The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 12, Issue 1, Article 17, pp. 1-9, 2015

ON THE HYERS-ULAM STABILITY OF HOMOMORPHISMS AND LIE DERIVATIONS

JAVAD IZADI AND BAHMANN YOUSEFI

Received 16 October, 2015; accepted 30 November, 2015; published 14 December, 2015.

DEPARTMENT OF MATHEMATICS, PAYAME NOOR UNIVERSITY, P.O. BOX: 19395-3697, TEHRAN, IRAN.
javadie2003@yahoo.com, b_yousefi@pnu.ac.ir

ABSTRACT. Let A be a Lie Banach*-algebra. For each elements (a, b) and (c, d) in $A^2 := A \times A$, by definitions

$$(a,b)(c,d) = (ac, bd),$$
$$\|(a,b)\| = \|a\| + \|b\|,$$
$$(a,b)^* = (a^*, b^*),$$

A^2 can be considered as a Banach*-algebra. This Banach*-algebra is called a Lie Banach*-algebra whenever it is equipped with the following definitions of Lie product:

$$[(a,b),(c,d)] = \left(\frac{ac-ca}{2}, \frac{bd-db}{2}\right)$$

for all a, b, c, d in A. Also, if A is a Lie Banach*-algebra, then $D : A^2 \longrightarrow A^2$ satisfying

$$D([(a,b),(c,d)]) = [D(a,b),(c,d)] + [(a,b), D(c,d)]$$

for all $a, b, c, d \in A$, is a Lie derivation on A^2. Furthermore, if A is a Lie Banach*-algebra, then D is called a Lie* derivation on A^2 whenever D is a Lie derivation with $D(a,b)^* = D(a^*, b^*)$ for all $a, b \in A$. In this paper, we investigate the Hyers-Ulam stability of Lie Banach*-algebra homomorphisms and Lie*derivations on the Banach*-algebra A^2.

Key words and phrases: Hyers-Ulam stability; Banach*-algebra; Fixed point method; Lie derivation.

2010 *Mathematics Subject Classification.* Primary 39B82. Secondary 47H10, 46L05.

ISSN (electronic): 1449-5910

1. INTRODUCTION

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?".

If the problem accepts a solution, we say that the equation is *stable*. The notion of stability of mathematical theorems considered from a rather general point of view: When is it true that by changing a little in the hypothesis of a theorem, one can still assert that the theorem remains true or approximately true? When is it true that the solution of an equation differing slightly from a given one, should be necessarily close to the solution of the given equation? Similarly, if we replace a given functional equation by a functional inequality, when can one assert that the of the inequality lie near to the exact solutions of that equation?

The stability problem of functional equations, concerning group homomorphisms, had been first raised by Ulam in 1940 ([16]). In 1941, this problem solved by Hyers for additive groups under the assumption that the groups are Banach spaces ([8]). In 1978, Th. M. Rassias provided a generalization of the Hypers'theorem by proving the existence of unique linear mappings near approximate to additive mappings ([13]). The result of Rassias has provided a lot of influence during the last three decades in the development of generalization of the Hyers-Ulam stability concept which is now called the Hyers-Ulam-Rassias stability theory for functional equations. Furthermore, in 1994, a generalization of Rassias, theorem was obtained by Găvruta by replacing the bound $\in (\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$ ([7]).

The stability problems of several functional equations have been extensively investigated by a number of authors (see [1]-[18]).

Theorem 1.1. *Let (X, d) be a complete generalized metric space and $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

: *(a) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
: *(b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
: *(c) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
: *(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Definition 1.1. By a Banach*-algebra A, we mean a Banach*-algebra over the field of complex numbers, together with a map $* : A \longrightarrow A$ called involution which has the following properties:

: 1. $(x + y)^* = x^* + y^*$ for all x, y in A.
: 2. $(\lambda x)^* = \bar{\lambda} x$ for every λ in $\mathbb{C}$ and every x in A.
: 3. $(x, y)^* = y^* x^*$ for all x, y in A.
: 4. $(x^*)^* = x$ for all x in A.

Recall that if in addition, the condition $\|xx^*\| = \|x\|\|x^*\|$ holds for all x in A, then A is called a C^*-algebra.

Definition 1.2. A $\mathbb{C}-$linear mapping H from a Banach*-algebra A in to a Banach*-algebra B is called a homomorphism in Banach*-algebras if it satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = (H(x))^*$ for all x, y in A.

Definition 1.3. A $\mathbb{C}$-linear self mapping D on a Banach*-algebra B is called a *derivation* on B if D satisfies

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in B$.

Definition 1.4. A Banach*-algebra A, endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

on A, is called a Lie Banach*-algebra.

Definition 1.5. Let A and B be Lie Banach*-algebras. A $\mathbb{C}$-linear mapping $H : A \longrightarrow B$ is called a Lie Banach*-algebra homomorphism if

$$H([x, y]) = [H(x), H(y)]$$

and $H(x^*) = (H(x))^*$ for all x, y in A.

Definition 1.6. A linear self mapping D on a Lie Banach*-algebra A is called a Lie derivation of

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in A$.

2. STABILITY OF HOMOMORPHISMS ON BANACH*-ALGEBRAS

In this section we want to investigate the Hyers-Ulam stability of homomorphisms for special functional equations on Banach*-algebras.

Let A be a Lie Banach*-algebra. For each element (a, b) and (c, d) in $A^2 := A \times A$ define

$$(a, b)(c, d) = (ac, bd),$$
$$\|(a, b)\| = \|a\| + \|b\|,$$
$$(a, b)^* = (a^*, b^*),$$

Then A^2 is a Banach*-algebra with the above norm and product, and involution. Note that A^2 is not necessarily a C^*-algebra. This Banach*-algebra is called a Lie Banach*-algebra with the following definition of Lie product:

$$[(a, b), (c, d)] = \left(\frac{ac - ca}{2}, \frac{bd - db}{2}\right)$$

for all a, b, c, d in A.

Theorem 2.1. *Suppose that A and B are Lie Banach*-algebras and $f : A^2 \longrightarrow B$ is a mapping such that $f(0, 0) = 0$. Let $\varphi : A^2 \longrightarrow [0, \infty)$ be a function satisfying:*

$$\left\|2\mu f\left(\tfrac{a+c}{2}, \tfrac{b+d}{2}\right) - f(\mu a, \mu b) - f(\mu c, \mu d)\right\| \le \varphi(a + c, b + d), \tag{2.1}$$

$$\|f([(a, b), (c, d)] - [f(a, b), f(c, d)])\| \le \varphi(ac, bd), \tag{2.2}$$

$$\|f(a^*, b^*) - (f(a, b))^*\| \le \varphi(a, b), \tag{2.3}$$

for all scalars μ with $|\mu| = 1$, and all a, b, c, d in A. Also, suppose that there exists $0 < r < 1$ such that for all $a, b \in A$,

$$\varphi(a, b) \le \frac{r}{2}\varphi(2a, 2b). \tag{2.4}$$

Then there exists a unique Lie Banach*-algebra homomorphism $H : A^2 \longrightarrow B$ satisfying

$$\|f(a, b) - H(a, b)\| \le \frac{1}{1 - r}\varphi(a, b). \tag{2.5}$$

Proof. Consider X as the set of all functions $g : A^2 \longrightarrow B$ such that $g(0,0) = 0$, and define a generalized metric d on X by

$$d(f,g) = \inf\{c \in [0,\infty) : \|f(a,b) - g(a,b)\| \leq c\varphi(a,b) \text{ for all } a,b \in A\}.$$

It is easy to see that (X,d) is a complete generalized metric on X. Now define $J : X \longrightarrow X$ by

$$Jg(a,b) = 2g\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a,b \in A$. Note that

$$\begin{aligned} \|Jg(a,b) - Jh(a,b)\| &\leq 2d(g,h)\varphi\left(\tfrac{a}{2}, \tfrac{b}{2}\right) \\ &\leq rd(g,h)\varphi(a,b) \end{aligned}$$

for all $g,h \in X$ and all $a,b \in A$. Hence

$$d(Jg, Jh) \leq rd(g,h)$$

for all g,h in X. Therefore, J is a contraction with constant at most r. Note that $f(0,0) = 0$, so if $\mu = 1$ and $c = d = 0$ in (2.1), we get

$$\left\|2f\left(\frac{a}{2}, \frac{b}{2}\right) - f(a,b)\right\| \leq \varphi(a,b) \tag{2.6}$$

for all $a,b \in A$. It follows from (2.6) that $d(Jf,f) \leq 1$. By Theorem 1.1, there exists a unique fixed point function $H : A^2 \longrightarrow B$ of J in the set

$$\Omega = \{g \in X : d(f,g) < \infty\}$$

(here f is the function defined in the hypothesis of the theorem). Since $JH = H$, thus we get $H\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{1}{2}H(a,b)$ for all $a,b \in A$. Also, clearly

$$\|f(a,b) - H(a,b)\| \leq s\varphi(a,b)$$

where $s = d(f,H) \in (0,\infty)$. Furthermore, by Theorem 1.1, $d(J^n f, H) \longrightarrow 0$ as $n \longrightarrow \infty$. Since

$$J^n f(a,b) = 2^n f\left(\frac{a}{2^n}, \frac{b}{2^n}\right),$$

we have

$$H(a,b) = \lim_{n\longrightarrow\infty} 2^n f\left(\frac{a}{2^n}, \frac{b}{2^n}\right). \tag{2.7}$$

Again by Theorem 1.1, $d(f,H) \leq \frac{1}{1-r}d(f,Jf)$ from which we conclude that $d(f,H) \leq \frac{1}{1-r}$. Since

$$\|f(a,b) - H(a,b)\| \leq d(f,H)\varphi(a,b),$$

we get $d(f,H) \leq \frac{1}{1-r}$. So the inequality (2.5) holds. Now, it follows from (2.1), (2.4) and (2.7) that:

$$\begin{aligned} \left\|2H\left(\tfrac{a+c}{2}, \tfrac{b+d}{2}\right) - H(a,b) - H(c,d)\right\| &\leq \lim_{n\longrightarrow\infty} 2^n \|2f\left(\tfrac{a+c}{2^n}, \tfrac{b+d}{2^n}\right) \\ &\quad - f\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right) - f\left(\tfrac{c}{2^n}, \tfrac{d}{2^n}\right)\| \\ &\leq \lim_{n\longrightarrow\infty} 2^n \varphi\left(\tfrac{a+c}{2^n}, \tfrac{b+d}{2^n}\right) \\ &\leq \lim_{n\longrightarrow\infty} r^n \varphi(a+c, b+d) \\ &= 0 \end{aligned}$$

for all $a, b, c, d \in A$. Thus we have

$$2H\left(\frac{a+c}{2}, \frac{b+d}{2}\right) = H(a,b) + H(c,d)$$

for all $a, b, c, d \in A$. Therefore, the mapping $H : A^2 \longrightarrow B$ is Jensen additive. Now, let $a = c$ and $b = d$ in (2.1). So we obtain

$$\|2\mu f(a,b) - 2f(\mu a, \mu b)\| \leq \varphi(2a, 2b)$$

for all μ with $|\mu| = 1$ and all $a, b \in A$. Now, by substituting a by $\frac{a}{2^n}$ and b by $\frac{b}{2^n}$ in the above relation, we get

$$\left\|2\mu f\left(\frac{a}{2^n}, \frac{b}{2^n}\right) - 2f\left(\frac{\mu a}{2^n}, \frac{\mu b}{2^n}\right)\right\| \leq \frac{r^{n-1}}{2^{n-1}}\varphi(a,b).$$

So

$$\|\mu H(a,b) - H(\mu a, \mu b)\| = \lim_{n\longrightarrow\infty}\left\|2^n\mu f\left(\frac{a}{2^n}, \frac{b}{2^n}\right) - 2^n f\left(\frac{\mu a}{2^n}, \frac{\mu b}{2^n}\right)\right\| = 0.$$

Thus, $\mu H(a,b) = H(\mu a, \mu b)$ for all μ with $|\mu| = 1$ and all $a, b \in A$. Now, clearly we can see that $H : A^2 \longrightarrow B$ is $\mathbb{C}$-linear. Note that by (2.2) and (2.4) we have

$$\begin{aligned}\|H([(a,b),(c,d)]) - [H(a,b), H(c,d)]\| &= \lim_{n\longrightarrow\infty} 4^n \|f\left(\left[\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right), \left(\tfrac{c}{2^n}, \tfrac{d}{2^n}\right)\right]\right) \\ &\quad - \left[f\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right), f\left(\tfrac{c}{2^n}, \tfrac{d}{2^n}\right)\right]\| \\ &\leq \lim_{n\longrightarrow\infty} 4^n\varphi\left(\tfrac{ac}{4^n}, \tfrac{bd}{4^n}\right) \\ &\leq \lim_{n\longrightarrow\infty}(r^n)^2\varphi(ac, bd) \\ &= 0\end{aligned}$$

for all a, b, c, d in A. Also, by (2.3), we obtain

$$\begin{aligned}\|H(a^*, b^*) - (H(a,b))^*\| &= \lim_{n\longrightarrow\infty} 2^n \left\|f\left(\tfrac{a^*}{2^n}, \tfrac{b^*}{2^n}\right) - \left(f\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right)\right)^*\right\| \\ &\leq \lim_{n\longrightarrow\infty} 2^n\varphi\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right) \\ &\leq \lim_{n\longrightarrow\infty}(r^n)\varphi(a,b) \\ &= 0\end{aligned}$$

for all a, b in A. So $H : A^2 \longrightarrow B$ is indeed a Lie Banach*-algebra homomorphism satisfying the desired conditions. This complete the proof. ∎

Theorem 2.2. *Suppose that A is a Lie Banach*-algebra. Let $f : A^2 \longrightarrow B$ and $\varphi : A^2 \longrightarrow [0,\infty)$ be two mappings satisfying the condition (2.2) of Theorem 2.1, and also let for all $a, b, c, d \in A$ and $\mu \in \mathbb{C}$ with $|\mu| = 1$,*

$$\|\mu f(a+c, b+d) - f(\mu a, \mu b) - f(\mu c, \mu d)\| \leq \varphi(a+c, b+d). \tag{2.8}$$

If there exists $0 < r < 1$ such that $\varphi(2a, 2b) \leq 2r\varphi(a,b)$ for all $a, b \in A$, then there exists a unique Lie Banach*-algebra homomorphism $H : A^2 \longrightarrow B$ such that condition (2.5) of Theorem 2.1 holds.

Proof. Put $X = \{g : A^2 \longrightarrow B\}$ and consider the generalized metric d defined on X as in Theorem 2.1. Define $J : X \longrightarrow X$ by

$$Jh(a,b) = \frac{1}{2}h(2a, 2b)$$

and note that

$$\|Jg(a,b) - Jh(a,b)\| \leq rd(g,h)\varphi(a,b)$$

for all $a, b \in A$. Thus

$$\|Jg - Jh\| \leq rd(g, h)$$

and so J is a contraction. Now, by a similar method used in the proof of Theorem 2.1 together with (2.8), we can see that the function $H : A^2 \longrightarrow B$ defined by

$$H(a, b) = \lim_{n \longrightarrow \infty} \frac{1}{2^n} f(2^n a, 2^n b); \qquad a, b \in A,$$

is additive and indeed it is the unique Lie Banach* homomorphism that holds in the condition (2.5). This completes the proof. ∎

3. STABILITY OF LIE DERIVATIONS ON BANACH*-ALGEBRAS

In this section we want to investigate the Hyers-Ulam stability of Lie derivations for special functional equations acting on Banach*-algebras.

If A is a Lie Banach*-algebra, then $D : A^2 \longrightarrow A^2$ satisfying

$$D([(a, b), (c, d)]) = [D(a, b), (c, d)] + [(a, b), D(c, d)]$$

for all $a, b, c, d \in A$, is called a Lie derivation on A^2. Also, if A is a Lie Banach*- algebra, then D is called a Lie* derivation on A^2 whenever D is a Lie derivation with $D(a, b)^* = D(a^*, b^*)$ for all $a, b \in A$.

Theorem 3.1. *Suppose that A is an Banach*-algebra and $f : A^2 \longrightarrow A^2$ is a mapping such that $f(0, 0) = (0, 0)$. Also, let $\varphi : A^2 \longrightarrow [0, \infty)$ be a function satisfying the condition (2.1) of Theorem 2.1 and*

$$\|(f(a, b))^* - f(a^*, b^*)\| \leq \varphi(a, b) \tag{3.1}$$

for all a, b in A. Also, let

$$\|f([(a, b), (c, d)]) - [f(a, b), (c, d)] - [(a, b), f(c, d)]\| \leq \varphi(ac, bd) \tag{3.2}$$

for all a, b, c, d in A. Suppose that there exists $0 < r < 1$ such that

$$\varphi(a, b) \leq \frac{r}{2} \varphi(2a, 2b) \tag{3.3}$$

for all a, b in A. Then there exists a unique Lie* derivative $D : A^2 \longrightarrow A^2$ satisfying the following condition:

$$\|f(a, b) - D(a, b)\| \leq \frac{1}{1 - r} \varphi(a, b) \tag{3.4}$$

for all $a, b \in A$.

Proof. By a similar method used in the proof of Theorem 2.1, for $B = A^2$, we can that there exists a unique $\mathbb{C}-$linear mapping $D : A^2 \longrightarrow A^2$ satisfying the relation (3.4). Note that D is given by

$$D(a, b) = \lim_{n \longrightarrow \infty} 2^n f\left(\frac{a}{2^n}, \frac{b}{2^n}\right)$$

for all $a, b \in A$. Now, by using the relation (3.2), (3.3) we get

$$\begin{aligned}
&\|D([(a,b),(c,d)]) - [D(a,b),(c,d)] - [(a,b),D(c,d)]\| \\
&= \lim_{n\longrightarrow\infty} 4^n \|f\left[\tfrac{(a,b)}{4^n}, \tfrac{(c,d)}{4^n}\right) \\
&- \left[f\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right), \left(\tfrac{c}{2^n}, \tfrac{d}{2^n}\right)\right] - \left[\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right), f\left(\tfrac{c}{2^n}, \tfrac{d}{2^n}\right)\right]\| \\
&\leq \lim_{n\longrightarrow\infty} 4^n \varphi\left(\tfrac{ac}{4^n}, \tfrac{bd}{4^n}\right) \\
&\leq \lim_{n\longrightarrow\infty} (r^n)^2 \varphi(ac, bd) \\
&= 0
\end{aligned}$$

for all $a, b, c, d \in A$. Also,

$$\begin{aligned}
\|D(a,b)^* - D(a^*, b^*)\| &= \lim_{n\longrightarrow\infty} (2^n) \|f\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right)^* - f\left(\tfrac{a^*}{2^n}, \tfrac{b^*}{2^n}\right)\| \\
&\leq \lim_{n\longrightarrow\infty} 2^n \varphi\left(\tfrac{a}{2^n}, \tfrac{b}{2^n}\right) \\
&\leq \lim_{n\longrightarrow\infty} 2^n \varphi(a,b) \\
&= 0
\end{aligned}$$

Hence $D : A^2 \longrightarrow A^2$ is the unique Lie* derivative satisfying the desired conditions. Now the proof is complete. ∎

Theorem 3.2. *Suppose that A is a Banach*-algebra and $f : A^2 \longrightarrow A^2$, and $\varphi : A^2 \longrightarrow [0, \infty)$ be two mappings satisfying*

$$\|\mu f(a+c, b+d) - f(\mu a, \mu b) - f(\mu c, \mu d)\| \leq \varphi(a+c, b+d),$$
$$\|f([(a,b),(c,d)]) - [f(a,b),(c,d)] - [(a,b), f(c,d)]\| \leq \varphi(ac, bd)$$

and

$$\|f(a,b)^* - f(a^*, b^*)\| \leq \varphi(a,b)$$

for all scalars μ with $|\mu| = 1$ and all a, b, c, d in A. Suppose that there exists $0 < r < 1$ such that

$$\varphi(2a, 2b) \leq 2r\varphi(a,b),$$

for all a, b in A. Then there exists a unique Lie* derivative $D : A^2 \longrightarrow A^2$ satisfying the condition

$$\|f(a,b) - D(a,b)\| \leq \frac{1}{1-r}\varphi(a,b)$$

for all $a, b \in A$.

Proof. By a similar method used in the proof of Theorem 2.2, for $B = A^2$, we can see that the mapping $D : A^2 \longrightarrow A^2$ defined by

$$D(a,b) = \lim_{n\longrightarrow\infty} \frac{1}{2^n} f(2^n a, 2^n b); \qquad a, b \in A,$$

is the unique $\mathbb{C}-$linear mapping such that $\|f(a,b)-D(a,b)\| \leq \frac{1}{1-r}\varphi(a,b)$ for all $a,b \in A$. Also, we note that

$$\begin{aligned}
&\|D([(a,b),(c,d)]) - [D(a,b),(c,d)] - [(a,b),D(c,d)]\| \\
&= \lim_{n\longrightarrow\infty} \tfrac{1}{4^n}\|f([4^n(a,b),4^n(c,d)]) - [f(2^n a,2^n b),\left(\tfrac{c}{2^n},\tfrac{d}{2^n}\right)] \\
&\quad - \left[\left(\tfrac{a}{2^n},\tfrac{b}{2^n}\right), f(2^n c,2^n d)\right]\| \\
&\leq \lim_{n\longrightarrow\infty} \tfrac{1}{4^n}\varphi(4^n ac,4^n bd) \\
&\leq \lim_{n\longrightarrow\infty}(r^n)^2\varphi(ac,bd) \\
&= 0
\end{aligned}$$

for all a,b,c,d in A. Also, clearly $(D(a,b))^* = D(a^*,b^*)$ for all $a,b \in A$. Thus D is indeed a Lie* derivative on A^2. This completes the proof. ∎

REFERENCES

[1] H. AZADI KENARY, and Y. JE CHO, Stability of mixed additive-quadratic Jensen type functional equation in varius spaces, *Computers and Mathematics with Applications,* **61,** (2011), pp. 2407–2724.

[2] H. AZADI KENARY, Random approximation of an additive functional equation m-Appollonius type, *Acta Mathematica Scientia,* **32B,** (5) (2012), pp. 1813–1825.

[3] Y. J. CHO, C. PARK, T. M. RASSIAS, and R. SAADATI, Stability of functional equations in Banach*-algebras, Springer, (2015).

[4] Z. DEHVARI, and S. M. MOSHTAGHIOUN, Fixed point method in the stability of Cauchy Functional equations, *Journal of Mathematical Extension,* **7**, (1) (2013), pp. 81–99.

[5] A. FOSNER, R. GER, A. GILANYI, and M. S. MOSLEHIAN, On linear functional equations and completeness of normed spaces, *Banach Journal of Mathematical Analysis,* **184**, (3) (1994), pp. 431–436.

[6] A. FOSNER, B. MARCEN, N. SIROVNIK, and J. VUKMAN, On some functional equations related to derivations and bicircular projections in rings, *Glasnik Matematicki,* **49**, (2) (2014), pp. 313–331.

[7] P. GAVRUTA, A generalization of the Hypers-Ulam-Rassias stability of approximately additive mappings, *Journal of Mathematical Analysis and Applications* **7**, (1) (2013), pp. 196–200.

[8] D. H. HYERS, On the stability of the linear functional equation, *Proceedings of the National Academy of Sciences of the United States of America,* (1941), pp. 222–224.

[9] Z. PALES, Generalized stability of the Cauchy functional equations, *Aequations Math.,* **56**, (3) (1998), pp. 222–232.

[10] C. PARK, On an approximate automorphisms on a C^*-algebra, *Proc. Amer. Math. Soc.,* **132**, (2004), pp. 1739–1745.

[11] C. PARK, and T. M. RASSIAS, Fixed points and stability of the cauchy functional equations, *Aust. J. Math. Anal. Appl.*, **6**, (2009), pp. 1–9.

[12] V. RADU, The fixed point alternative and the stability of functional equations, *Fixed Points Theory,* 4 (2003), pp. 91–96.

[13] T. M. RASSIAS, On the stability of the linear mapping in Banach spaces, *Procedings of the American Mathematical Society,* **72**, (2) (1978), pp. 297–300.

[14] T. M. RASSIAS, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.*, **62**, (2000), pp. 23–130.

[15] T. M. RASSIAS, On the stability of functional equations in Banach spaces, *J. Math. Appl. Math.*, **251,** (2000), pp. 264–284.

[16] S. M. ULAM, *Problems in Modern Mathematics*, John Wiely Sons, New York, NY, USA, (1994).

[17] A. YADEGARNEGAD, S. JAHEDI, B. YOUSEFI, and S. M. VAEZPOUR, Semistability of iterations in cone spaces, *Fixed Point Theory Appl.*, 2011, doi:10.1186/1687-1812-2011-70, 2011:70.

[18] B. YOUSEFI, A. YADEGARANEJAD, H. AZADI, and C. PARK, Equivalence of semistability of Picaed, Mann, Krasnoselski and Ishikawa iterations, *Fixed Point Theory Appl.*, 2014, doi:10.1186/1687-1812-2014-5, 2014:5.