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A SHORT PROOF OF AN OPEN INEQUALITY WITH POWER-EXPONENTIAL FUNCTIONS

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ABSTRACT. V. Cîrtoaje conjectured that $a^{3b} + b^{3a} + ((a-b)/2)^4 \leq 2$ holds for all nonnegative numbers a and b with $a + b = 2$. In this short note, we give a proof of the Cîrtoaje's conjecture with power-exponential functions.

Key words and phrases: Open inequality, Power-exponential function.

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1. INTRODUCTION

V. Cîrtoaje [1] proved that the inequality

$$a^{2b} + b^{2a} + \left(\frac{a-b}{2}\right)^2 \leq 2$$

holds for all nonnegative real numbers a and b with $a + b = 2$. Moreover, in [1], he posted several conjectures on inequalities with power-exponential functions. The inequality

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \leq 2 \tag{1.1}$$

is Conjecture 4.7 in [1]. In this short note, we give a proof of the inequality (1.1) holds for all $a, b \geq 0$ with $a + b = 2$.

Theorem 1.1. *If a and b are nonnegative real numbers with $a + b = 2$, then the inequality* (1.1) *holds.*

2. PRELIMINARIES AND PROOF OF THEOREM 1.1

We need the following lemma to prove our main theorem.

Lemma 2.1. *If $-1 \leq x \leq 1$, then*

$$(1+x)^{1-x} \leq \frac{1}{4}(1+x)^2(2-x^2)(2-2x+x^2).$$

Proof of Lemma 2.1. We set

$$f(x) = \ln\left(\frac{1}{4}(1+x)^2(2-x^2)(2-2x+x^2)\right) - \ln(1+x)^{1-x}.$$

Then we have

$$f(x) = \ln\frac{1}{4} + (1+x)\ln(1+x) + \ln(2-x^2) + \ln(2-2x+x^2)$$

and the derivatives

$$f'(x) = 1 - \frac{2x}{2-x^2} + \frac{-2+2x}{2-2x+x^2} + \ln(1+x),$$

$$f''(x) = \frac{x^2(16-16x-20x^2+8x^3+12x^4-8x^5+x^6)}{(1+x)(-2+x^2)^2(2-2x+x^2)^2}.$$

Here, we set $g(x) = 16 - 16x - 20x^2 + 8x^3 + 12x^4 - 8x^5 + x^6$.

(1) We first assume that $0 \leq x \leq 1$. We have the derivative $g'(x) = -16 - 40x + 24x^2 + 48x^3 - 40x^4 + 6x^5 = 2h_1(x) - 8xh_2(x)$, where $h_1(x) = -8 + 3x^5$ and $h_2(x) = 5 - 3x - 6x^2 + 5x^3$. Since $h_1(x) \leq h_1(1) = -5 < 0$, we get $h_1(x) \leq 0$ for any $0 \leq x \leq 1$. Since $h_2'(x) = -3 - 12x + 15x^2 < 0$, h_2 is strictly decreasing on the interval $(0, 1)$. Since $h_2(x) \geq h_2(1) = 1 > 0$, we get $h_2(x) \geq 0$ for any $0 \leq x \leq 1$. Therefore, the derivative $g'(x) < 0$ and hence $g(x)$ is strictly decreasing on the interval $(0, 1)$. Since $g(0) = 16$ and $g(1) = -7$, there exists uniquely a number $c \in (0, 1)$ such that $g(x) > 0$ for any $0 < x < c$ and $g(x) < 0$ for any $c < x < 1$. Hence, $f'(x)$ is strictly increasing on the interval $(0, c)$ and strictly decreasing on the interval $(c, 1)$. Since $f'(0) = 0$, $f'(c) > 0$ and $f'(1) = -1 + \ln 2 < 0$, there exists uniquity a number $\hat{c} \in (c, 1)$ such that $g(\hat{c}) = 0$. Thus, $f'(x) > 0$ for any $0 < x < \hat{c}$ and $f'(x) < 0$ for any $\hat{c} < x < 1$. Hence, $f(x)$ is strictly increasing on the interval $(0, \hat{c})$ and strictly decreasing on the interval $(\hat{c}, 1)$. Since $f(0) = 0$ and $f(1) = 0$, we can get $f(x) > 0$ for any $0 < x < 1$.

(2) We next assume that $-1 \le x \le 0$. We have $g(x) = 4h_3(x) + 12x^4 - 8x^5 + x^6$, where $h_3(x) = 4 - 4x - 5x^2 + 2x^3$. Since $h_3'(x) = -4 - 10x + 6x^2 = 2(-2+x)(1+3x)$, $h_3'(x) > 0$ for any $-1 < x < -\frac{1}{3}$ and $h_3'(x) < 0$ for any $-\frac{1}{3} < x < 0$. Therefore, $h_3(x)$ is strictly increasing on the interval $(-1, -\frac{1}{3})$ and strictly decreasing on the interval $(-\frac{1}{3}, 0)$. Since $h_3(-1) = 1$ and $h_3(0) = 4$, we can get $h_3(x) > 0$ for any $-1 < x < 0$. Since $12x^4 - 8x^5 + x^6 > 0$ and $h_3(x) > 0$ for any $-1 < x < 0$, we have $g(x) > 0$ for any $-1 < x < 0$. Hence, $f''(x) > 0$, so $f'(x)$ is strictly increasing on the interval $(-1, 0)$. Since $f'(0) = 0$, $f(x)$ is strictly decreasing on the interval $(-1, 0)$. Since $f(0) = 0$, we can get $f(x) > 0$ for any $-1 < x < 0$.

∎

By using Lemma 2.1, we can prove our main theorem immediately.

Proof of Theorem 1.1. Without loss of generality, we may assume that $a = 1+x$ and $b = 1-x$, where $0 \le x \le 1$. Here, the inequality (1.1) is equivalent to

$$(1+x)^{3(1-x)} + (1-x)^{3(1+x)} + x^4 - 2 \le 0.$$

By substituting $-x$ for x in Lemma 2.1, we notice that the inequality

$$(1-x)^{1+x} \le \frac{1}{4}(1-x)^2(2-x^2)(2+2x+x^2)$$

holds for any $0 \le x \le 1$. Therefore, by this inequality and Lemma 2.1, we can get the following inequality;

$$\begin{aligned}
&(1+x)^{3(1-x)} + (1-x)^{3(1+x)} + x^4 - 2 \\
&\le \left(\frac{1}{4}(1+x)^2(2-x^2)(2-2x+x^2)\right)^3 \\
&\qquad + \left(\frac{1}{4}(1-x)^2(2-x^2)(2+2x+x^2)\right)^3 + x^4 - 2 \\
&= \frac{1}{32}(1-x)x^4(1+x)(-16 - 32x^2 + 76x^4 - 68x^6 + 31x^8 - 8x^{10} + x^{12}).
\end{aligned}$$

We may show that

$$\begin{aligned}
&-16 - 32x^2 + 76x^4 - 68x^6 + 31x^8 - 8x^{10} + x^{12} \\
&= -2(8 + 16x^2 - 38x^4 + 17x^6) + x^6(-34 + 31x^2 - 8x^4 + x^6) \\
&\le 0
\end{aligned}$$

for any $0 \le x \le 1$. It suffices to show that $f(x) = 8 + 16x - 38x^2 + 17x^3 > 0$ and $g(x) = -34 + 31x - 8x^2 + x^3 < 0$. Since $f'(x) = 16 - 76x + 51x^2$, $f'(x) > 0$ for any $0 \le x \le \frac{2}{51}(19 - \sqrt{157})$ and $f'(x) < 0$ for any $\frac{2}{51}(19 - \sqrt{157}) \le x \le 1$. Thus, $f(x)$ is strictly increasing on the interval $(0, \frac{2}{51}(19 - \sqrt{157}))$ and $f(x)$ is strictly decreasing on the interval $(\frac{2}{51}(19 - \sqrt{157}), 1)$ Since $f(0) = 8$ and $f(1) = 3$, we can get $f(x) > 0$ for any $0 \le x \le 1$. Since $g'(x) = 31 - 16x + 3x^2 > 0$ for any $0 \le x \le 1$, $g(x)$ is strictly increasing on the interval $(0, 1)$. Since $g(1) = -10 < 0$, we can get $g(x) < 0$ for any $0 \le x \le 1$. Therefore, we completed the proof of Theorem 1.1. ∎

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