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ON THE BIHARMONIC EQUATION WITH NONLINEAR BOUNDARY INTEGRAL CONDITIONS

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ABSTRACT. In the present work, we deal with the biharmonic problems in a bounded domain in the plane with the nonlinear boundary integral conditions. After applying the Boundary integral method, a system of nonlinear boundary integral equations is obtained. The result show that when the nonlinearity satisfies some conditions lead the existence and uniqueness of the solution.

Key words and phrases: Boundary integral equations methods; Monotone operators; Nonlinear boundary integral conditions; Pseudo-differential operator.

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1. INTRODUCTION

Problems involving nonlinearities form a basis of mathematical models of various phenomena and processes in mechanics, physics and many other areas of science.
In the paper [7] and [8]), a nonlinear boundary conditions for the Laplace equation is considered. Using the boundary integral method the problem is converted into a nonlinear integral equation for the unknown data of the boundary. For studying the solvability of the nonlinear equation the authors give some assumption on the nonlinearity part.
The purpose of the present paper is to see the feasibility of extending the approach in [7] and [8]), for the Laplace equation to the biharmonic equation.

The biharmonic equations are an important class of equations in both physics and engineering. In fluid dynamics, the so-called stream function satisfies the biharmonic equation. Many problems in elasticity can also be formulated in terms of the biharmonic equation where the fundamental physical quantities such as displacement, stress, and strain all satisfy the biharmonic equation. There have been extensive research activities on the biharmonic equation both theoretically and computationally (see, for example, [5], [9]).

In the present paper, we look for the solution of the Bilaplacian equation with nonlinear data of the form:

$$\Delta^2 u(x) = 0 \quad , \quad x \in \Omega, \tag{1.1}$$

$$\Delta u(x) = -\int_\Gamma K_1\left(x, u(y), \partial_n u(y)\right) ds_y + f(x) \ \ , \ x \in \Gamma, \tag{1.2}$$

$$-\partial_n \Delta u(x) = -\int_\Gamma K_2\left(x, u(y), \partial_n u(y)\right) ds_y + g(x) \ \ , \ x \in \Gamma. \tag{1.3}$$

We denote by $n = (n_1, n_2)$ the unit outward normal vector to Γ, and the normal derivative by $\partial_n(.) := \frac{\partial(.)}{\partial_n}$.
The given functions $f \in H^{-\frac{1}{2}}(\Gamma)$ and $g \in H^{-\frac{3}{2}}(\Gamma)$ are defined on Γ .
The boundary operators (Δ) , $(-\partial_n \Delta)$ and (Δ^2) are defined by

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

$$\Delta^2 u = \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_1^2}\right)$$

$$-\partial_n \Delta u = -n_1 \frac{\partial \Delta u}{\partial x_1} - n_2 \frac{\partial \Delta u}{\partial x_2}$$

physically, (Δu) is the bending moment and $-\partial_n \Delta u$ is the transverse force consisting of the shear force and twisting moment . For $u \in H^2(\Omega)$ we have that $u|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$ and $\partial_n u|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$.
In (1.1), we assume Ω is an open bounded region in $\mathbb{R}^2$ with a smooth boundary $\Gamma = \partial\Omega$, and

$$f : \Gamma \to \mathbb{R} \quad , \quad K_1 : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$g : \Gamma \to \mathbb{R} \quad , \quad K_2 : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

are given real value functions with some precise assumptions on K_i , $i = 1, 2$ that we will state below.

Now integration by parts leads to the first Green formula in the form

$$\int_\Omega (\Delta^2 u v dx = a(u, v) - \int_\Gamma \left\{\partial_n v . \Delta u - v . \partial_n \Delta u\right\} ds, \qquad \text{x} \in \Omega \tag{1.4}$$

where the bilinear form $a(u,v)$ is defined by

$$a(u,v) = \int_\Omega \Delta u \Delta v dx. \tag{1.5}$$

We note that the bilinear form $a(.,.)$ is well defined for functions in $H^2(\Omega)$. Now let $u \in H^2(\Omega, \triangle^2)$ where

$$H^2(\Omega,\triangle^2) = \{u \in H^2(\Omega); \triangle^2 u \in \widetilde{H}^{-2}(\Omega)\}$$

with $\widetilde{H}^{-2}(\Omega)$ denoting the dual space of $H^2(\Omega)$ and choose $v \in H^2(\Omega)$. Then the above Green formula holds and by duality argument one shows that $\Delta u \in H^{-\frac{1}{2}}(\Gamma)$ and $-\partial_n \Delta u \in H^{-\frac{3}{2}}(\Gamma)$ are well defined, where $H^{-\frac{1}{2}}(\Gamma)$ and $H^{-\frac{3}{2}}(\Gamma)$ are the dual spaces of $H^{\frac{1}{2}}(\Gamma)$ and $H^{\frac{3}{2}}(\Gamma)$, respectively.

By the integral equation method, we formulate a system of nonlinear integral equations on the boundary Γ of the domain Ω. Under some assumptions on the Kernel of the nonlinear integral equations $K_i\,(x,u,\partial_n u)$ we prove the existence and uniqueness of the solution.

Definition 1.1. (see [1], [3], [5], and [10]).
Let $m \in \mathbb{N}$, we denote by $H^m(\Omega)$ the Sobolev space

$$H^m(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), |\alpha| \ \leq m\}.$$

Definition 1.2. (see [1], [3], [5], and [10]).
Let $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^n)$ the Sobolev space :

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); (1+|\xi|^2)^{\frac{s}{2}}|F[u]| \in L^2(\mathbb{R}^n)\},$$

and the associated norm:

$$\|u\|_{H^s} = (\int_{\mathbb{R}^n}(1+|\xi|^2)^s|F[u]|^2 d\xi)^{\frac{1}{2}},$$

with $F[.]$ the Fourier transform.

Definition 1.3. ,(see [1], [3], [5], and [10]).
Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $\Gamma := \partial\Omega$, we defined

$$H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^n)\}, s \in \mathbb{R}$$

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+\frac{1}{2}}(\mathbb{R}^n)\}, s > 0 \\ L^2(\Gamma), \quad s = 0 \\ (H^{-s}(\Gamma))' \ \text{(dual space)}, s < 0 \end{cases}.$$

2. THE BOUNDARY INTEGRAL METHOD

For reformulate the problem (1.1, 1.2 and 1.3) as a system of nonlinear boundary integral equations, we start with the Green representation formula of a weak solution in $H^2(\Omega)$

$$u(x) = \mathbb{V}(\Delta u, -\partial_n \Delta u)(x) - \mathbb{W}(u, \partial_n u)(x) \ \ , x \in \Omega, \tag{2.1}$$

in term of simple and double layer potentiels [3], and [5]. Here

$$\mathbb{V} : H^{-\frac{1}{2}}(\Gamma) \times H^{-\frac{3}{2}}(\Gamma) \to H^2(\Omega) \quad \text{and } \mathbb{W} : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^2(\Omega)$$

are continuous operators defined by

$$\mathbb{V}(\Delta u, -\partial_n \Delta u)(x) = \int_\Gamma \left\{E(x,y)(-\partial_n \Delta u(y)) + \partial_{n_y} E(x,y)(\Delta u(y))\right\} ds_y, \ \ x \in \mathbb{R}^2 \setminus \Gamma,$$

$$\mathbb{W}(u,\partial_n u)(x)=\int_\Gamma \left\{\Delta_y E(x,y)\partial_n u(y)-\partial_{n_y}\Delta E(x,y)u(y)\right\}ds_y,\ \ x\in\mathbb{R}^2\setminus\Gamma$$

where

$$E\left(x,y\right):=\frac{1}{8\pi}\left|x-y\right|^2\log\left|x-y\right| \tag{2.2}$$

is the fundamental solution of the biharmonic equation.
Letting $x\to\Gamma$ from inside Ω, and following the standard procedure in potential theory involving jump relations, we obtain the following integral equations on Γ (see [3], and [5]):

$$u(x)=\int_\Gamma \left\{E(x,y).(-\partial_n\Delta u(y))+\partial_{n_y}E(x,y).\Delta u(y)\right\}ds_y+\frac{1}{2}u(x)$$

$$-\int_\Gamma \left\{(\Delta_y E(x,y))\partial_{n_y}u+(-\partial_{n_y}\Delta E(x,y)).u(y)\right\}ds_y, \tag{2.3}$$

$$\partial_n u(x)=\int_\Gamma \left\{\frac{\partial E(x,y)}{\partial n_x}(-\partial_n\Delta u(y))+(\frac{\partial^2 E}{\partial n_x\partial n_y}(x,y)\Delta u(y)\right\}ds_y+\frac{1}{2}\partial_n u(x)$$

$$-\int_\Gamma \left\{(\frac{\partial}{\partial n_x}\Delta_y E(x,y))\partial_{n_y}u(y)+(\frac{\partial}{\partial n_x}(-\partial_n\Delta)_y E(x,y))u(y)\right\}ds_y. \tag{2.4}$$

In present time, in order to formulate the integral equations, we define the following operators at the boundary:

Definition 2.1. Let $u\in C^\infty\left(\Gamma\right)$. We define the following operators for $x\in\Gamma$:

$$\begin{aligned}
K_{11}u(x) &= \int_\Gamma(-\partial_n\Delta_y)E(x,y))u(y)ds_y\\
V_{12}\partial_n u(x) &= -\int_\Gamma \Delta_y E(x,y)\frac{\partial u}{\partial n_y}ds_y\\
D_{21}u(x) &= \int_\Gamma \frac{\partial}{\partial n_x}(-\partial_n\Delta_y)E(x,y))u(y)ds_y\\
K_{22}\partial_n u(x) &= \partial_n V_{12}\partial_n u(x)\\
V_{13}\Delta u(x) &= \int_\Gamma \frac{\partial E}{\partial n_y}(x,y)\Delta u(y)ds_y\\
V_{14}(-\partial_n\Delta u(x)) &= \int_\Gamma E(x,y)(-\partial_n\Delta)u(y)ds_y\\
V_{23}\Delta u(x) &= \partial_n V_{13}\Delta u(x)\\
V_{24}(-\partial_n\Delta)u(x) &= \partial_n V_{14}(-\partial_n\Delta)u(x)\\
A_i(x,u(x),\partial_n u(x)) &= \int_\Gamma K_i\left(x,u(y),\partial_n u(y)\right)ds_y\ \ ,\ \ i=1,2
\end{aligned}$$

.

The mapping properties of the integrals operators are collected in the following lemma.

Lemma 2.1. *(see* [1], [3], [5], *and* [10]*)* . *The operators defined by :*

$$K_{11}:H^s\left(\Gamma\right)\longrightarrow H^s\left(\Gamma\right),\ \ D_{21}:H^s\left(\Gamma\right)\longrightarrow H^{s-1}\left(\Gamma\right)$$

$$V_{12}:H^s\left(\Gamma\right)\longrightarrow H^{s+1}\left(\Gamma\right),\ \ V_{13}:H^s\left(\Gamma\right)\longrightarrow H^{s+3}\left(\Gamma\right)$$

$$V_{14}: H^{s}\left(\Gamma\right) \longrightarrow H^{s+3}\left(\Gamma\right), \;\; V_{23}: H^{s}\left(\Gamma\right) \longrightarrow H^{s+1}\left(\Gamma\right)$$

$$V_{24}: H^{s}\left(\Gamma\right) \longrightarrow H^{s+3}\left(\Gamma\right), \;\; K_{22}: H^{s}\left(\Gamma\right) \longrightarrow H^{s}\left(\Gamma\right)$$

are continuous.

For the problem (1.1, 1.2 and 1.3) as integral equations at the boundary Γ, we consider $u \in H^{2}\left(\Omega, \Delta^{2}\right)$ satisfying the boundary conditions of the problem (1.1, 1.2 and 1.3).
If we introduce in the system (2.3) and (2.4) the given functions and the unknown functions on Γ such that:

$$\begin{cases} u\left(x\right) = v\left(x\right) \;\; (unknown \;\; function) \\ \partial_{n} u\left(x\right) = w\left(x\right) \;\; (unknown \;\; function) \end{cases}, x \in \Gamma,$$

and

$$\begin{cases} \Delta u(x) = A_1(x, u(x), \partial_n u(x)) + f(x) \;, \; x \in \Gamma \\ -\partial_n \Delta u(x) = A_2(x, u(x), \partial_n u(x)) + g(x) \;, \;\; x \in \Gamma \end{cases},$$

the equation (2.3)and (2.4) may be written as

$$\begin{cases} (\frac{1}{2}I + K_{11})v - V_{12}w + V_{14}A_2(x,v,w) + V_{13}A_1(x,v,w) = V_{14}g + V_{13}f \\ -D_{21}v + (\frac{1}{2}I - K_{22})w + V_{24}A_2(x,v,w) + V_{23}A_1(x,v,w) = V_{24}g + V_{23}f \end{cases}.$$

This system of equations can be written in a matrix form as follows

$$\begin{bmatrix} \frac{1}{2}I + K_{11} & -V_{12} \\ -D_{21} & \frac{1}{2}I - K_{22} \end{bmatrix} \begin{bmatrix} v(x) \\ w(x) \end{bmatrix} + \begin{bmatrix} V_{14} & V_{13} \\ V_{24} & V_{23} \end{bmatrix} \begin{bmatrix} A_2(x,v,w) \\ A_1(x,v,w) \end{bmatrix} =$$

$$\begin{bmatrix} V_{14} & V_{13} \\ V_{24} & V_{23} \end{bmatrix} \begin{bmatrix} g(x) \\ f(x)) \end{bmatrix}$$

(2.5) $$L(U) + V(A(U)) = V(F) \text{ on } \Gamma,$$

where

$$L = \begin{bmatrix} \frac{1}{2}I + K_{11} & -V_{12} \\ -D_{21} & \frac{1}{2}I - K_{22} \end{bmatrix}, V = \begin{bmatrix} V_{14} & V_{13} \\ V_{24} & V_{23} \end{bmatrix}, A = \begin{bmatrix} A_2(x,v,w) \\ A_1(x,v,w) \end{bmatrix}$$

and

$$U = \begin{pmatrix} v \\ w \end{pmatrix}, F = \begin{pmatrix} g \\ f \end{pmatrix}.$$

For studying the solvability of the nonlinear equation (2.5) , we give some assumptions to be made here.
(**H1**) The Kernels $K_1(.,.,.)$ and $K_2(.,.,.)$ are a Caratheodory functions.
(**H2**) We assume that for all $x \in \Gamma$,

$$K_1(x,.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \;, K_2(x,.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

are differentiable and the derivatives are bounded satisfying

$$0 < \alpha \leq \frac{\partial K_1}{\partial v} \leq l_1 < +\infty \;, |\frac{\partial K_1}{\partial w}| \leq \beta$$
$$0 < \alpha \leq \frac{\partial K_2}{\partial w} \leq l_2 < +\infty \;, |\frac{\partial K_2}{\partial v}| \leq \beta$$

for some constants α, β and l_i, $i = 1, 2$ with $\alpha > \beta$.

Remark 2.1. 1) The functions $K_1(.,.,.)$ and $K_2(.,.,.)$ are Caratheodory functions (**H1**) (i.e) $K_1(.,v,w)$ and $K_2(.,v,w)$ are measurable for all $(v,w) \in \mathbb{R} \times \mathbb{R}$ and $K_1(x,.,.), K_2(x,.,.)$ are continuous for almost all $x \in \Gamma$.

2) The assumption (**H2**) implies that the Nemytski operator

$$A : L^2(\Gamma) \times L^2(\Gamma) \to L^2(\Gamma) \times L^2(\Gamma)$$

is Lipschitz continuous and strongly monotonous.
From the Lagrange's mean value theorem, there exists ζ_1, ζ_2 such that:

$$\begin{aligned} K_1(v,w) - K_1(v',w') &= (K_1(v,w) - K_1(v',w)) + (K_1(v',w) - K_1(v',w')) \\ &= \frac{\partial K_1(\zeta_1,w)}{\partial v}(v-v') + \frac{\partial K_1(v',\zeta_2)}{\partial w}(w-w'), \end{aligned}$$

then

$$\begin{aligned} (v-v')(A_1(v,w) - A_1(v',w')) &\geq mes(\Gamma)(\alpha\,|v-v'|^2 - \beta\,|v-v'|\,|w-w'|) \\ &\geq mes(\Gamma)(\alpha\,|v-v'|^2 - \frac{1}{2}\beta\,|v-v'|^2 - \frac{1}{2}\beta\,|w-w'|^2), \end{aligned}$$

such that

$$|a.b| \leq \frac{1}{2}\,|a|^2 + \frac{1}{2}\,|b|^2\,.$$

In the same manner it can be shown that

$$(w-w')(A_2(v,w) - A_2(v',w')) \geq mes(\Gamma)(\alpha\,|w-w'|^2 - \frac{1}{2}\beta\,|v-v'|^2 - \frac{1}{2}\beta\,|w-w'|^2).$$

Finally combining these inequalities, we obtain

$$\begin{aligned} (U-U', A(U) - A(U')) &= ((v-v'), A_1(v,w) - A_1(v',w')) \\ &+ ((w-w'), A_2(v,w) - A_2(v',w')) \\ &\geq mes(\Gamma)(\alpha-\beta)(\|v-v'\|^2 + \|w-w'\|^2) \\ &\geq mes(\Gamma)(\alpha-\beta)\,\|U-U'\|_0^2\,, \end{aligned}$$

with $U = (v,w), U' = (v',w') \in L^2(\Gamma) \times L^2(\Gamma)$ Hence the operator $A(U)$ is strongly monotonous.
For the continuity of the Nemytski operator $A(U)$

$$\begin{aligned} |K_1(v,w) - K_1(v',w')| &= |(K_1(v,w) - K_1(v',w)) + (K_1(v',w) - K_1(v',w'))| \\ &= |\frac{\partial K_1(\zeta_1,w)}{\partial v}(v-v') + \frac{\partial K_1(v',\zeta_2)}{\partial w}(w-w')| \\ &\leq l_1\,|v-v'| + \beta\,|w-w'| \\ &\leq Max\{l_1,\beta\}(|v-v'| + |w-w'|) \\ &\leq l_1(|v-v'| + |w-w'|), \end{aligned}$$

then we have

$$\|A_1(U) - A_1(U')\| \;\leq\; mes(\Gamma) l_1\,\|U-U'\|_0\,.$$

In the same manner, it can be shown that

$$\|A_2(U) - A_2(U')\| \leq mes(\Gamma) l_2 \|U - U'\|_0 ,$$

then we get

$$\|A(U) - A(U')\|_0 \leq mes(\Gamma)(l_1 + l_2) \|U - U'\|_0 ,$$

which proves that the operator A is Lipschitz continuous for $U = (v, w), U' = (v', w') \in L^2(\Gamma) \times L^2(\Gamma)$.

Based on this property we can consider the solvability of (2.5).

Theorem 2.2. *Let assumptions (**H1**) and (**H2**) hold. Then, for every* $F = (g, f) \in H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ *there exists a unique* $U = (v, w) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ *such that*

$$L(U) + V(A(U)) = V(F) \text{ on } \Gamma.$$

Proof. The proof follows from the well-known theorem by Browder and Minty on monotone operators (see [2] and [6]).
Since the simple layer potential operator on Γ

$$V : H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$$

is an isomorphism it is sufficient to consider the unique solvability of equation

$$TU := V^{-1}LU + AU = F \text{ on } \Gamma. \tag{2.6}$$

We shall prove that the operator

$$T : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

is continuous and strongly monotonous.
i- in the first we show that T is continuous:
It is clear from the continuity of the mapping properties of the simple layer operator V and L by the Lemma 2.1 , that

$$V^{-1}L : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

is continuous. And from (H2)

$$A : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

is continuous. Hence the boundary integral operator

$$T : H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

is continuous.

ii- In the second we show that T is strongly monotonous operator.
Let $\mu = (\mu_1, \mu_2) \in H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ defined by

$$\mu(x) := V^{-1}LU(x),$$

for all $U(x) = (v, w) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, is the $(\Delta\varphi, -\partial_n \Delta\varphi)$ of the biharmonic function

$$\begin{aligned}\varphi(x) &= \mathbb{V}(\Delta\varphi(x), -\partial_n \Delta\varphi(x)) - \mathbb{W}(\varphi(x), \frac{\partial\varphi}{\partial n}) \\ &= \mathbb{V}\mu(x) - \mathbb{W}U(x)\end{aligned}$$

for $x \in \Omega$, this means that $\varphi(x)$ satisfies the problem

$$\begin{cases} \Delta^2\varphi(x) = 0 \quad , \quad x \in \Omega \\ \varphi(x) = v \;\; , \;\; x \in \Gamma \\ \frac{\partial\varphi}{\partial n} = w \;\; , \;\; x \in \Gamma \end{cases}.$$

Then Green's theorem yields

$$\begin{aligned} &(V^{-1}LU, U) = \int_\Gamma \mu_1 v ds + \int_\Gamma \mu_2 w ds \\ &= -\int_\Gamma \partial_n \Delta\varphi\varphi ds + \int_\Gamma \Delta\varphi \frac{\partial\varphi}{\partial n} ds \\ &= a(\varphi, \varphi). \end{aligned}$$

Hence, the linearity of $V^{-1}L$ implies that for all $U, U' \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$

$$\begin{aligned} (V^{-1}L(U-U'), U-U') &= a(\varphi - \varphi', \varphi - \varphi') \\ &= \int_\Omega (\Delta(\varphi - \varphi'))^2 dx. \end{aligned}$$

In an other hand, we have

$$\|\varphi - \varphi'\|_{H^2(\Omega)} \leq c \|U - U'\|_{H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)} \leq c \|U - U'\|_0 .$$

Hence,

$$\begin{aligned} (TU - TU', U - U') &= (AU - AU', U - U') + (V^{-1}L(U - U'), U - U') \\ &\geq mes(\Gamma)(\alpha - \beta)\|U - U'\|_0^2 + a(\varphi - \varphi', \varphi - \varphi') \\ &\geq mes(\Gamma)(\alpha - \beta)\|U - U'\|_0^2 \\ &\geq \frac{mes(\Gamma)}{c^2}(\alpha - \beta)\|\varphi - \varphi'\|_{H^2(\Omega)}^2 \\ &\geq \frac{mes(\Gamma)}{c^2}(\alpha - \beta)\|U - U'\|_{H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)}^2 \end{aligned}$$

by the trace theorem (see [3] and [5]). Which completes the proof. ∎

Conclusion 1. *Finding weak solution $u \in H^2(\Omega)$ of problem (1.1 , 1.2, and 1.3) is composed in three steps.*
- Firstly, a system of nonlinear boundary integral equations 2.5 is solved for $(u, \frac{\partial u}{\partial n})$ on the boundary Γ .
- In the next step, boundary differential operators $(\Delta u, -\partial_n \Delta u)$ are deduced from formula (1.1and 1.2).
- The last step consists on determining $u(x)$ at any point $x \in \Omega$ by formula 2.1.

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