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HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR HARMONICALLY s-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, some Hermite-Hadamard-Fejer type integral inequalities for harmonically s-convex functions in fractional integral forms have been obtained.

Key words and phrases: Hermite-Hadamard inequality; Hermite-Hadamard-Fejer inequality; Riemann-Liouville fractional integral; Harmonically s-convex function.

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1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [4].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [3], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a + b)/2$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [5, 6, 16, 18].

We recall the following inequality and special functions which are known as Beta and hypergeometric function respectively

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x,y>0,$$

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \\ c &> b>0, |z|<1 \text{ (see [13])}. \end{aligned}$$

Lemma 1.2. [15, 20]. *For $0 < \alpha \leq 1$ and $0 \leq a < b$ we have*

$$|a^\alpha - b^\alpha| \leq (b-a)^\alpha.$$

Following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 1.1. [13]. Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of oder $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x>a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^\infty e^{-t}t^{\alpha-1}dt$ and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [2, 7, 8, 17, 19, 20].

Definition 1.2. [11]. Let $I \subset (0,\infty)$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically s-convex, if

$$f\left(\frac{xy}{tx+(1-t)\,y}\right) \le t^s f(y) + (1-t)^s f(x)$$

for all $x, y \in I$,$t \in [0,1]$ and for some fixed $s \in (0,1]$.

In [10], İşcan gave definition of harmonically convex functions and established following Hermite-Hadamard type inequality for harmonically convex functions as follows:

Definition 1.3. Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)\,y}\right) \le tf(y) + (1-t)\,f(x) \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.3) is reversed, then f is said to be harmonically concave.

Theorem 1.3. [10]. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

In [9], İşcan and Wu represented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral forms as follows:

Theorem 1.4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\le \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J^{\alpha}_{1/a-}(f\circ h)(1/b) \\ +J^{\alpha}_{1/b+}(f\circ h)(1/a) \end{array}\right\} \\ &\le \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.5}$$

with $\alpha > 0$ and $h(x) = 1/x$.

In [14] Latif et. al. gave the following definition:

Definition 1.4. A function $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $2ab/a+b$ if

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x \in [a,b]$.

In [1] Chan and Wu represented Hermite-Hadamard-Fejer inequality for harmonically convex functions as follows:

Theorem 1.5. *Let $f : I \subseteq \mathbb{R}\backslash \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ and $g : [a, b] \subseteq \mathbb{R}\backslash \{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a + b$, then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx \leq \int_a^b \frac{f(x)g(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx. \tag{1.6}$$

In [12] İşcan and Kunt represented Hermite–Hadamard-Fejér type inequality for harmonically convex functions in fractional integral forms and established following identity as follows:

Theorem 1.6. *Let $f : [a, b] \to \mathbb{R}$ be harmonically convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{1/b+}(g\circ h)(1/a) + J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ \leq & \left[J^{\alpha}_{1/b+}(fg\circ h)(1/a) + J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \\ \leq & \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a) + J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \end{aligned} \tag{1.7}$$

with $\alpha > 0$ and $h(x) = 1/x$, $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

Lemma 1.7. [12]. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ and $a < b$. If $g : [a, b] \to \mathbb{R}$ is integrable and harmonically symmetric with respect to $2ab/a + b$, then the following equality for fractional integrals hold:*

$$\begin{aligned} & \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a) + J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ & -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a) + J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \\ = & \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left[\begin{array}{c} \int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}(g\circ h)(s)ds \\ -\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds \end{array}\right](f\circ h)'(t)dt \end{aligned} \tag{1.8}$$

with $\alpha > 0$ and $h(x) = 1/x$, $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

In this paper, we obtained some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for harmonically s-convex function in fractional integrals.

2. Main Results

Throughout this section, we take $\|g\|_{\infty} = \sup_{t\in[a,b]}|g(x)|$, for the continuous function $g : [a, b] \to \mathbb{R}$.

Theorem 2.1. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ and $a < b$. If $|f'|$ is harmonically s-convex on $[a, b]$, $g : [a, b] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, then the following inequality* *for fractional integrals hold:*

$$\begin{aligned}
&\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}\left(g\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(g\circ h\right)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}\left(fg\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(fg\circ h\right)(1/b)\right] \end{array} \right| \\
&\leq \frac{\|g\|_{\infty}ab\left(b-a\right)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[C_1\left(\alpha\right)\left|f'\left(a\right)\right|+C_2\left(\alpha\right)\left|f'\left(b\right)\right|\right]
\end{aligned} \tag{2.1}$$

where

$$C_1\left(\alpha\right)=\left[\begin{array}{c} \frac{b^{-2}}{\alpha+s+1}\ {}_2F_1\left(2,1;\alpha+s+2;1-\frac{a}{b}\right) \\ -\frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,\alpha+1;\alpha+s+2;1-\frac{a}{b}\right) \\ +\frac{\beta(\alpha+1,s+1)}{2^{s-2}(a+b)^2}\ {}_2F_1\left(2,\alpha+1;\alpha+s+2;\frac{b-a}{b+a}\right) \end{array} \right]$$

$$C_2\left(\alpha\right)=\left[\begin{array}{c} \frac{b^{-2}}{\alpha+s+1}\ {}_2F_1\left(2,\alpha+s+1;\alpha+s+2;1-\frac{a}{b}\right) \\ -\frac{\beta(s+1,\alpha+1)}{b^2}\ {}_2F_1\left(2,s+1;\alpha+s+2;1-\frac{a}{b}\right) \\ +\frac{1}{2^s}\frac{\beta(s+1,-\alpha+1)}{b^2}\ {}_2F_1\left(2,s+1;-\alpha+s+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right) \end{array} \right]$$

with $0<\alpha\leq 1$ *and* $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. From Lemma 1.7 we have

$$\begin{aligned}
&\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}\left(g\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(g\circ h\right)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}\left(fg\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(fg\circ h\right)(1/b)\right] \end{array} \right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left| \begin{array}{c} \int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}\left(g\circ h\right)(s)ds \\ -\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)ds \end{array} \right|\left|\left(f\circ h\right)'(t)\right|dt
\end{aligned} \tag{2.2}$$

Since g is harmonically symmetric with respect to $2ab/a+b$, using Definition 1.4 we have $g\left(\frac{1}{x}\right)=g(\frac{1}{(\frac{1}{a})+(\frac{1}{b})-x})$, for all $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

$$\begin{aligned}
&\left|\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}\left(g\circ h\right)(s)ds-\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)ds\right| \\
&= \left|\int_{\frac{1}{a}+\frac{1}{b}-t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)ds+\int_{\frac{1}{a}}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)ds\right| \\
&= \left|\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)ds\right| \\
&\leq \begin{cases} \int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)\right|ds & , \quad t\in\left[\frac{1}{b},\frac{a+b}{2ab}\right] \\ \int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)\right|ds & , \quad t\in\left[\frac{a+b}{2ab},\frac{1}{a}\right] \end{cases}
\end{aligned} \tag{2.3}$$

If we use (2.3) in (2.2), we have

$$\begin{aligned}
&\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right] \\ -\left[J_{1/b+}^{\alpha}\left(fg\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(fg\circ h\right)(1/b)\right] \end{array}\right| \\
&\le \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)\right|ds\right)\left|\left(f\circ h\right)'(t)\right|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)\right|ds\right)\left|\left(f\circ h\right)'(t)\right|dt \end{array}\right] \\
&\le \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\left|\left(f\circ h\right)'(t)\right|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\left|\left(f\circ h\right)'(t)\right|dt \end{array}\right] \\
&= \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\frac{1}{t^2}\left|f'(\frac{1}{t})\right|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\frac{1}{t^2}\left|f'(\frac{1}{t})\right|dt \end{array}\right]
\end{aligned}$$

Setting $t=\frac{ub+(1-u)a}{ab}$, and $dt=\left(\frac{b-a}{ab}\right)du$ gives

$$\begin{aligned}
&\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right] \\ -\left[J_{1/b+}^{\alpha}\left(fg\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(fg\circ h\right)(1/b)\right] \end{array}\right| \\
&\le \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \int_{0}^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \end{array}\right]
\end{aligned} \tag{2.4}$$

Since $|f'|$ is harmonically s-convex on $[a,b]$, we have

$$\left|f'(\frac{ab}{ub+(1-u)a})\right| \le u^s\left|f'(a)\right|+(1-u)^s\left|f'(b)\right| \tag{2.5}$$

If we use (2.5) in (2.4), we have

$$\begin{aligned}
&\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right] \\ -\left[J_{1/b+}^{\alpha}\left(fg\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(fg\circ h\right)(1/b)\right] \end{array}\right| \\
&\le \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
&\times\left[\begin{array}{c} \left[\begin{array}{c} \int_{0}^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}u^s du \\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}u^s du \end{array}\right]\left|f'(a)\right| \\ +\left[\begin{array}{c} \int_{0}^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)^s du \\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)^s du \end{array}\right]\left|f'(b)\right| \end{array}\right]
\end{aligned} \tag{2.6}$$

Calculating following integrals by Lemma 1.2, we have

$$\begin{aligned}
(2.7)\quad & \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} u^s du + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} u^s du \\
= & \int_0^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} u^s du + 2\int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} u^s du \\
\leq & \int_0^1 \frac{u^{\alpha+s}}{(ub+(1-u)a)^2} du - \int_0^1 \frac{u^s (1-u)^\alpha}{(ub+(1-u)a)^2} du + 2\int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub+(1-u)a)^2} u^s du \\
= & \int_0^1 \frac{(1-u)^{\alpha+s}}{(ua+(1-u)b)^2} du - \int_0^1 \frac{(1-u)^s u^\alpha}{(ua+(1-u)b)^2} du + \frac{1}{2^s}\int_0^1 \frac{u^s (1-u)^\alpha}{\left(\frac{u}{2}b + \left(1-\frac{u}{2}\right)a\right)^2} du \\
= & \int_0^1 (1-u)^{\alpha+s} b^{-2} \left(1 - u\left(1-\frac{a}{b}\right)\right)^{-2} du \\
& - \int_0^1 (1-u)^s u^\alpha b^{-2} \left(1 - u\left(1-\frac{a}{b}\right)\right)^{-2} du \\
& + \frac{1}{2^s}\int_0^1 (1-v)^s v^\alpha \left(\frac{a+b}{2}\right)^{-2} \left(1 - v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv \\
= & \left[\begin{array}{c}
\frac{b^{-2}}{\alpha+s+1}\ {}_2F_1\left(2,1;\alpha+s+2;1-\frac{a}{b}\right) \\
-\frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,\alpha+1;\alpha+s+2;1-\frac{a}{b}\right) \\
+\frac{\beta(\alpha+1,s+1)}{2^{s-2}(a+b)^2}\ {}_2F_1\left(2,\alpha+1;\alpha+s+2;\frac{b-a}{b+a}\right)
\end{array}\right] \\
= & C_1(\alpha)
\end{aligned}$$

and similarly we get

$$\begin{aligned}
(2.8)\quad & \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} (1-u)^s du + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} (1-u)^s du \\
\leq & \int_0^1 \frac{u^{\alpha+s}}{(ua+(1-u)b)^2} du - \int_0^1 \frac{u^s (1-u)^\alpha}{(ua+(1-u)b)^2} du \\
& - \frac{1}{2^s}\int_0^1 \frac{u^s (1-u)^{-\alpha}}{\left(\frac{u}{2}a + \left(1-\frac{u}{2}\right)b\right)^2} du \\
= & \int_0^1 u^{\alpha+s} b^{-2} \left(1 - u\left(1-\frac{a}{b}\right)\right)^{-2} du \\
& - \int_0^1 u^s (1-u)^\alpha b^{-2} \left(1 - u\left(1-\frac{a}{b}\right)\right)^{-2} du \\
& - \frac{1}{2^s}\int_0^1 u^s (1-u)^{-\alpha} b^{-2} \left(1 - \frac{u}{2}\left(1-\frac{a}{b}\right)\right)^{-2} du \\
= & \left[\begin{array}{c}
\frac{b^{-2}}{\alpha+s+1}\ {}_2F_1\left(2,\alpha+s+1;\alpha+s+2;1-\frac{a}{b}\right) \\
-\frac{\beta(s+1,\alpha+1)}{b^2}\ {}_2F_1\left(2,s+1;\alpha+s+2;1-\frac{a}{b}\right) \\
+\frac{1}{2^s}\frac{\beta(s+1,-\alpha+1)}{b^2}\ {}_2F_1\left(2,s+1;-\alpha+s+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)
\end{array}\right] \\
= & C_2(\alpha)
\end{aligned}$$

If we use (2.7) and (2.8) in (2.6), we have (2.1). This completes the proof. ∎

Corollary 2.2. *In Theorem 2.1;*

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer inequality for harmonically s-convex functions which is related the right-hand side of (1.6):

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx - \int_a^b \frac{f(x)g(x)}{x^2} dx \right| \\ \leq & \frac{\|g\|_\infty (b-a)^2}{2} \left[C_1(1) |f'(a)| + C_2(1) |f'(b)| \right], \end{aligned}$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard type inequality for harmonically s-convex function in fractional integral forms which is related the right-hand side of (1.5):

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a} \right)^\alpha \left\{ \begin{array}{c} J^\alpha_{1/a-} (f \circ h)(1/b) \\ +J^\alpha_{1/b+} (f \circ h)(1/a) \end{array} \right\} \right| \\ \leq & \frac{ab(b-a)}{2} \left[C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(b)| \right], \end{aligned}$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for harmonically s-convex function which is related the right-hand side of (1.4):

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left[C_1(1) |f'(a)| + C_2(1) |f'(b)| \right].$$

Theorem 2.3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|f'|^q, q \geq 1$, is harmonically s-convex on $[a,b]$, $g : [a,b] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} & \left| \begin{array}{c} \frac{f(a)+f(b)}{2} \left[J^\alpha_{1/b+} (g \circ h)(1/a) + J^\alpha_{1/a-} (g \circ h)(1/b) \right] \\ - \left[J^\alpha_{1/b+} (fg \circ h)(1/a) + J^\alpha_{1/a-} (fg \circ h)(1/b) \right] \end{array} \right| \\ \leq & \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)} \left(\frac{b-a}{ab} \right)^\alpha \\ & \times \left[\begin{array}{c} C_3^{1-\frac{1}{q}}(\alpha) \left[\left(\begin{array}{c} C_4(\alpha) |f'(a)|^q \\ +C_5(\alpha) |f'(b)|^q \end{array} \right) \right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(\alpha) \left[\left(\begin{array}{c} C_7(\alpha) |f'(a)|^q \\ +C_8(\alpha) |f'(b)|^q \end{array} \right) \right]^{\frac{1}{q}} \end{array} \right] \end{aligned} \tag{2.9}$$

where

$$
\begin{aligned}
C_3(\alpha) &= \frac{2(a+b)^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;\tfrac{b-a}{b+a}\right)\ , \\
C_4(\alpha) &= \frac{(a+b)^{-2}}{2^{s-1}(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;\tfrac{b-a}{b+a}\right)\ , \\
C_5(\alpha) &= \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+s+1)}\ {}_2F_1\left(2,\alpha+s+1;\alpha+s+2;1-\frac{a}{b}\right) \\ -\frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,s+1;\alpha+s+2;1-\frac{a}{b}\right) \\ \frac{1}{2^{s+1}}\frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,s+1;\alpha+s+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right) \end{array}\right], \\
C_6(\alpha) &= \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\ +C_3(\alpha) \end{array}\right] \\
C_7(\alpha) &= \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+s+1)}\ {}_2F_1\left(2,1;\alpha+s+2;1-\frac{a}{b}\right) \\ -\frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,\alpha+1;\alpha+s+2;1-\frac{a}{b}\right)\ +C_4(\alpha) \end{array}\right] \\
C_8(\alpha) &= \left[\begin{array}{c} \frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,s+1;\alpha+s+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+s+1)}\ {}_2F_1\left(2,\alpha+s+1;\alpha+s+2;1-\frac{a}{b}\right)\ +C_5(\alpha) \end{array}\right],
\end{aligned}
$$

with $0<\alpha\leq 1$ *and* $h(x)=1/x$*,* $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. Using (2.4), power mean inequality and the harmonically s-convexity of $|f'|^q$, it follows that

$$
\begin{aligned}
&\left|\begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \end{array}\right| \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du \\ +\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|du \end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^q du\right)^{\frac{1}{q}} \end{array}\right. \\
&\left.\begin{array}{c} +\left(\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{ub+(1-u)a}\right)\right|^q du\right)^{\frac{1}{q}} \end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
(2.10)\quad & \leq \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
& \times\left[\begin{array}{c}
\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\
\times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left[u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q\right]du\right)^{\frac{1}{q}} \\
+\left(\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\
\times\left(\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left[u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q\right]du\right)^{\frac{1}{q}}
\end{array}\right] \\
& = \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
& \times\left[\begin{array}{c}
\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\
\times\left(\begin{array}{c}\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}u^s du\left|f'(a)\right|^q \\ +\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)^s du\left|f'(b)\right|^q\end{array}\right)^{\frac{1}{q}} \\
+\left(\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\
\times\left(\begin{array}{c}\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}u^s du\left|f'(a)\right|^q \\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)^s du\left|f'(b)\right|^q\end{array}\right)^{\frac{1}{q}}
\end{array}\right]
\end{aligned}
$$

Calculating following integrals by Lemma 1.2, we have

$$
\begin{aligned}
(2.11)\quad \int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du & \leq \int_0^{\frac{1}{2}}\frac{(1-2u)^{\alpha}}{(ub+(1-u)a)^2}du \\
& = \frac{1}{2}\int_0^1\frac{(1-u)^{\alpha}}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^2}du \\
& = 2(a+b)^{-2}\int_0^1 v^{\alpha}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv \\
& = \frac{2(a+b)^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\tfrac{b-a}{b+a}\right) \\
& = C_3(\alpha)
\end{aligned}
$$

$$
\begin{aligned}
(2.12)\quad \int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}u^s du & \leq \int_0^{\frac{1}{2}}\frac{(1-2u)^{\alpha}}{(ub+(1-u)a)^2}u^s du \\
& = \frac{1}{2^{s+1}}\int_0^1\frac{u^s(1-u)^{\alpha}}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^2}du \\
& = \frac{(a+b)^{-2}}{2^{s-1}}\int_0^1(1-v)v^{\alpha}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}du \\
& = \frac{(a+b)^{-2}}{2^{s-1}(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;\tfrac{b-a}{b+a}\right) \\
& = C_4(\alpha)
\end{aligned}
$$

$$\begin{aligned}(2.13)\qquad &\int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}(1-u)^s\,du \\ \le\; &\int_0^1 \frac{u^{\alpha+s}}{(ua+(1-u)b)^2}du - \int_0^1 \frac{u^s(1-u)^\alpha}{(ua+(1-u)b)^2}du \\ &+\frac{1}{2^{s+1}}\int_0^1 \frac{u^s(1-u)^\alpha}{\left(\frac{u}{2}a+(1-\frac{u}{2})b\right)^2}du \\ =\; &\left[\begin{array}{c} \frac{b^{-2}}{(\alpha+s+1)}\ {}_2F_1\left(2,\alpha+s+1;\alpha+s+2;1-\frac{a}{b}\right) \\ -\frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,s+1;\alpha+s+2;1-\frac{a}{b}\right) \\ \frac{1}{2^{s+1}}\frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,s+1;\alpha+s+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\end{array}\right] \\ =\; &C_5(\alpha)\end{aligned}$$

$$\begin{aligned}(2.14)\int_{\frac{1}{2}}^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}du \;&=\; \int_0^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}du + \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha-u^\alpha}{(ub+(1-u)a)^2}du \\ &\le\; \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) \ +C_3(\alpha)\end{array}\right] \\ &=\; C_6(\alpha)\end{aligned}$$

$$\begin{aligned}(2.15)\qquad &\int_{\frac{1}{2}}^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}u^s du \\ =\; &\int_0^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}u^s du + \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha-u^\alpha}{(ub+(1-u)a)^2}u^s du \\ \le\; &\left[\begin{array}{c} \frac{b^{-2}}{(\alpha+s+1)}\ {}_2F_1\left(2,1;\alpha+s+2;1-\frac{a}{b}\right) \\ -\frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,\alpha+1;\alpha+s+2;1-\frac{a}{b}\right) \ +C_4(\alpha)\end{array}\right] \\ =\; &C_7(\alpha)\end{aligned}$$

$$\begin{aligned}(2.16)\qquad &\int_{\frac{1}{2}}^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}(1-u)^s du \\ =\; &\int_0^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}(1-u)^s du + \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha-u^\alpha}{(ub+(1-u)a)^2}(1-u)^s du \\ \le\; &\left[\begin{array}{c} \frac{\beta(\alpha+1,s+1)}{b^2}\ {}_2F_1\left(2,s+1;\alpha+s+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+s+1)}\ {}_2F_1\left(2,\alpha+s+1;\alpha+s+2;1-\frac{a}{b}\right) \ +C_5(\alpha)\end{array}\right] \\ =\; &C_8(\alpha)\end{aligned}$$

If we use $(2.11-2.16)$ in (2.10), we have (2.9).This completes the proof. ∎

Corollary 2.4. *In Theorem 2.3;*

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer inequality for harmonically s-convex functions which is related the right-hand side of (1.6):

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right| \\
&\leq \frac{\|g\|_\infty (b-a)^2}{2}\left[\begin{array}{c} C_3^{1-\frac{1}{q}}(1)\left[\left(\begin{array}{c} C_4(1)|f'(a)|^q \\ +C_5(1)|f'(b)|^q \end{array}\right)\right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(1)\left[\left(\begin{array}{c} C_7(1)|f'(a)|^q \\ +C_8(1)|f'(b)|^q \end{array}\right)\right]^{\frac{1}{q}} \end{array}\right],
\end{aligned}$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard type inequality for harmonically s-convex function in fractional integral forms which is related the right-hand side of (1.5):

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J_{1/a-}^{\alpha}(f\circ h)(1/b) \\ +J_{1/b+}^{\alpha}(f\circ h)(1/a) \end{array}\right\}\right| \\
&\leq \frac{ab(b-a)}{2}\left[\begin{array}{c} C_3^{1-\frac{1}{q}}(\alpha)\left[\left(\begin{array}{c} C_4(\alpha)|f'(a)|^q \\ +C_5(\alpha)|f'(b)|^q \end{array}\right)\right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(\alpha)\left[\left(\begin{array}{c} C_7(\alpha)|f'(a)|^q \\ +C_8(\alpha)|f'(b)|^q \end{array}\right)\right]^{\frac{1}{q}} \end{array}\right],
\end{aligned}$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for s-harmonically convex function which is related the right-hand side of (1.4):

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\left[\begin{array}{c} C_3^{1-\frac{1}{q}}(1)\left[\left(\begin{array}{c} C_4(1)|f'(a)|^q \\ +C_5(1)|f'(b)|^q \end{array}\right)\right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(1)\left[\left(\begin{array}{c} C_7(1)|f'(a)|^q \\ +C_8(1)|f'(b)|^q \end{array}\right)\right]^{\frac{1}{q}} \end{array}\right].
\end{aligned}$$

We can state another inequality for $q > 1$ as follows:

Theorem 2.5. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|f'|^q, q > 1$, is harmonically s-convex on $[a,b]$, $g : [a,b] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, then the following inequality for fractional integrals hold:*

$$\begin{aligned}
&\left|\begin{array}{c} \frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}(g\circ h)(1/a) + J_{1/a-}^{\alpha}(g\circ h)(1/b)\right] \\ -\left[J_{1/b+}^{\alpha}(fg\circ h)(1/a) + J_{1/a-}^{\alpha}(fg\circ h)(1/b)\right] \end{array}\right| \\
&\leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
&\quad\times\left[\begin{array}{c} C_9^{\frac{1}{p}}(\alpha)\left[\frac{|f'(a)|^q+\left(2^{s+1}-1\right)|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}} \\ +C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{\left(2^{s+1}-1\right)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}} \end{array}\right]
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned} C_9(\alpha) &= \left(\frac{a+b}{2}\right)^{-2p} \frac{1}{2(\alpha p+1)} \; {}_2F_1\left(2p, \alpha p+1; \alpha p+2; \tfrac{b-a}{b+a}\right) , \\ C_{10}(\alpha) &= b^{-2p} \frac{1}{2(\alpha p+1)} \; {}_2F_1\left(2p, 1; \alpha p+2; \tfrac{1}{2}\left(1-\tfrac{a}{b}\right)\right) , \end{aligned}$$

with $0<\alpha\leq 1$, $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$ *and* $1/p+1/q=1$.

Proof. Using (2.4), Hölder's inequality and the harmonically s-convexity of $|f'|^q$, it follows that

$$\begin{aligned} &\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \end{array} \right| \\ &\leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \left[\begin{array}{c} \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha-u^\alpha}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \\ +\int_{\frac{1}{2}}^1 \frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \end{array} \right] \\ &\leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \left[\begin{array}{c} \left(\int_0^{\frac{1}{2}} \frac{[(1-u)^\alpha-u^\alpha]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{1}{2}} \left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}} \end{array} \right. \\ &\qquad \left. \begin{array}{c} +\left(\int_{\frac{1}{2}}^1 \frac{[u^\alpha-(1-u)^\alpha]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{1}{2}}^1 \left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}} \end{array} \right] \\ &\leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \left[\begin{array}{c} \left(\int_0^{\frac{1}{2}} \frac{[(1-u)^\alpha-u^\alpha]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{1}{2}} u^s|f'(a)|^q+(1-u)^s|f'(b)|^q du\right)^{\frac{1}{q}} \end{array} \right. \\ &\qquad \left. \begin{array}{c} +\left(\int_{\frac{1}{2}}^1 \frac{[u^\alpha-(1-u)^\alpha]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{1}{2}}^1 u^s|f'(a)|^q+(1-u)^s|f'(b)|^q du\right)^{\frac{1}{q}} \end{array} \right] \\ &= \frac{\|g\|_\infty ab(b-a)}{2\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\ &\quad \times\left[\left(\int_0^{\frac{1}{2}} \frac{[(1-u)^\alpha-u^\alpha]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \left[\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\int_{\frac{1}{2}}^1 \frac{[u^\alpha-(1-u)^\alpha]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \left[\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\right] \end{aligned} \tag{2.18}$$

Calculating following integrals by Lemma 1.2, we have

$$(2.19)\quad \begin{aligned} \int_0^{\frac{1}{2}} \frac{[(1-u)^\alpha - u^\alpha]^p}{(ub+(1-u)a)^{2p}} du &\le \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ub+(1-u)a)^{2p}} du \\ &= \frac{1}{2}\int_0^1 \frac{(1-u)^{\alpha p}}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^{2p}} du \\ &= \frac{1}{2}\int_0^1 v^{\alpha p}\left(\frac{a+b}{2}\right)^{-2p}\left[1-v\left(\frac{b-a}{b+a}\right)\right]^{-2p} dv \\ &= \left(\frac{a+b}{2}\right)^{-2p} \frac{1}{2(\alpha p+1)} \\ &\quad \times\ {}_2F_1\left(2p, \alpha p+1; \alpha p+2; \tfrac{b-a}{b+a}\right) \\ &= C_9(\alpha) \end{aligned}$$

Similarly, we have

$$(2.20)\quad \begin{aligned} \int_{\frac{1}{2}}^1 \frac{[u^\alpha - (1-u)^\alpha]^p}{(ub+(1-u)a)^{2p}} du &\le \int_{\frac{1}{2}}^1 \frac{(2u-1)^{\alpha p}}{(ub+(1-u)a)^{2p}} du \\ &= \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ua+(1-u)b)^{2p}} du \\ &= \frac{1}{2}\int_0^1 \frac{(1-v)^{\alpha p}}{\left(\frac{v}{2}a+(1-\frac{v}{2})b\right)^{2p}} dv \\ &= \frac{1}{2}\int_0^1 (1-v)^{\alpha p} b^{-2p}\left(1-\frac{v}{2}\left(1-\frac{a}{b}\right)\right)^{-2p} dv \\ &= b^{-2p}\frac{1}{2(\alpha p+1)}\ {}_2F_1\left(2p, 1; \alpha p+2; \tfrac{1}{2}\left(1-\tfrac{a}{b}\right)\right) \\ &= C_{10}(\alpha) \end{aligned}$$

If we use (2.19) and (2.20) in (2.18), we have (2.17).This completes the proof. ∎

Corollary 2.6. *In Theorem 2.5;*

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer inequality for harmonically s-convex functions which is related the right-hand side of (1.6):

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx \right| \\ \le\ & \frac{\|g\|_\infty (b-a)^2}{2} \left[\begin{array}{c} C_9^{\frac{1}{p}}(1)\left[\frac{|f'(a)|^q+\left(2^{s+1}-1\right)|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}} \\ +C_{10}^{\frac{1}{p}}(1)\left[\frac{\left(2^{s+1}-1\right)|f'(a)|^q+|f'(b)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}} \end{array} \right] \end{aligned}$$

(2) If we take $g(x)=1$ we have following Hermite-Hadamard type inequality for harmonically s-convex function in fractional integral forms which is related the right-hand side of (1.5)*:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J_{1/a-}^{\alpha}(f\circ h)(1/b) \\ +J_{1/b+}^{\alpha}(f\circ h)(1/a)\end{array}\right\}\right|$$
$$\leq \frac{ab(b-a)}{2}\left[\begin{array}{c} C_{9}^{\frac{1}{p}}(\alpha)\left[\frac{|f'(a)|^{q}+\left(2^{s+1}-1\right)|f'(b)|^{q}}{2^{s+1}(s+1)}\right]^{\frac{1}{q}} \\ +C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{\left(2^{s+1}-1\right)|f'(a)|^{q}+|f'(b)|^{q}}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\end{array}\right]$$

(3) If we take $\alpha=1$ and $g(x)=1$ we have the following Hermite-Hadamard type inequality for harmonically s-convex function which is related the right-hand side of (1.4)*:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_{a}^{b}\frac{f(x)}{x^{2}}dx\right|$$
$$\leq \frac{ab(b-a)}{2}\left[\begin{array}{c} C_{9}^{\frac{1}{p}}(1)\left[\frac{|f'(a)|^{q}+\left(2^{s+1}-1\right)|f'(b)|^{q}}{2^{s+1}(s+1)}\right]^{\frac{1}{q}} \\ +C_{10}^{\frac{1}{p}}(1)\left[\frac{\left(2^{s+1}-1\right)|f'(a)|^{q}+|f'(b)|^{q}}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\end{array}\right]$$

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