The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 12, Issue 1, Article 1, pp. 1-16, 2015

ASYMPTOTIC ANALYSIS OF POSITIVE DECREASING SOLUTIONS OF A CLASS OF SYSTEMS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS IN THE FRAMEWORK OF REGULAR VARIATION

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Received 9 July, 2014; accepted 18 November, 2014; published 16 March, 2015.

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ABSTRACT. The system of nonlinear differential equations

(A) $$x'' = p_1(t)x^{\alpha_1} + q_1(t)y^{\beta_1}, \qquad y'' = p_2(t)x^{\alpha_2} + q_2(t)y^{\beta_2}$$

is under consideration, where α_i and β_i are positive constants and $p_i(t)$ and $q_i(t)$ are continuous regularly varying functions on $[a, \infty)$. Two kinds of criteria are established for the existence of strongly decreasing regularly varying solutions with negative indices of (A) with precise asymptotic behavior at infinity. Fixed point techniques and basic theory of regular variation are utilized for this purpose.

Key words and phrases: Systems of nonlinear differential equations, Positive solutions, Decreasing solutions, Asymptotic behavior, Regularly varying functions.

1991 *Mathematics Subject Classification.* Primary 34C11, Secondary 26A12.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In this paper we consider two-dimensional nonlinear differential systems of the form

$$x'' = p_1(t)x^{\alpha_1} + q_1(t)y^{\beta_1}, \qquad y'' = p_2(t)x^{\alpha_2} + q_2(t)y^{\beta_2}, \tag{A}$$

under the assumption that

(a) α_i and β_i, $i = 1, 2$, are positive constants;

(b) $p_i(t)$ and $q_i(t)$, $i = 1, 2$, are continuous regularly varying functions on $[a, \infty)$, $a > 0$.

By a positive solution of system (A) we mean a vector function $(x(t), y(t))$ both components of which are positive and satisfy (A) in some neighborhood of infinity, say for $t \geq T$. We focus our attention on those positive solutions of (A) both components of which are decreasing and tend to zero as $t \to \infty$. Such solutions are referred to as *strongly decreasing solutions* of (A). It is clear that a strongly decreasing solution $(x(t), y(t))$ of (A) satisfies the system of integral equations

$$\begin{aligned} x(t) &= \int_t^\infty \int_s^\infty \Big\{ p_1(r)x(r)^{\alpha_1} + q_1(r)y(r)^{\beta_1} \Big\} dr ds, \\ y(t) &= \int_t^\infty \int_s^\infty \Big\{ p_2(r)x(r)^{\alpha_2} + q_2(r)y(r)^{\beta_2} \Big\} dr ds \end{aligned} \tag{1.1}$$

for $t \geq T$.

The aim of this paper is to establish the existence of strongly decreasing solutions of (A) with precise asymptotic behavior as $t \to \infty$ by solving (1.1) in the class of regularly varying functions of negative indices. (For the definition of regularly varying functions see Section 2.) Fixed point techniques and basic theory of regular variation are utilized for this purpose.

We notice that there are two special cases of (A), the *diagonal system* (A_d) and the *cyclic system* (A_c), whose strongly decreasing solutions can be thoroughly analyzed in the framework of regular variation:

$$x'' = p_1(t)x^{\alpha_1}, \qquad y'' = q_2(t)y^{\beta_2}, \tag{A_d}$$

$$x'' = q_1(t)y^{\beta_1}, \qquad y'' = p_2(t)x^{\alpha_2}. \tag{A_c}$$

In fact necessary and sufficient conditions have recently been found for these two systems to have regularly varying solutions with negative indices. Then one would expect that system (A) which can be regarded as small perturbations of (A_d) or of (A_c) in a certain sense may possess strongly decreasing solutions belonging to the same class of regularly varying functions. The truth of this expectation will be verified in Sections 3 and 4 which are devoted to perturbed systems of (A_d) and (A_c), respectively. Examples illustrating the main results will be presented in Section 5.

2. REGULARLY VARYING FUNCTIONS

For the reader's convenience we summarize here the definition and some basic properties of regularly varying functions (in the sense of Karamata).

Definition 2.1. A measurable function $f : [0, \infty) \to (0, \infty)$ is called *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all} \;\; \lambda > 0.$$

The totality of regularly varying functions of index ρ is denoted by RV(ρ). We often use the symbol SV to denote RV(0) and call members of SV *slowly varying functions*. Any function $f \in$ RV(ρ) is expressed as $f(t) = t^{\rho} g(t)$ with $g \in$ SV, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. One of the most important properties of regularly varying functions is the following *representation theorem*.

Proposition 2.1. *$f(t) \in$ RV(ρ) if and only if $f(t)$ is represented in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s} ds \right\}, \quad t \geq t_0 \tag{2.1}$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad and \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If in particular $c(t) \equiv c_0$ in (2.1), then $f(t)$ is referred to as a *normalized* regularly varying function of index ρ.

Typical examples of slowly varying functions are: all functions tending to some positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \;\; \alpha_n \in \mathbb{R}, \quad \text{and} \quad \exp\left\{ \prod_{n=1}^{N} (\log_n t)^{\beta_n} \right\}, \;\; \beta_n \in (0, 1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function

$$L(t) = \exp\left\{ (\log t)^{\theta} \cos\, (\log t)^{\theta} \right\}, \quad \theta \in \left(0, \frac{1}{2}\right),$$

is a slowly varying function which is oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} L(t) = 0.$$

The following result illustrates operations which preserve slow variation.

Proposition 2.2. *Let $L(t)$, $L_1(t)$, $L_2(t)$ be slowly varying. Then, $L(t)^{\alpha}$ for any $\alpha \in \mathbb{R}$, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$) are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following

Proposition 2.3. *Let $f(t) \in$ SV. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{\varepsilon} f(t) = \infty, \qquad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0.$$

A simple criterion for determining the regularity of differentiable positive functions follows.

Proposition 2.4. *A differentiable positive function $f(t)$ is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t \frac{f'(t)}{f(t)} = \rho.$$

The following result called Karamata's integration theorem is of highest importance in handling slowly and regularly varying functions analytically, and will be used throughout Sections 3 and 4.

Proposition 2.5. *Let* $L(t) \in$ SV. *Then,*

(i) if $\alpha > -1$,

$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) if $\alpha < -1$,

$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) if $\alpha = -1$,

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathit{SV} \quad and \quad m(t) = \int_t^\infty \frac{L(s)}{s} ds \in \mathit{SV},$$

provided $L(t)/t$ *is integrable near the infinity in the latter case.*

Here the symbol $\sim$ is used to mean the asymptotic equivalence between two positive functions

$$f(t) \sim g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

If $f(t) \sim g(t)$, $t \to \infty$, and $g \in$ RV(ρ), then $f \in$ RV(ρ).

Given two positive functions $f(t)$ and $g(t)$, we write $f(t) \asymp g(t)$, $t \to \infty$, to denote that there exist positive constants k and K such that $kf(t) \leq g(t) \leq Kf(t)$ for all large t. It is clear that $f(t) \sim g(t)$, $t \to \infty$, implies $f(t) \asymp g(t)$, $t \to \infty$, but not conversely.

Definition 2.2. If $f(t) \asymp g(t)$, $t \to \infty$, and $g \in$ RV(ρ), then $f(t)$ is said to be a *nearly regularly varying function of index* ρ.

It is known that $2 + \sin(\log\log t)$ is slowly varying, but $2 + \sin(\log t)$ is not. The latter is nearly slowly varying since it holds that $2 + \sin(\log t) \asymp 2 + \sin(\log\log t)$, $t \to \infty$. It follows that, for any $\rho \in \mathbb{R}$, $t^\rho(2 + \sin(\log t))$ is nearly regularly varying, but not regularly varying, of index ρ.

A vector function $(x(t), y(t))$ is said to be *regularly varying of index* (ρ, σ) if $x(t)$ and $y(t)$ are regularly varying of indices ρ and σ, respectively. Nearly regularly varying vector functions can be defined analogously.

For the most complete exposition of theory of regular variation and its applications we refer to the book of Bingham, Goldie and Teugels [1]. See also Seneta [16]. A comprehensive survey of results up to 2000th on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [15]. Since the publication of [15] there has been an increasing interest in the analysis of ordinary differential equations by means of regularly varying functions, and thus theory of regular variation has proved to be a powerful tool of determining the accurate asymptotic behavior of positive solutions for a variety of nonlinear differential equations of Emden-Fowler and Thomas-Fermi types. See, for example, the papers [4] - [14].

3. PERTURBATIONS OF THE DIAGONAL SYSTEM (A_d)

We first show that useful information about the existence of strongly decreasing solutions can be drawn for system (A) which is viewed as a small perturbation of the diagonal system

$$x'' = p_1(t)x^{\alpha_1}, \qquad y'' = q_2(t)y^{\beta_2}, \tag{A_d}$$

which is in fact a set of two independent Thomas-Fermi differential equations. It is assumed that

$$\alpha_1 < 1, \qquad \beta_2 < 1, \tag{3.1}$$

and that $p_1(t)$ and $q_2(t)$ are continuous regularly varying functions of indices λ_1 and μ_2, respectively, expressed as

$$p_1(t) = t^{\lambda_1} l_1(t), \qquad q_2(t) = t^{\mu_2} m_2(t), \qquad l_1, m_2 \in \text{SV}. \tag{3.2}$$

Our discussions essentially depend on the fact that complete analysis can be made of the existence and asymptotic behavior of strongly decreasing reglarly varying solutions of the diagonal system (A_d).

Proposition 3.1. *Let conditions* (3.1) *and* (3.2) *be satisfied. Then, system* (A_d) *has regularly varying solutions* $(x(t), y(t))$ *with negative index* (ρ, σ) *if and only if*

$$\lambda_1 < -2, \qquad \mu_2 < -2, \tag{3.3}$$

in which case ρ *and* σ *are defined by*

$$\rho = \frac{\lambda_1 + 2}{1 - \alpha_1}, \qquad \sigma = \frac{\mu_2 + 2}{1 - \beta_2}, \tag{3.4}$$

and any such solution enjoys one and the same asymptotic behavior

$$x(t) \sim X(t), \qquad y(t) \sim Y(t), \qquad t \to \infty, \tag{3.5}$$

where $X(t) \in$ *RV(ρ) and* $Y(t) \in$ *RV(σ) are given by*

$$\begin{aligned} X(t) &= \left[\frac{t^2 p_1(t)}{(-\rho)(1-\rho)}\right]^{\frac{1}{1-\alpha_1}} = t^{\rho}\left[\frac{l_1(t)}{(-\rho)(1-\rho)}\right]^{\frac{1}{1-\alpha_1}}, \\ Y(t) &= \left[\frac{t^2 q_2(t)}{(-\sigma)(1-\sigma)}\right]^{\frac{1}{1-\beta_2}} = t^{\sigma}\left[\frac{m_2(t)}{(-\sigma)(1-\sigma)}\right]^{\frac{1}{1-\beta_2}}. \end{aligned} \tag{3.6}$$

The proof of this proposition can be found in the paper [14].

Of particular importance in proving the main result is the fact that $X(t)$ and $Y(t)$ satisfy the asymptotic relations

$$\int_t^{\infty}\int_s^{\infty} p_1(r)X(r)^{\alpha_1} dr ds \sim X(t), \qquad \int_t^{\infty}\int_s^{\infty} q_2(r)Y(r)^{\beta_2} dr ds \sim Y(t), \qquad t \to \infty. \tag{3.7}$$

The elementary proof of (3.7) is left to the reader. Here and throughout we assume that

$$p_i \in \text{RV}(\lambda_i), \qquad q_i \in \text{RV}(\mu_i), \qquad i = 1, 2, \tag{3.8}$$

and

$$p_i(t) = t^{\lambda_i} l_i(t), \qquad q_i(t) = t^{\mu_i} m_i(t), \qquad l_i, m_i \in \text{SV}, \qquad i = 1, 2. \tag{3.9}$$

One of our main results now follows.

Theorem 3.2. *Assume that* (3.8), (3.1) *and* (3.3) *hold. Let the negative constants* ρ *and* σ *be given by* (3.4) *and consider the functions* $X(t)$ *and* $Y(t)$ *defined by* (3.7). *Suppose that*

$$\lim_{t\to\infty} \frac{q_1(t)Y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1}} = 0, \qquad \lim_{t\to\infty} \frac{p_2(t)X(t)^{\alpha_2}}{q_2(t)Y(t)^{\beta_2}} = 0. \tag{3.10}$$

Then, system (A) *possesses strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of index* (ρ, σ) *whose asymptotic behavior is governed by the unique formula* (3.5).

Proof. In view of (3.7) there exists $T > a$ such that

$$\begin{aligned} \frac{1}{2}X(t) \le \int_t^\infty \int_s^\infty p_1(r)X(r)^{\alpha_1} drds \;&\le\; 2X(t), \\ \frac{1}{2}Y(t) \le \int_t^\infty \int_s^\infty q_2(r)Y(r)^{\beta_2} drds \;&\le\; 2Y(t), \end{aligned} \tag{3.11}$$

for $t \ge T$. Let h, H, k and K be positive constants such that

$$h \le 2^{-\frac{1}{1-\alpha_1}}, \qquad H \ge 4^{\frac{1}{1-\alpha_1}}, \qquad k \le 2^{-\frac{1}{1-\beta_2}}, \qquad K \ge 4^{\frac{1}{1-\beta_2}}, \tag{3.12}$$

and choose $T > a$ large enough so that in addition to (3.11) the following inequlaities hold

$$\frac{q_1(t)Y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1}} \le \frac{h^{\alpha_1}}{K^{\beta_1}}, \qquad \frac{p_2(t)X(t)^{\alpha_2}}{q_2(t)Y(t)^{\beta_2}} \le \frac{k^{\beta_2}}{H^{\alpha_2}}, \qquad t \ge T. \tag{3.13}$$

which is possible because of (3.10).

Let us now define the integral operators

$$\begin{aligned} \mathcal{F}(x,y)(t) &= \int_t^\infty \int_s^\infty \Big\{ p_1(r)x(r)^{\alpha_1} + q_1(r)y(r)^{\beta_1} \Big\} drds, \\ \mathcal{G}(x,y)(t) &= \int_t^\infty \int_s^\infty \Big\{ p_2(r)x(r)^{\alpha_2} + q_2(r)y(r)^{\beta_2} \Big\} drds, \end{aligned} \tag{3.14}$$

for $t \ge T$, and let it act on the set

$$\mathcal{W} = \Big\{ (x,y) \in C[T,\infty)^2 : hX(t) \le x(t) \le HX(t), \;\; kY(t) \le y(t) \le KY(t), \quad t \ge T \Big\}. \tag{3.15}$$

Finally we consider the mapping $\Phi : \mathcal{W} \to C[T,\infty)^2$ defined by

$$\Phi(x,y)(t) = \Big(\mathcal{F}(x,y)(t), \mathcal{G}(x,y)(t) \Big), \quad t \ge T. \tag{3.16}$$

It can be shown that Φ is a self-map on $\mathcal{W}$ and sends $\mathcal{W}$ into a relatively compact subset of $C[T,\infty)^2$.

(i) $\Phi(\mathcal{W}) \subset \mathcal{W}$. Let $(x,y) \in \mathcal{W}$. Using (3.13), we see that for $t \ge T$

$$\begin{aligned} p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1} &= p_1(t)x(t)^{\alpha_1} \left(1 + \frac{q_1(t)y(t)^{\beta_1}}{p_1(t)x(t)^{\alpha_1}} \right) \\ &\le p_1(t)x(t)^{\alpha_1} \left(1 + \frac{K^{\beta_1} q_1(t)Y(t)^{\beta_1}}{h^{\alpha_1} p_1(t)X(t)^{\alpha_1}} \right) \le 2p_1(t)x(t)^{\alpha_1} \end{aligned}$$

and

$$p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2} = q_2(t)y(t)^{\beta_2}\left(1 + \frac{p_2(t)x(t)^{\alpha_2}}{q_2(t)y(t)^{\beta_2}}\right)$$
$$\leq q_2(t)y(t)^{\beta_2}\left(1 + \frac{H^{\alpha_2}p_2(t)X(t)^{\alpha_2}}{k^{\beta_2}q_2(t)Y(t)^{\beta_2}}\right) \leq 2q_2(t)y(t)^{\beta_2}.$$

and hence that

$$\mathcal{F}(x,y)(t) \leq 2\int_t^\infty \int_s^\infty p_1(r)x(r)^{\alpha_1}drds$$
$$\leq 2H^{\alpha_1}\int_t^\infty \int_s^\infty p_1(r)X(r)^{\alpha_1}drds \leq 4H^{\alpha_1}X(t) \leq HX(t),$$

and

$$\mathcal{G}(x,y)(t) \leq 2\int_t^\infty \int_s^\infty q_2(r)y(r)^{\beta_2}drds$$
$$\leq 2K^{\beta_2}\int_t^\infty \int_s^\infty q_2(r)Y(r)^{\beta_2}drds \leq 4K^{\beta_2}Y(t) \leq KY(t)$$

for $t \geq T$, where (3.11) and (3.12) have been used. On the other hand, we easily find that

$$\mathcal{F}(x,y)(t) \geq \int_t^\infty \int_s^\infty p_1(r)x(r)^{\alpha_1}drds$$
$$\geq h^{\alpha_1}\int_t^\infty \int_s^\infty p_1(r)X(r)^{\alpha_1}drds \geq \frac{1}{2}h^{\alpha_1}X(t) \geq hX(t),$$

and

$$\mathcal{G}(x,y)(t) \geq \int_t^\infty \int_s^\infty q_2(r)y(r)^{\beta_2}drds$$
$$\geq k^{\beta_2}\int_t^\infty \int_s^\infty q_2(r)Y(r)^{\beta_2}drds \geq \frac{1}{2}k^{\beta_2}Y(t) \geq kY(t)$$

for $t \geq T$. It follows that $hX(t) \leq \mathcal{F}(x,y)(t) \leq HX(t)$ and $kY(t) \leq \mathcal{G}(x,y)(t) \leq KY(t)$ for $t \geq T$, which implies that $\Phi(x,y) \in \mathcal{W}$, that is, Φ maps $\mathcal{W}$ into itself.

(ii) $\Phi(\mathcal{W})$ is relatively compact. The inclusion $\Phi(\mathcal{W}) \subset \mathcal{W}$ implies that $\Phi(\mathcal{W})$ is uniformly bounded on $[T,\infty)$. The inequalities

$$0 \geq (\mathcal{F}(x,y))'(t) \geq -\int_t^\infty \Big\{H^{\alpha_1}p_1(s)X(s)^{\alpha_1} + K^{\beta_1}q_1(s)Y(s)^{\beta_1}\Big\}ds,$$
$$0 \geq (\mathcal{G}(x,y))'(t) \geq -\int_t^\infty \Big\{H^{\alpha_2}p_2(s)X(s)^{\alpha_2} + K^{\beta_2}q_2(s)Y(s)^{\beta_2}\Big\}ds,$$

holding for $t \geq T$ and for all $(x,y) \in \mathcal{W}$ ensure that $\Phi(\mathcal{W})$ is equicontinuous on $[T,\infty)$. The relative compactness of $\Phi(\mathcal{W})$ follows from the Arzela-Ascoli lemma.

(iii) Φ is a continuous map. Let $\{(x_n,y_n)\}$ be a sequence in $\mathcal{W}$ converging to $(x,y) \in \mathcal{W}$ in the topology of $C[T,\infty)^2$, which means that $\{(x_n(t),y_n(t))\}$ converges to $(x(t),y(t))$ as $n \to \infty$ uniformly on any compact subinterval of $[T,\infty)$. We need to prove that

$$\mathcal{F}(x_n,y_n)(t) \to \mathcal{F}(x,y)(t) \qquad \text{and} \qquad \mathcal{G}(x_n,y_n)(t) \to \mathcal{G}(x,y)(t)$$

uniformly on compact subintervals of $[T,\infty)$. But this follows readily from the Lebesgue dominated convergence theorem applied to the right-hand side of the following inequalities

$$|\mathcal{F}(x_n,y_n)(t)-\mathcal{F}(x,y)(t)| \le \int_t^\infty s\Big(p_1(s)|x_n(s)^{\alpha_1}-x(s)^{\alpha_1}| + q_1(s)|y_n(s)^{\beta_1}-y(s)^{\beta_1}|\Big)ds,$$

$$|\mathcal{G}(x_n,y_n)(t)-\mathcal{G}(x,y)(t)| \le \int_t^\infty s\Big(p_2(s)|x_n(s)^{\alpha_2}-x(s)^{\alpha_2}| + q_2(s)|y_n(s)^{\beta_2}-y(s)^{\beta_2}|\Big)ds.$$

The details may be omitted.

Therefore, by the Schauder-Tychonoff fixed point theorem (see, e.g., Chapter I of Coppel [2]) there exists $(x,y)\in\mathcal{W}$ such that $(x,y)=\Phi(x,y)=(\mathcal{F}(x,y),\mathcal{G}(x,y))$, i.e.,

$$\begin{aligned} x(t) &= \int_t^\infty\int_s^\infty\Big\{p_1(r)x(r)^{\alpha_1}+q_1(r)y(r)^{\beta_1}\Big\}drds, \\ y(t) &= \int_t^\infty\int_s^\infty\Big\{p_2(r)x(r)^{\alpha_2}+q_2(r)y(r)^{\beta_2}\Big\}drds, \end{aligned} \tag{3.17}$$

for $t\ge T$, This means that $(x(t),y(t))$ gives a solution of the system (A) on $[T,\infty)$, and the membership $(x,y)\in\mathcal{W}$ shows that it is nearly regularly varying of index (ρ,σ).

It remains to verify that $(x(t),y(t))$ is actually a regularly varying solution of index (ρ,σ) having the asymptotic behavior (3.5). This can be done with the help of the generalized L'Hospital's rule stated below. For the proof see e.g. Haupt and Aumann [3]. ∎

Lemma 3.3. *Let $f(t),g(t)\in C^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t)=\lim_{t\to\infty} g(t)=\infty \qquad and \qquad g'(t)>0 \quad for\ all\ large\ \ t,$$

or

$$\lim_{t\to\infty} f(t)=\lim_{t\to\infty} g(t)=0 \qquad and \qquad g'(t)<0 \quad for\ all\ large\ \ t.$$

Then,

$$\liminf_{t\to\infty}\frac{f'(t)}{g'(t)}\le\liminf_{t\to\infty}\frac{f(t)}{g(t)}, \qquad \limsup_{t\to\infty}\frac{f(t)}{g(t)}\le\limsup_{t\to\infty}\frac{f'(t)}{g'(t)}.$$

Let $(x(t),y(t))$ be the nearly regularly varying solution of (A) constructed as a solution of the system of integral equations (3.17). Define $u(t)$ and $v(t)$ on $[a,\infty)$ by

$$\begin{aligned} u(t) &= \int_t^\infty\int_s^\infty\Big\{p_1(r)X(r)^{\alpha_1}+q_1(r)Y(r)^{\beta_1}\Big\}drds, \\ v(t) &= \int_t^\infty\int_s^\infty\Big\{p_2(r)X(r)^{\alpha_2}+q_2(r)Y(r)^{\beta_2}\Big\}drds. \end{aligned} \tag{3.18}$$

Since (3.10) implies

$$p_1(t)X(t)^{\alpha_1}+q_1(t)Y(t)^{\beta_1}\sim p_1(t)X(t)^{\alpha_1}, \quad p_2(t)X(t)^{\alpha_2}+q_2(t)Y(t)^{\beta_2}\sim q_2(t)Y(t)^{\beta_2}, \tag{3.19}$$

as $t\to\infty$, from (3.7) it follows that

$$u(t)\sim X(t), \qquad v(t)\sim Y(t), \qquad t\to\infty. \tag{3.20}$$

It is clear that

$$p_1(t)x(t)^{\alpha_1}+q_1(t)y(t)^{\beta_1}\sim p_1(t)x(t)^{\alpha_1}, \quad p_2(t)x(t)^{\alpha_2}+q_2(t)y(t)^{\beta_2}\sim q_2(t)y(t)^{\beta_2}, \tag{3.21}$$

as $t \to \infty$. Let us consider the following superior and inferior limits:

(3.22) $$l = \liminf_{t\to\infty} \frac{x(t)}{u(t)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{u(t)}, \quad m = \liminf_{t\to\infty} \frac{y(t)}{v(t)}, \quad M = \limsup_{t\to\infty} \frac{y(t)}{v(t)}.$$

Clearly, these are finite positive constants. Applying Lemma 3.3 to l repeatedly, we find that

$$\begin{aligned} l &\geq \liminf_{t\to\infty} \frac{\displaystyle\int_t^\infty \{p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_2}\}ds}{\displaystyle\int_t^\infty \{p_1(s)X(s)^{\alpha_1} + q_1(s)Y(s)^{\beta_2}\}ds} \\ &\geq \liminf_{t\to\infty} \frac{p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1} + q_1(t)Y(t)^{\beta_1}} \\ &= \liminf_{t\to\infty} \frac{p_1(t)x(t)^{\alpha_1}}{p_1(t)X(t)^{\alpha_1}} \\ &= \left(\liminf_{t\to\infty} \frac{x(t)}{X(t)}\right)^{\alpha_1} = \left(\liminf_{t\to\infty} \frac{x(t)}{u(t)}\right)^{\alpha_1} = l^{\alpha_1}, \end{aligned}$$

where (3.19) and (3.21) have been used in the middle and (3.20) has been used in the last step. Thus, we have

$$l \geq l^{\alpha_1} \qquad \Longrightarrow \qquad l \geq 1 \quad \text{because} \quad \alpha_1 < 1.$$

Similarly, applying Lemma 3.3 to L, we have

$$L \leq L^{\alpha_1} \qquad \Longrightarrow \qquad L \leq 1 \quad \text{because} \quad \alpha_1 < 1.$$

It follows therefore that $l = L = 1$, which implies that

$$\lim_{t\to\infty} \frac{x(t)}{u(t)} = 1 \qquad \Longrightarrow \qquad x(t) \sim u(t) \sim X(t) \quad \text{as} \quad t \to \infty.$$

On the other hand, Lemma 3.3 applied to m and M shows that

$$\lim_{t\to\infty} \frac{y(t)}{v(t)} = 1 \qquad \Longrightarrow \qquad y(t) \sim v(t) \sim Y(t) \quad \text{as} \quad t \to \infty,$$

and so it is concluded that $(x(t), y(t))$ is regularly varying of negative index (ρ, σ) enjoying the asymptotic behavior (3.5) as $t \to \infty$. This completes the proof of Theorem 3.2.

Remark 3.1. Let us take a close look at condition (3.10) in Theorems 3.2. Using (3.9) we see that

$$\frac{q_1(t)Y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1}} = t^{\mu_1+\beta_1\sigma-\lambda_1-\alpha_1\rho}L(t), \qquad \frac{p_2(t)X(t)^{\alpha_2}}{q_2(t)Y(t)^{\beta_2}} = t^{\lambda_2+\alpha_2\rho-\mu_2-\beta_2\sigma}M(t),$$

for some $L, M \in$ SV, so that (3.10) is satisfied regardless of $L(t)$ and $M(t)$ if

(3.23) $$\mu_1 + \beta_1\sigma < \lambda_1 + \alpha_1\rho \quad \text{and} \quad \lambda_2 + \alpha_2\rho < \mu_2 + \beta_2\sigma,$$

(cf. Proposition 2.3, or equivalently, if

$$\mu_1 + \beta_1\sigma < \rho - 2 \quad \text{and} \quad \lambda_2 + \alpha_2\rho < \sigma - 2.$$

This remark combined with Theorem 3.2 yields the following practical criteria for system (A) to have strongly decreasing solutions.

Corollary 3.4. *Assume that* (3.8), (3.1) *and* (3.3) *hold. Let* $\rho < 0$ *and* $\sigma < 0$ *be defined by* (3.4)*. If* (3.23) *holds, then system* (A) *possesses strongly decreasing regularly varying solutions of index* (ρ, σ) *whose asymptotic behavior is governed by the formula* (3.5)*.*

4. PERTURBATIONS OF THE CYCLIC SYSTEM (A_c)

We will show that different information about strongly decreasing solutions of system (A) can be acquired by regarding (A) as a small perturbation of the cyclic system

$$x'' = q_1(t)y^{\beta_1}, \qquad y'' = p_2(t)x^{\alpha_2}, \tag{A$_c$}$$

whose in-depth analysis from the angle of regular variation has recently been attempted by Jaroš and Kusano [6]. Our development is based on their result (Proposition 4.1 below) characterizing the existence of regularly varying solutions of negative indices for (A_c).

As in the preceding section we assume that $p_i(t)$ and $q_i(t)$, $i = 1, 2$, are regularly varying functions satisfying (3.8) and (3.9). One of our essential requirements is that

$$\alpha_2\beta_1 < 1. \tag{4.1}$$

Proposition 4.1. *Let (4.1) hold. System (A_c) has strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of negative index* (ρ, σ) *if and only if*

$$\mu_1 + 2 + \beta_1(\lambda_2 + 2) < 0, \qquad \alpha_2(\mu_1 + 2) + \lambda_2 + 2 < 0, \tag{4.2}$$

in which case ρ *and* σ *are given by*

$$\rho = \frac{\mu_1 + 2 + \beta_1(\lambda_2 + 2)}{1 - \alpha_2\beta_1}, \qquad \sigma = \frac{\alpha_2(\mu_1 + 2) + \lambda_2 + 2}{1 - \alpha_2\beta_1}, \tag{4.3}$$

and any such solution enjoys one and the same asymptotic behavior

$$x(t) \sim X(t), \qquad y(t) \sim Y(t), \qquad t \to \infty, \tag{4.4}$$

where the functions $X \in \mathrm{RV}(\rho)$ *and* $Y \in \mathrm{RV}(\sigma)$ *are defined by*

$$\begin{aligned} X(t) &= \left[\left(\frac{t^2 q_1(t)}{(-\rho)(1-\rho)}\right)\left(\frac{t^2 p_2(t)}{(-\sigma)(1-\sigma)}\right)^{\beta_1}\right]^{\frac{1}{1-\alpha_2\beta_1}}, \\ Y(t) &= \left[\left(\frac{t^2 q_1(t)}{(-\rho)(1-\rho)}\right)^{\alpha_2}\left(\frac{t^2 p_2(t)}{(-\sigma)(1-\sigma)}\right)\right]^{\frac{1}{1-\alpha_2\beta_1}}. \end{aligned} \tag{4.5}$$

In the subsequent discussions a crucial role is played by the cyclic system of asymptotic relations

$$\int_t^\infty \int_s^\infty q_1(r)Y(r)^{\beta_1} dr ds \sim X(t), \qquad \int_t^\infty \int_s^\infty p_2(r)X(r)^{\alpha_2} dr ds \sim Y(t) \tag{4.6}$$

as $t \to \infty$. In fact, rewriting $X(t)$ and $Y(t)$ as

$$X(t) = t^\rho \left[\left(\frac{m_1(t)}{(-\rho)(1-\rho)}\right)\left(\frac{l_2(t)}{(-\sigma)(1-\sigma)}\right)^{\beta_1}\right]^{\frac{1}{1-\alpha_2\beta_1}},$$

$$Y(t) = t^\sigma \left[\left(\frac{m_1(t)}{(-\rho)(1-\rho)}\right)^{\alpha_2}\left(\frac{l_2(t)}{(-\sigma)(1-\sigma)}\right)\right]^{\frac{1}{1-\alpha_2\beta_1}},$$

and using the relations

$$\rho = \mu_1 + \beta_1\sigma + 2, \qquad \sigma = \lambda_2 + \alpha_2\rho + 2,$$

we compute

$$\begin{aligned}\int_t^\infty q_1(s)Y(s)^{\beta_1}ds &= \int_t^\infty s^{\mu_1+\beta_1\sigma}m_1(s)\left[\left(\frac{m_1(s)}{(-\rho)(1-\rho)}\right)^{\alpha_2}\left(\frac{l_2(s)}{(-\sigma)(1-\sigma)}\right)\right]^{\frac{\beta_1}{1-\alpha_2\beta_1}}ds \\ &= \int_t^\infty s^{\rho-2}m_1(s)\left[\left(\frac{m_1(s)}{(-\rho)(1-\rho)}\right)^{\alpha_2}\left(\frac{l_2(s)}{(-\sigma)(1-\sigma)}\right)\right]^{\frac{\beta_1}{1-\alpha_2\beta_1}}ds \\ &= \frac{t^{\rho-1}m_1(t)}{(1-\rho)}\left[\left(\frac{m_1(t)}{(-\rho)(1-\rho)}\right)^{\alpha_2}\left(\frac{l_2(t)}{(-\sigma)(1-\sigma)}\right)\right]^{\frac{\beta_1}{1-\alpha_2\beta_1}},\end{aligned}$$

and

$$\begin{aligned}\int_t^\infty\int_s^\infty q_1(r)Y(r)^{\beta_1}drds &\sim t^\rho\frac{m_1(t)}{(-\rho)(1-\rho)}\left[\left(\frac{m_1(t)}{(-\rho)(1-\rho)}\right)^{\alpha_2}\left(\frac{l_2(t)}{(-\sigma)(1-\sigma)}\right)\right]^{\frac{\beta_1}{1-\alpha_2\beta_1}} \\ &= t^\rho\left[\left(\frac{m_1(t)}{(-\rho)(1-\rho)}\right)\left(\frac{l_2(t)}{(-\sigma)(1-\sigma)}\right)^{\beta_1}\right]^{\frac{1}{1-\alpha_2\beta_1}} = X(t), \quad t\to\infty.\end{aligned}$$

This establishes the first relation in (4.6), and the second relation can be verified analogously.

We state and prove the main result of this section.

Theorem 4.2. *Assume that* (4.1) *and* (4.2) *hold. Define the negative constants* ρ *and* σ *by* (4.3) *and consider the functions* $X(t)$ *and* $Y(t)$ *defined by (4.5). Suppose that*

$$\lim_{t\to\infty}\frac{p_1(t)X(t)^{\alpha_1}}{q_1(t)Y(t)^{\beta_1}} = 0, \qquad \lim_{t\to\infty}\frac{q_2(t)Y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2}} = 0. \tag{4.7}$$

Then, system (A) *possesses strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of negative index* (ρ,σ) *whose asymptotic behavior is governed by the unique formula* (4.4).

Proof. By (4.6) there exists $T > a$ such that

$$\begin{aligned}\frac{1}{2}X(t) &\le \int_t^\infty\int_s^\infty q_1(r)Y(r)^{\beta_1}drds \le 2X(t), \\ \frac{1}{2}Y(t) &\le \int_t^\infty\int_s^\infty p_2(r)X(r)^{\alpha_2}drds \le 2Y(t).\end{aligned}\tag{4.8}$$

for $t \ge T$. Let h, H, k and K denote the constants

$$h = 2^{-\frac{1+\beta_1}{1-\alpha_2\beta_1}}, \quad H = 4^{\frac{1+\beta_1}{1-\alpha_2\beta_1}}, \quad k = 2^{-\frac{1+\alpha_2}{1-\alpha_2\beta_1}}, \quad K = 4^{\frac{1+\alpha_2}{1-\alpha_2\beta_1}}. \tag{4.9}$$

Because of (4.7) one can choose $T > a$ large enough so that in addition to (4.8) the following inequalities hold for $t \ge T$:

$$\frac{p_1(t)X(t)^{\alpha_1}}{q_1(t)Y(t)^{\beta_1}} \le \frac{k^{\beta_1}}{H^{\alpha_1}}, \qquad \frac{q_2(t)Y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2}} \le \frac{h^{\alpha_2}}{K^{\beta_2}}. \tag{4.10}$$

Let us define the set $\mathcal{W}$ by

$$\mathcal{W} = \Big\{(x,y)\in C[T,\infty)^2 : hX(t)\le x(t)\le HX(t), \;\; kY(t)\le y(t)\le KY(t), \;\; t\ge T\Big\} \tag{4.11}$$

and consider the mapping $\Phi : \mathcal{W} \to C[T,\infty)^2$ defined by

$$\Phi(x,y)(t) = \Big(\mathcal{F}(x,y)(t), \mathcal{G}(x,y)(t)\Big), \quad t \geq T, \tag{4.12}$$

where $\mathcal{F}(x,y)$ and $\mathcal{G}(x,y)$ are the integral operators given in (3.14).

Let $(x,y) \in \mathcal{W}$. Then, using (4.10) we have

$$\frac{p_1(t)x(t)^{\alpha_1}}{q_1(t)y(t)^{\beta_1}} \leq \frac{H^{\alpha_1}}{k^{\beta_1}}\frac{p_1(t)X(t)^{\alpha_1}}{q_1(t)Y(t)^{\beta_1}} \leq 1, \qquad \frac{q_2(t)y(t)^{\beta_2}}{p_2(t)x(t)^{\alpha_2}} \leq \frac{K^{\beta_2}}{h^{\alpha_2}}\frac{q_2(t)Y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2}} \leq 1,$$

whence we see that

$$\begin{aligned} p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1} &= q_1(t)y(t)^{\beta_1}\left(1 + \tfrac{p_1(t)x(t)^{\alpha_1}}{q_1(t)y(t)^{\beta_1}}\right) \leq 2q_1(t)y(t)^{\beta_1}, \\ p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2} &= p_2(t)x(t)^{\alpha_2}\left(1 + \tfrac{q_2(t)y(t)^{\beta_2}}{p_1(t)x(t)^{\alpha_2}}\right) \leq 2p_2(t)x(t)^{\alpha_2}. \end{aligned} \tag{4.13}$$

Combining (4.8), (4.9), (4.11) and (4.13), we obtain for $t \geq T$

$$\begin{aligned} \mathcal{F}(x,y)(t) &\leq 2\int_t^\infty \int_s^\infty q_1(r)y(r)^{\beta_1}drds \\ &\leq 2K^{\beta_1}\int_t^\infty \int_s^\infty q_1(r)Y(r)^{\beta_1}drds \leq 4K^{\beta_1}X(t) = HX(t), \\ \mathcal{G}(x,y)(t) &\leq 2\int_t^\infty \int_s^\infty p_2(r)x(r)^{\alpha_2}drds \\ &\leq 2H^{\alpha_2}\int_t^\infty \int_s^\infty p_2(r)X(r)^{\alpha_2}drds \leq 4H^{\alpha_2}Y(t) = KY(t). \end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned} \mathcal{F}(x,y)(t) &\geq \int_t^\infty \int_s^\infty q_1(r)y(r)^{\beta_1}drds \\ &\geq k^{\beta_1}\int_t^\infty \int_s^\infty q_1(r)Y(r)^{\beta_1}drds \geq \frac{1}{2}k^{\beta_1}X(t) = hX(t), \\ \mathcal{G}(x,y)(t) &\geq \int_t^\infty \int_s^\infty p_2(r)x(r)^{\alpha_2}drds \\ &\geq h^{\alpha_2}\int_t^\infty \int_s^\infty p_2(r)X(r)^{\alpha_2}drds \geq \frac{1}{2}h^{\alpha_2}Y(t) = kY(t) \end{aligned}$$

for $t \geq T$. This shows that $\Phi(x,y) \in \mathcal{W}$, ensuring that Φ is a self-map on $\mathcal{W}$.

Furthermore, one can verify in a routine manner that Φ is continuous and sends $\mathcal{W}$ into a relatively compact subset of $C[T,\infty)^2$. Consequently, the Schauder-Tychonoff fixed point theorem guarantees the existence of a fixed point $(x,y) \in \mathcal{W}$ of Φ, which gives birth to a strongly decreasing solution $(x(t), y(t))$ of system (A) which is nearly regularly varying of index (ρ,σ).

To prove that $(x(t),y(t))$ is really regularly varying of index (ρ,σ) we proceed as follows. Note that $(x(t),y(t))$ solves the system of integral equations (3.17) on $[T,\infty)$. Define $u(t)$

and $v(t)$ by (3.18), where we have to use the functions $X(t), Y(t)$ given by (4.5). Since (4.7) implies that

$$p_1(t)X(t)^{\alpha_1}+q_1(t)Y(t)^{\beta_1}\sim q_1(t)Y(t)^{\beta_1},\quad p_2(t)X(t)^{\alpha_2}+q_2(t)Y(t)^{\beta_2}\sim p_2(t)X(t)^{\alpha_2} \tag{4.14}$$

as $t\to\infty$, from (4.6) we obtain

$$u(t)\sim X(t),\qquad v(t)\sim Y(t),\qquad t\to\infty. \tag{4.15}$$

Use is made of the relations

$$p_1(t)x(t)^{\alpha_1}+q_1(t)y(t)^{\beta_1}\sim q_1(t)y(t)^{\beta_1},\quad p_2(t)x(t)^{\alpha_2}+q_2(t)y(t)^{\beta_2}\sim p_2(t)x(t)^{\alpha_2}, \tag{4.16}$$

which follow immeditately from (4.14). Consider the superior and inferior limits defined by

$$l=\liminf_{t\to\infty}\frac{x(t)}{u(t)},\quad L=\limsup_{t\to\infty}\frac{x(t)}{u(t)},\quad m=\liminf_{t\to\infty}\frac{y(t)}{v(t)},\quad M=\limsup_{t\to\infty}\frac{y(t)}{v(t)}. \tag{4.17}$$

It is clear that all of them are finite positive constants. We now apply Lemma 3.3 to l and m. Taking (4.15) and (4.16) into accout, we obtain

$$\begin{aligned}
l &\geq \liminf_{t\to\infty}\frac{\displaystyle\int_t^\infty\{p_1(s)x(s)^{\alpha_1}+q_1(s)y(s)^{\beta_1}\}ds}{\displaystyle\int_t^\infty\{p_1(s)X(s)^{\alpha_1}+q_1(s)Y(s)^{\beta_1}\}ds}\\
&\geq \liminf_{t\to\infty}\frac{p_1(t)x(t)^{\alpha_1}+q_1(t)y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1}+q_1(t)Y(t)^{\beta_1}}=\liminf_{t\to\infty}\frac{q_1(t)y(t)^{\beta_1}}{q_1(t)Y(t)^{\beta_1}}\\
&=\left(\liminf_{t\to\infty}\frac{y(t)}{Y(t)}\right)^{\beta_1}=\left(\liminf_{t\to\infty}\frac{y(t)}{v(t)}\right)^{\beta_1}=m^{\beta_1},
\end{aligned}$$

and

$$\begin{aligned}
m &\geq \liminf_{t\to\infty}\frac{\int_t^\infty\{p_2(s)x(s)^{\alpha_2}+q_2(s)y(s)^{\beta_2}\}ds}{\int_t^\infty\{p_2(s)X(s)^{\alpha_2}+q_2(s)Y(s)^{\beta_2}\}ds}\\
&\geq \liminf_{t\to\infty}\frac{p_2(t)x(t)^{\alpha_2}+q_2(t)y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2}+q_2(t)Y(t)^{\beta_2}}=\liminf_{t\to\infty}\frac{p_2(t)x(t)^{\alpha_2}}{p_2(t)X(t)^{\alpha_2}}\\
&=\left(\liminf_{t\to\infty}\frac{x(t)}{X(t)}\right)^{\alpha_2}=\left(\liminf_{t\to\infty}\frac{x(t)}{u(t)}\right)^{\alpha_2}=l^{\alpha_2}.
\end{aligned}$$

Thus, we have

$$l\geq m^{\beta_1}\quad\text{and}\quad m\geq l^{\alpha_2},$$

which implies that

$$l\geq l^{\alpha_2\beta_1}\quad\text{and}\quad m\geq m^{\alpha_2\beta_1}\quad\Longrightarrow\quad l\geq 1\quad\text{and}\quad m\geq 1\quad\text{because}\quad \alpha_2\beta_1<1. \tag{4.18}$$

Likewise, application of Lemma 3.3 to L and M yields

$$L\leq M^{\beta_1}\quad\text{and}\quad M\leq L^{\alpha_2},$$

which leads to

$$L\leq L^{\alpha_2\beta_1}\quad\text{and}\quad M\leq M^{\alpha_2\beta_1}\quad\Longrightarrow\quad L\leq 1\quad\text{and}\quad M\leq 1\quad\text{because}\quad \alpha_2\beta_1<1. \tag{4.19}$$

From (4.18) and (4.19) it follows that $l=L=1$ and $m=M=1$, that is,

$$\lim_{t\to\infty}\frac{x(t)}{u(t)}=1,\qquad \lim_{t\to\infty}\frac{y(t)}{v(t)}=1.$$

Therefore we conclude from (4.15) that

$$x(t) \sim u(t) \sim X(t), \qquad y(t) \sim v(t) \sim Y(t), \qquad t \to \infty,$$

confirming that $x \in$ RV(ρ) and $y \in$ RV(σ). This completes the proof of Theorem 4.2. ∎

Remark 4.1. In view of (3.9) the functions in (4.7) are expressed in the form

$$\frac{p_1(t)X(t)^{\alpha_1}}{q_1(t)Y(t)^{\beta_1}} = t^{\lambda_1+\alpha_1\rho-\mu_1-\beta_1\sigma}L(t), \quad \frac{q_2(t)Y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2}} = t^{\mu_2+\beta_2\sigma-\lambda_2-\alpha_2\rho}M(t),$$

for some $L, M \in$ SV, and so condition (4.7) is satisfied if

$$\lambda_1 + \alpha_1\rho < \mu_1 + \beta_1\sigma, \qquad \mu_2 + \beta_2\sigma < \lambda_2 + \alpha_2\rho. \tag{4.20}$$

This can be used as a useful criterion for the existence of strongly decreasing solutions for system (A).

Corollary 4.3. *Assume that* (4.1) *and* (4.2) *hold. Define* $\rho < 0$ *and* $\sigma < 0$ *by* (4.3)*. If* (4.20) *holds, then system* (A) *possesses strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of index* (ρ, σ) *whose asymptotic behavior is governed by the formula* (4.4)*.*

5. EXAMPLES

In the final section we give examples illustrating our main results: Theorems 3.2 and 4.2.

Example 5.1. *Consider the system*

$$\begin{aligned} x'' &= 3t^{-3}\exp\{-(\log t)^{\frac{1}{3}}\}x^{\frac{2}{3}} + 2t^{-\frac{5}{3}}\frac{\log t}{\log\log t}y^{\frac{4}{3}}, \\ y'' &= 2t^{-\frac{12}{5}}\exp\{(\log t)^{\frac{1}{3}}\}x^{\frac{3}{5}} + 5t^{-3}\Big(2+\sin(\log\log t)\Big)y^{\frac{3}{4}}. \end{aligned} \tag{5.1}$$

This system is a special case of (A) in which

$$\alpha_1 = \frac{2}{3}, \quad \beta_1 = \frac{4}{3}, \quad \alpha_2 = \frac{3}{5}, \quad \beta_2 = \frac{3}{4},$$
$$\lambda_1 = -3, \quad \mu_1 = -\frac{5}{3}, \quad \lambda_2 = -\frac{12}{5}, \quad \mu_2 = -3,$$

and

$$l_1 = 3\exp\{-(\log t)^{\frac{1}{3}}\}, \quad m_1 = 2\frac{\log t}{\log\log t}, \quad l_2 = 2\exp\{(\log t)^{\frac{1}{3}}\}, \quad m_2 = 5\Big(2+\sin(\log\log t)\Big).$$

(i) *First we consider* (5.1) *as a perturbation of the cyclic system*

$$x'' = 2t^{-\frac{5}{3}}\frac{\log t}{\log\log t}y^{\frac{4}{3}}, \qquad y'' = 2t^{-\frac{12}{5}}\exp\Big\{(\log t)^{\frac{1}{3}}\Big\}x^{\frac{3}{5}}. \tag{5.2}$$

Since

$$\rho_c = \frac{(\mu_1+2)+\beta_1(\lambda_2+2)}{1-\alpha_2\beta_1} = -1, \qquad \sigma_c = \frac{\alpha_2(\mu_1+2)+(\lambda_2+2)}{1-\alpha_2\beta_1} = -1,$$

Proposition 4.1 implies that (5.2) *has strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of index* $(-1, -1)$ *such that*

$$x(t) \sim X_c(t) = t^{-1}\left(\frac{m_1(t)}{2}\right)^5\left(\frac{l_2(t)}{2}\right)^{\frac{20}{3}}, \quad y(t) \sim Y_c(t) = t^{-1}\left(\frac{m_1(t)}{2}\right)^3\left(\frac{l_2(t)}{2}\right)^5$$

as $t \to \infty$. *It is elementary to check that*

$$\lambda_1 + \alpha_1 \rho_c = -\frac{11}{3} < -3 = \mu_1 + \beta_1 \sigma_c, \qquad \mu_2 + \beta_2 \sigma_c = -\frac{15}{4} < -3 = \lambda_2 + \alpha_2 \rho_c,$$

which shows that (5.1) *is a sufficiently "small"perturbation of* (5.2) *to which Theorem 4.2* (*or Corollary 4.3*) *is applicable. It follows that system* (5.1) *possesses strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of index* $(-1,-1)$ *all of which enjoy the unique asymptotic behavior*

$$x(t) \sim t^{-1}\Big(\frac{\log t}{\log\log t}\Big)^5 \exp\Big\{\frac{20}{3}(\log t)^{\frac{1}{3}}\Big\}, \qquad y(t) \sim t^{-1}\Big(\frac{\log t}{\log\log t}\Big)^3 \exp\Big\{5(\log t)^{\frac{1}{3}}\Big\},$$

as $t \to \infty$.

(ii) *One may consider* (5.1) *as a perturbation of the diagonal system*

$$x'' = 3t^{-3}\exp\{-(\log t)^{\frac{1}{3}}\}x^{\frac{2}{3}}, \qquad y'' = 5t^{-3}\Big(2 + \sin(\log\log t)\Big)y^{\frac{3}{4}}. \tag{5.3}$$

Since

$$\rho_d = \frac{\lambda_1 + 2}{1-\alpha_1} = -3, \qquad \sigma_d = \frac{\mu_2 + 2}{1-\beta_2} = -4, \tag{5.4}$$

by Proposition 3.1 implies that (5.3) *has strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of index* $(-3,-4)$ *such that*

$$x(t) \sim X_d(t) = t^{-3}\Big(\frac{l_1(t)}{12}\Big)^3, \qquad y(t) \sim Y_d(t) = t^{-4}\Big(\frac{m_2(t)}{20}\Big)^4, \qquad t \to \infty. \tag{5.5}$$

It is easily confirmed that

$$\mu_1 + \beta_1 \sigma_d < \lambda_1 + \alpha_1 \rho_d \qquad \textit{but} \qquad \lambda_2 + \alpha_2 \rho_d > \mu_2 + \beta_2 \sigma_d,$$

which violates the condition (3.10). *Therefore, one cannot apply Theorem 3.2 to* (5.3), *which means that nothing can be said at this stage about the existence or nonexistence of strongly decreasing solutions satisfying the asymptotic behavior* (5.5).

Example 5.2. *Consider the differential system*

$$\begin{aligned} x'' &= 3t^{-3}\exp\{-(\log t)^{\frac{1}{3}}\}x^{\frac{2}{3}} + 2t^{-\frac{5}{3}}\frac{\log t}{\log\log t}y^{\frac{4}{3}}, \\ y'' &= 2t^{-\frac{22}{5}}\exp\{(\log t)^{\frac{1}{3}}\}x^{\frac{3}{5}} + 5t^{-3}\Big(2+\sin(\log\log t)\Big)y^{\frac{3}{4}}. \end{aligned} \tag{5.6}$$

which is the same as system (5.1) *except that the regularity index* $-\frac{12}{5}$ *of* $p_2(t)$ *is replaced with* $-\frac{22}{5}$.

Using $\rho_d = -3$ *and* $\sigma_d = -4$, *we see that the condition* (3.10) *is satisfied:*

$$\mu_1 + \beta_1 \sigma_d = -7 < -5 = \lambda_1 + \alpha_1 \rho_d \qquad \textit{and} \qquad \lambda_2 + \alpha_2 \rho_d = -\frac{31}{5} < -6 = \mu_2 + \beta_2 \sigma_d.$$

Therefore, from Theorem 3.2 we conclude that system (5.6) *possesses strongly decreasing regularly varying solutions* $(x(t), y(t))$ *of index* $(-3,-4)$ *whose asymptotic behavior is governed by the unique formula* (5.5), *more precisely,*

$$x(t) \sim \frac{1}{64}t^{-3}\exp\Big\{-3(\log t)^{\frac{1}{3}}\Big\}, \qquad y(t) \sim \frac{1}{256}t^{-4}\Big(2+\sin(\log\log t)\Big)^4, \qquad t \to \infty.$$

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