k} |\nabla u|^p\, dx \leq c(k+1)^{\alpha p-1} \int_{A_k} |f|.|u|^m\, dx. \tag{4.5}$$

∎

Proof of Theorem 1.3. Let us now prove the following

Theorem 4.2. *Let u a minimum of I on $W^{1,q}(I\!R^N)$. Under the hypotheses of Theorem 1.3, the following assertions hold true :*

(1) **Assertion I:** *For all $\gamma \geq 1$, we have*

$$\int_{I\!R^N} |\nabla u|^p |u|^{p(\gamma-1)}\, dx \leq c_1 + c_2 \int_{I\!R^N} |f||u|^{\alpha p + p\gamma - p + m}\, dx \tag{4.6}$$

where c_1 and c_2 are two positive constants.

(2) **Assertion II:** *There exists a positive constant c_3 such that*

$$\int_{I\!R^N} |u|^s\, dx \leq c_3.$$

(3) **Assertion III:** *There exists a positive constant c_4 such that*

$$\int_{I\!R^N} |\nabla u|^p\, dx \leq c_4.$$

We cite the following (simple but fundamental) lemma.

Lemma 4.3. *Let $\gamma = \dfrac{r(N-p)\,[p(1-\alpha)-m]}{p(N-rp)}$, we have*

i) $s = \gamma p^* = r'(\alpha p + \gamma p - p + 1)$,

ii) $\gamma \geq 1$ *if and only if* $r \geq \left(\dfrac{p^*}{m+\alpha p}\right)'$,

iii) $\dfrac{p}{p^*} > \dfrac{1}{r'}$ *if and only if* $r < \dfrac{N}{p}$.

∎

Proof of Theorem 4.2. (1) **Proof of Assertion I:** Let $\gamma \geq 1$, we have by using (4.5)

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^p |u|^{p(\gamma-1)}\,dx &= \sum_{k=0}^{+\infty} \int_{B_k} |\nabla u|^p |u|^{p(\gamma-1)}\,dx \\ &\leq \sum_{k=0}^{+\infty} \int_{B_k} |\nabla u|^p (1+k)^{p(\gamma-1)}\,dx \\ &\leq c\sum_{k=0}^{+\infty} \int_{A_k} |f|.|u|^m (1+k)^{\alpha p+\gamma p-p-1}\,dx.\end{aligned} \tag{4.7}$$

Recalling that, for $k \in \mathbb{N}$, we have:

$$\int_{A_k} |f|.|u|^m\,dx = \sum_{h=k}^{+\infty} \int_{B_h} |f|.|u|^m\,dx, \tag{4.8}$$

hence

$$\begin{aligned}\sum_{k=0}^{+\infty}(1+k)^{\alpha p+\gamma p-p-1} \int_{A_k} |f|.|u|^m\,dx \\ = \sum_{k=0}^{+\infty}(1+k)^{\alpha p+\gamma p-p-1} \sum_{h=k}^{+\infty} \int_{B_h} |f|.|u|^m\,dx.\end{aligned} \tag{4.9}$$

Therefore, by changing the order of summation, and recalling that:

$$\sum_{k=0}^{h} k^l \leq c(1+h)^{l+1}, \tag{4.10}$$

with $c = c(l)$, we have:

$$\begin{aligned}\sum_{k=0}^{+\infty}(1+k)^{\alpha p+\gamma p-p-1} \int_{A_k} |f|.|u|^m\,dx \\ = \sum_{h=0}^{+\infty}\sum_{k=0}^{h}(1+k)^{\alpha p+\gamma p-p-1} \int_{B_h} |f|.|u|^m\,dx \\ = \sum_{h=0}^{+\infty}(1+h)^{\alpha p+\gamma p-p} \int_{B_h} |f|.|u|^m\,dx,\end{aligned} \tag{4.11}$$

using (4.7) we obtain:

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^p |u|^{p(\gamma-1)}\,dx &\leq c\sum_{k=0}^{+\infty}(1+k)^{\alpha p+\gamma p-p} \int_{B_k} |f|.|u|^m\,dx \\ &\leq c\sum_{k=0}^{+\infty} \int_{B_k} |f|.|u|^m (1+|u|)^{\alpha p+\gamma p-p}\,dx \\ &\leq c\int_{\Omega} |f|.|u|^m\,dx + c\int_{\Omega} |f|.|u|^{\alpha p+\gamma p-p+m}\,dx.\end{aligned}$$

Finally, by the fact that $\int_\Omega |f|.|u|^m\, dx < +\infty$, we deduce (4.6).

Proof of Assertion II: Let γ as in Lemma 4.2. We obtain, by Sobolev embedding and Lemma 4.2,

$$\begin{aligned}\left(\int_{I\!R^N} |u|^s\, dx\right)^{\frac{p}{p^*}} &= \left(\int_{I\!R^N} |u|^{\gamma p^*}\, dx\right)\frac{p}{p^*} \leq \int_{I\!R^N} |\nabla u|^p |u|^{p(\gamma-1)}\, dx \\ &\leq c + c\int_{I\!R^N} |f||u|^{\alpha p+\gamma p-p+m}\, dx.\end{aligned} \tag{4.12}$$

The above inequality and Hölder inequality implies

$$\left(\int_{I\!R^N} |u|^s\, dx\right)^{\frac{p}{p^*}} \leq \left(\int_{I\!R^N} |f|^r\, dx\right)^{\frac{1}{r}} \left(\int_{I\!R^N} |u|^{r'(\alpha p+\gamma p-p+m)}\, dx\right)^{\frac{1}{r'}}.$$

Recalling $i), iii)$ of Lemma 4.2, we deduce by Young's inequality that:

$$\int_{I\!R^N} |u|^s\, dx \leq c_3. \tag{4.13}$$

(2) **Proof of Assertion III:** From (4.12), we get

$$\int_{I\!R^N} |\nabla u|^p |u|^{p(\gamma-1)}\, dx \leq c_4, \tag{4.14}$$

on the other hand, we have:

$$\begin{aligned}\int_{I\!R^N} |\nabla u|^p\, dx &= \int_{\{|u|\leq 1\}} |\nabla u|^p\, dx + \int_{\{|u|\geq 1\}} |\nabla u|^p\, dx \\ &\leq c\int_{\{|u|\leq 1\}} |\nabla u|^p\, dx + \int_{I\!R^N} |\nabla u|^p |u|^{p(\gamma-1)}\, dx \\ &\leq c\int_{I\!R^N} |f||u|^m\, dx + \int_{I\!R^N} |\nabla u|^p |u|^{p(\gamma-1)}\, dx,\end{aligned}$$

thus, by (4.14) implies that:

$$\int_{I\!R^N} |\nabla u|^p\, dx \leq c_5. \tag{4.15}$$

Since $u \in L^s(I\!R^N)$, with $s = p^*\gamma \geq p^*$ and $u \in L^q(I\!R^N)$, this implies (since $q < p \leq p^*$),

$$u \in L^p(I\!R^N). \tag{4.16}$$

From $(4.13), (4.15)$ and (4.16), we deduce that:

$$u \in W^{1,p}(I\!R^N) \cap L^s(I\!R^N).$$

∎

Proof of Theorem 1.4.

Theorem 4.4. *Let u a minimum of I in $W^{1,q}(I\!R^N)$. Under the hypotheses of Theorem 1.4, the following assertions hold true :*

(1) **Assertion I:** *For all* $\lambda > 0$, *we have:*

$$\int_{I\!R^N} \frac{|\nabla u|^p}{(1+|u|)^\lambda}\, dx \le c_5 + c_6 \int_{I\!R^N} |f|(a(x)+|u|)^{\alpha p-\lambda+m}\, dx, \tag{4.17}$$

where c_5 *and* c_6 *are two positive constants.*

(2) **Assertion II:** *There exists a positive constant* c_7 *such that*

$$\int_{I\!R^N} |u|^s\, dx \le c_7.$$

(3) **Assertion III:** *There exists a positive constant* c_8 *such that*

$$\int_{I\!R^N} |\nabla u|^\rho\, dx \le c_8.$$

The following lemma is simple but fundamental for the proof of the previous theorem.

Lemma 4.5. *Let* $\lambda = \dfrac{p\,[N - r(m+\alpha p)] - rN\,[p(1-\alpha)-m]}{N-rp}$, *we have the following properties :*

i) $s = \dfrac{\lambda\rho}{p-\rho} = r'(\alpha p - \lambda + m)$,

ii) $\lambda > 0$ *if and only if* $r < \left(\dfrac{p^*}{m+\alpha p}\right)'$,

iii) $\dfrac{\rho}{pr'} + 1 - \dfrac{\rho}{p} < \dfrac{\rho}{s}$.

∎

Proof of Theorem 4.4. (1) **Proof of assertion I :** Let $\lambda > 0$ and let $u \in W^{1,q}(I\!R^N)$ a minimum of I, we have:

$$\begin{aligned} \int_{I\!R^N} \frac{|\nabla u|^p}{(a(x)+|u|)^\lambda}\, dx &= \int_{I\!R^N} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}(a(x)+|u|)^{\alpha p-\lambda}\, dx \\ &\le c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k} \frac{|\nabla u|^p}{(a(x)+|u|)^{\alpha p}}\, dx. \end{aligned} \tag{4.18}$$

By (4.1), inequality (4.19) can be written as

$$\int_{I\!R^N} \frac{|\nabla u|^p}{(a(x)+|u|)^\lambda}\, dx \le c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k} |f|.|u|^m\, dx. \tag{4.19}$$

on the other hand, by (4.8) one has

$$\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k} |f|.|u|^m\, dx$$
$$= \sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\sum_{h=k}^{+\infty}\int_{B_h} |f|.|u|^m\, dx.$$

Reasoning as in the proof of Theorem 4.2, we have:

$$\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k}|f|.|u|^m\,dx \leq \sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}|f|.|u|^m\,dx. \tag{4.20}$$

From (4.19) and (4.20), we deduce that:

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{(a(x)+|u|)^\lambda}\,dx &\leq c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}|f|.|u|\,dx\\ &\leq c\sum_{k=0}^{+\infty}\int_{B_k}|f|.|u|^m(1+|u|)^{\alpha p-\lambda}\,dx\\ &\leq c\int_{\mathbb{R}^N}|f|.|u|^m\,dx+\int_{\mathbb{R}^N}|f|.|u|^{\alpha p-\lambda+m}\,dx.\end{aligned}$$

Since $\int_{\mathbb{R}^N}|f|.|u|^m\,dx<+\infty$, the assertion I is proved.

(2) **Proof of assertion II :** Let $\lambda>0$ be as in Lemma 4.5. Since $\rho^*=s$, we have by the Sobolev embedding

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|u|^s\,dx\right)^{\frac{\rho}{s}} &= \left(\int_{\mathbb{R}^N}|u|^{\rho^*}\,dx\right)^{\frac{\rho}{\rho^*}}\leq c\int_{\mathbb{R}^N}|\nabla u|^\rho\,dx\\ &= c\int_{\mathbb{R}^N}\frac{|\nabla u|^\rho}{(a(x)+|u|)^{\frac{\lambda\rho}{p}}}(a(x)+|u|)^{\frac{\lambda\rho}{p}}\,dx.\end{aligned} \tag{4.21}$$

Applying Hölder inequality, we have:

$$\left(\int_{\mathbb{R}^N}|u|^s\,dx\right)^{\frac{\rho}{s}}\leq c\left[\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{(a(x)+|u|)^\lambda}\,dx\right]^{\frac{\rho}{p}}\left[\int_{\mathbb{R}^N}(a(x)+|u|)^{\frac{\lambda\rho}{p-\rho}}\,dx\right]^{1-\frac{\rho}{p}}. \tag{4.22}$$

On the other hand, by assertion I and Hölder inequality , we deduce that:

$$\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{(a(x)+|u|)^\lambda}\,dx\leq c_8+c_9\left[\int_{\mathbb{R}^N}|u|^{r'(\alpha p-\lambda+m)}\,dx\right]^{\frac{1}{r'}}. \tag{4.23}$$

From $(4.21), (4.22)$ and (4.23), we obtain:

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|u|^s\,dx\right)^{\frac{\rho}{s}} &\leq c\int_{\mathbb{R}^N}|\nabla u|^\rho\,dx\\ &\leq c\left[1+\int_{\mathbb{R}^N}|u|^{r'(\alpha p-\lambda+m)}\,dx\right]^{\frac{\rho}{pr'}}\\ &\quad\times\left[\int_{\mathbb{R}^N}(a(x))^{\frac{\lambda\rho}{p-\rho}}\,dx+\int_{\mathbb{R}^N}|u|^{\frac{\lambda\rho}{p-\rho}}\,dx\right]^{1-\frac{\rho}{p}}.\end{aligned} \tag{4.24}$$

From 4.5, (4.24) becomes

$$\left(\int_{\mathbb{R}^N}|u|^s\,dx\right)^{\frac{\rho}{s}}\leq c_9+c_{10}\left[\int_\Omega|u|^s\,dx\right]^{\frac{\rho}{pr'}+1-\frac{\rho}{p}}.$$

Finally, from $iii)$ of Lemma 4.5 we deduce by Young's inequality that

$$\int_{I\!R^N} |u|^s \, dx \leq c_{11}.$$

(3) **Proof of assertion III :** By assertion II and (4.24), we have:

$$\int_{I\!R^N} |\nabla u|^\rho \, dx \leq c_{12}.$$

hence, since $\rho^* = s$ and $\rho \geq q$, we have:

$$u \in L^\rho(I\!R^N).$$

Finally, we deduce that:

$$u \in W^{1,\rho}(I\!R^N).$$

∎

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