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NEW STOCHASTIC CALCULUS

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ABSTRACT. We present new stochastic differential equations, that are more general and simpler than the existing Ito-based stochastic differential equations. As an example, we apply our approach to the investment (portfolio) model.

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1. INTRODUCTION

The literature on stochastic processes relied mainly on Ito's rule to derive stochastic differential equations such as the Hamilton-Jacobi partial differential equation. Examples include Siu [4], Bayraktar and Young [2], Alghalith [1] and Focardi and Fabozzi [3], among others.

It is well-known that Ito's rule is based on the assumptions of normality and Levy processes such as Wiener processes (Brownian motions). In addition, Ito's rule assumes convergence in probability. Moreover, the coefficients in Ito's formula are not necessarily constant, which makes the formula less convenient.

In this paper, we provide general and simple stochastic differential equations that overcome the limitations of the existing differential equations. In doing so, we derive stochastic differential equations that, first, do not assume normality or any particular probability distribution. Secondly, we do not assume a Levy process or any particular process. Thirdly, we do not assume convergence. Moreover, these stochastic differential equations have constant coefficients. This provides analytical convenience.

2. THE MODEL

We consider the stochastic process $f(t, \xi(t))$, where ξ is stochastic and t denotes time. Without loss of generality, we define $\xi(t) = \bar{\xi}(t) + \sigma(t)\eta(t)$, where η is random with $E\eta(t) = 0$, while $\bar{\xi}$ is the mean and σ^2 is the volatility. Using Taylor series expansion of $f(t, \xi(t))$ around $\psi = (a, b)$ (and suppressing the notational dependence on t), we obtain

$$\begin{aligned} f(t,\xi) &= f(\psi) + f_t(\psi)(t-a) + f_\xi(\psi)(\xi-b) + \frac{1}{2}f_{\xi\xi}(\psi)(\xi-b)^2 + \frac{1}{2}f_{tt}(\psi)(t-a)^2 + \\ &\quad f_{t\xi}(\psi)(t-a)(\xi-b) + R(t,\xi), \end{aligned} \tag{2.1}$$

where $R(t,\xi)$ is the remainder and the subscripts denote derivatives. Our intermediate objective is to minimize the absolute value of the expected value of the remainder

$$\min_{\bar{\xi}} |ER(t,\xi)|,$$

where

$$ER(t,\xi_t) = E\left[\begin{array}{c} f(t,\xi(t)) - \\ \left\{\begin{array}{c} f(\psi) + f_t(\psi)(t-a) + f_\xi(\psi)(\xi-b) + \\ \frac{1}{2}f_{\xi\xi}(\psi)(\xi-b)^2 + \\ \frac{1}{2}f_{tt}(\psi)(t-a)^2 + f_{t\xi}(\psi)(t-a)(\xi-b) \end{array}\right\} \end{array}\right]. \tag{2.2}$$

The solution yields

$$Ef_{\bar{\xi}}\left(t, \bar{\xi}^* + \sigma\eta\right) - f_{\bar{\xi}}(\psi) - f_{\bar{\xi}\bar{\xi}}(\psi)\left(\bar{\xi}^* - b\right) - f_{t\bar{\xi}}(\psi)(t-a) = 0. \tag{2.3}$$

Since the solution $\bar{\xi}^*$ depends on the values of a and b, the optimization is equivalent to choosing a specific value of $a = \hat{a}$ such that $\bar{\xi}^*(t)$ is equal to the actual expected value of ξ obtained from historical data or numerical methods. Therefore the point of expansion $\hat{a}$ is not arbitrarily chosen (however b is arbitrarily chosen). Hence (2.3) can be rewritten as

$$Ef_{\bar{\xi}}(t,\xi) = f_{\bar{\xi}}(\psi) + f_{\bar{\xi}\bar{\xi}}(\psi)(E\xi - b) + f_{t\bar{\xi}}(\psi)(t-\hat{a}).$$

Thus

$$
\begin{aligned}
Ef(t,\xi) &= f(\psi) + f_t(\psi)(t-\hat{a}) + f_{\bar{\xi}}(\psi)\left(\bar{\xi}-b\right) + \\
&\quad \frac{1}{2} f_{\bar{\xi}\bar{\xi}}(\psi) E(\xi-b)^2 + \\
&\quad \frac{1}{2} f_{tt}(\psi)(t-\hat{a})^2 + f_{t\bar{\xi}}(\psi)(t-\hat{a})\left(\bar{\xi}-b\right). \qquad (2.4)
\end{aligned}
$$

Without loss of generality, we define $f(t,\xi_t) = Ef(t,\xi) + \delta(t)\varpi(t)$, where ϖ is random and δ^2 is the volatility of f. Adding $\delta(t)\varpi(t)$ to both sides of (2.4), we obtain

$$
\begin{aligned}
f(t,\xi) &= f(\psi) + \delta\varpi + f_t(\psi)(t-\hat{a}) + \\
&\quad f_{\bar{\xi}}(\psi)\left(\bar{\xi}-b\right) + \frac{1}{2} f_{\bar{\xi}\bar{\xi}}(\psi) E(\xi-b)^2 + \\
&\quad \frac{1}{2} f_{tt}(\psi)(t-\hat{a})^2 + f_{t\bar{\xi}}(\psi)(t-\hat{a})\left(\bar{\xi}-b\right). \qquad (2.5)
\end{aligned}
$$

Differentiating (2.5) with respect to t and ϖ, respectively, we obtain

$$
f_t(t,\xi) = f_t(\psi) + f_{tt}(\psi)(t-\hat{a}) + f_{t\bar{\xi}}(\psi)\left(\bar{\xi}(t)-b\right) + \delta_t(t)\varpi_t(t), \qquad (2.6)
$$

$$
f_\xi(t,\xi^*(t)) = \delta(t)\varpi_\xi(t), \qquad (2.7)
$$

and thus

$$
df(t,\xi) = \left[\begin{array}{c} f_t(\psi) + f_{tt}(\psi)(t-\hat{a}) + f_{t\bar{\xi}}(\psi)\left(\bar{\xi}(t)-b\right) + \\ \delta_t(t)\varpi_t(t) \end{array}\right] dt + \delta(t)\varpi_\xi(t)d\xi(t). \qquad (2.8)
$$

If $t=0$ (time-independence), (2.8) is reduced to

$$
df(\xi) = \delta\varpi_\xi d\xi. \qquad (2.9)
$$

Therefore, virtually, the dynamics of any stochastic process are given by (2.8) or (2.9). The extension to the multiple-variable case is straightforward.

3. PRACTICAL EXAMPLE (THE INVESTMENT MODEL)

We apply our new method to the standard investment model (a major model in finance). Below is a brief description of the investment model (see, for example, Focardi and Fabozzi [3]). The wealth process is given by

$$
X(T) = x + \int_s^T \mu\pi(t)\,ds + \int_s^T \pi(t)\sigma dW(t); s \le t \le T, \qquad (3.1)
$$

where x is the initial wealth, π is the risky portfolio (the value of the risky asset), W is a Brownian motion, μ is the risk premium (the rate of return of the risky asset minus the risk-free rate), and σ is the volatility of the risky asset. Hence,

$$
dX(t) = \mu(t)\pi(t)\,ds + \pi(t)\sigma(t)\,dW(t). \qquad (3.2)
$$

The investor invests a fraction of his/her wealth in the risky asset and the remainder in the risk free asset (such as a bank account). The investor's objective is to maximize the expected utility of the terminal wealth with respect to the risky portfolio

$$
V(s,x) = \underset{\pi}{Sup} E_s U(X(T)),
$$

where $V(.)$ is the value function, $U(.)$ is a differentiable and bounded utility function. Similar to previous literature, we define $E_s U(X(t)) = J(t, X(t))$.

Now we apply our new approach. Using (2.8) and choosing $b = 0$, we obtain

(3.3)
$$dJ(t, X) = [J_t(\psi) + J_{tt}(\psi)(t - \hat{a}) + J_{tX}(\psi) EX + \delta_t(t) dW(t)] dt + \varpi_X(t) \delta(t) dX(t).$$

Using Stein's lemma $\delta(t) = J_{XX}\pi^2(t)\sigma^2(t)$ and $\varpi_X(t) = 1/\pi(t)\sigma(t)$, and since the partial derivatives in (3.3) are constant, we can rewrite (3.3) as

$$dJ(t, X) = (c_1 + c_2(t - \hat{a}) + c_3 EX) dt + c_4 \pi(t) \sigma(t) dX(t), \tag{3.4}$$

where c_i is a constant.

To solve for the optimal risky portfolio π^*, using the Feynman-Kac approach, we obtain

$$c_1 + c_2(s - \hat{a}) + c_3 x + \underset{\pi}{Sup}\left\{c_3 \mu(s) \pi(s) + c_4 \pi^2(s) \sigma(s) \mu(s)\right\} = 0. \tag{3.5}$$

Differentiating the above formula with respect $\pi(s)$ and arranging yields

$$\pi^*(s) = \frac{c}{2\sigma(s)}, \tag{3.6}$$

where $c = -c_3/c_4$. It is worth emphasizing that this solution is far simpler than the solution based on Ito's calculus.

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