The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 11, Issue 1, Article 3, pp. 1-7, 2014

AN IMPROVEMENT OF THE HERMITE-HADAMARD INEQUALITY FOR FUNCTIONS CONVEX ON THE COORDINATES

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Received 21 August, 2013; accepted 14 February, 2014; published 26 June 2014.

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ABSTRACT. An improvement of the Hermite-Hadamard inequality for functions convex on the coordinates is given.

Key words and phrases: Functions convex on the coordinates, Hermite-Hadamard inequality.

2000 *Mathematics Subject Classification.* Primary 26D15. Secondary 26B25.

ISSN (electronic): 1449-5910

1. INTRODUCTION

A function $\varphi : C \to \mathbb{R}$ is said to be convex on a convex subset C of a real linear space X if

$$\varphi\left(t\boldsymbol{x} + (1-t)\,\boldsymbol{y}\right) \le t\varphi\left(\boldsymbol{x}\right) + (1-t)\,\varphi\left(\boldsymbol{y}\right) \tag{1.1}$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in C$ and $0 \le t \le 1$. It is said to be concave if (1.1) is reversed.

A function $\varphi : I \times J \to \mathbb{R}$, where $I \times J \subset \mathbb{R}^2$ and $I,\ J$ are convex sets, is called *convex on the coordinates* if the partial mappings $\varphi_y : I \to \mathbb{R}$ defined by $\varphi_y(u) = \varphi(u, y)$, and $\varphi_x : J \to \mathbb{R}$ defined by $\varphi_x(v) = \varphi(x, v)$, are convex for all $y \in J$ and $x \in I$. Analogously we define functions which are concave on the coordinates. Obviously, if $\varphi : I \times J \to \mathbb{R}$ is convex (concave), then it is also convex (concave) on the coordinates, but functions which are convex (concave) on the coordinates are not necessarily convex (concave) in the standard sense. For instance, the function $\varphi : [0, \infty\rangle^2 \to \mathbb{R}$ defined by

$$\varphi(x, y) = x^p y^q,$$

where $p, q \ge 1,$ is convex on the coordinates, but it is not convex in the standard sense. This means that the class of convex functions is a proper subclass of the class of functions which are convex on the coordinates. Analogously, for $0 < p < 1$ and $0 < q < 1$ φ is concave on the coordinates, but it is not concave in the standard sense unless $p + q \le 1.$

The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : I \longrightarrow \mathbb{R}$ is a convex function and $I = [a, b]$, $-\infty < a < b < \infty$, is known in the literature as the Hermite-Hadamard inequality (see for example [4, p. 137]) and beside the Jensen inequality is one of the two most famous inequalities for convex functions. In paper [1] Dragomir considered an inequality of Hadamard's type for functions convex on the coordinates defined on a rectangle from the plane $\mathbb{R}^2$. He proved the following theorem.

THEOREM A. *Suppose that* $\varphi : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$, *where* $-\infty < a < b < \infty$ *and* $-\infty < c < d < \infty$, *is convex on the coordinates on* $[a, b] \times [c, d]$. *Then*

$$\begin{aligned}
& \varphi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\le\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \varphi\left(x, \frac{c+d}{2}\right)\mathrm{d}x + \frac{1}{d-c}\int_c^d \varphi\left(\frac{a+b}{2}, y\right)\mathrm{d}y\right] \\
\le\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \varphi(x, y)\,\mathrm{d}x\mathrm{d}y \\
\le\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b \varphi(x, c)\,\mathrm{d}x + \frac{1}{b-a}\int_a^b \varphi(x, d)\,\mathrm{d}x + \right. \\
& \left. \frac{1}{d-c}\int_c^d \varphi(a, y)\,\mathrm{d}y + \frac{1}{d-c}\int_c^d \varphi(b, y)\,\mathrm{d}y\right] \\
\le\ & \frac{\varphi(a, c) + \varphi(a, d) + \varphi(b, c) + \varphi(b, d)}{4}.
\end{aligned} \tag{1.3}$$

These inequalities are sharp.

The resultas established in [1] were generalized and expanded in [2] but the goal of this paper is to improve the result stated in Theorem A.

2. IMPROVEMENT

Let E be a nonempty set and L a *linear class* of functions $f : E \to \mathbb{R}$ having the properties:

(L1) $(\forall f, g \in L)\,(\forall \alpha, \beta \in \mathbb{R})\ af + \beta g \in L$;
(L2) $\mathbf{1} \in L$ (that is if $(\forall t \in E)\, f(t) = 1$ then $f \in L$);
(L3) $(\forall f, g \in L)\,(\min\{f, g\} \in L \wedge \max\{f, g\} \in L)$ (*lattice* property).

We consider *positive linear functionals* $A : L \to \mathbb{R}$, that is, we assume:

(A1) $(\forall f, g \in L)\,(\forall \alpha, \beta \in \mathbb{R})\ \ A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ (linearity)
(A2) $(\forall f \in L)\,(f \geq 0 \longrightarrow A(f) \geq 0)$ (positivity).

If additionally the condition $A(\mathbf{1}) = 1$ is satisfied, we say that A is a *positive normalized linear functional*.

Obviously, $\left(\mathbb{R}^E, \leq\right)$ (with standard ordering) is a lattice. Also, it can be easily verified that a subspace $X \subseteq \mathbb{R}^E$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of the fact that for every $x \in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|(t) = |x(t)|, \quad x^+(t) = \max\{0, x(t)\}, \quad x^-(t) = -\min\{0, x(t)\}, \qquad t \in E,$$

and

$$x^+ + x^- = |x|, \qquad x^+ - x^- = x,$$
$$\min\{x, y\} = \frac{1}{2}(x + y - |x - y|), \quad \max\{x, y\} = \frac{1}{2}(x + y + |x - y|).$$

Recently, in [3], the following theorem was proved.

THEOREM B. *Let L satisfy $(L1)$, $(L2)$ and $(L3)$ on a nonempty set E and let A be a positive normalized linear functional. If $f : I \to \mathbb{R}$ is a continuous convex function and $[a, b] \subseteq I$ then for all $g \in L$ such that $g(E) \subseteq [a, b]$ and $f(g) \in L$ we have $A(g) \in [a, b]$ and*

$$f\left(\frac{pa + qb}{p + q}\right) \leq A(f(g)) \leq \frac{pf(a) + qf(b)}{p + q} - A(\tilde{g})\,\delta_f, \tag{2.1}$$

where p and q are any nonnegative real numbers such that

$$A(g) = \frac{pa + qb}{p + q} \tag{2.2}$$

and $\tilde{g}$, δ_f are defined by

$$\tilde{g} = \frac{1}{2}\mathbf{1} - \frac{\left|g - \frac{a+b}{2}\mathbf{1}\right|}{b - a}, \quad \delta_f = f(a) + f(b) - 2f\left(\frac{a + b}{2}\right).$$

It can be easily seen that an improvement of the Hermite-Hadamard inequality can be obtained as a special case of Theorem B. Set $E = [a, b]$ and $L = \mathcal{R}([a, b])$, where $\mathcal{R}(E)$ denotes the subspace of all (bounded) R-integrable functions on $[a, b]$ (note here that $\mathcal{R}([a, b])$ is a lattice since $f \in \mathcal{R}([a, b])$ implies $|f| \in \mathcal{R}([a, b])$). If we define

$$A(f) = \frac{1}{b - a}\int_a^b f(x)\mathrm{d}x$$

and $g = id_E$ we can easily see that A is a positive normalized linear functional and

$$A(g) = A(id_E) = \frac{1}{b - a}\int_a^b x\mathrm{d}x = \frac{a + b}{2}$$

which means that for $p = q = 1$ we have

$$A(g) = \frac{a+b}{2} = \frac{pa+qb}{p+q}.$$

If $f : [a,b] \to \mathbb{R}$ is a convex function than we have $f(g) = f \in \mathcal{R}([a,b]) = L$ and all the conditions of Theorem B are satisfied, hence we obtain

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2} - A(\tilde{g})\,\delta_f,$$

where δ_f is defined as in Theorem B and

$$\begin{aligned} A(\tilde{g}) &= A\left(\frac{1}{2}\mathbf{1} - \frac{\left|id_E - \frac{a+b}{2}\mathbf{1}\right|}{b-a}\right) = \frac{1}{b-a}\int_a^b \left(\frac{1}{2} - \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\right)\mathrm{d}x \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left(\frac{1}{2} - \frac{\frac{a+b}{2} - x}{b-a}\right)\mathrm{d}x + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b \left(\frac{1}{2} - \frac{x - \frac{a+b}{2}}{b-a}\right)\mathrm{d}x = \frac{1}{4} \end{aligned}$$

The condition that f has to be continuous on $[a,b]$, which is for an arbitrary A required in Theorem B for the same reasons as in Jessen's inequality (see [4, p. 45]), can be omitted in this special case.

In other words, as a special case of Theorem B we have the following corollary.

Corollary 2.1. *If $f : [a,b] \to \mathbb{R}$, $-\infty < a < b < \infty$, is a convex function then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{4} + \frac{1}{2}f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} \tag{2.3}$$

In the following we use (2.3) to obtain an improvement of Theorem A.

Theorem 2.2. *Suppose that $\varphi : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$, where $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$, is convex on the coordinates on $[a,b] \times [c,d]$. Then*

$$\begin{aligned} &\varphi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \le\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \varphi\left(x, \frac{c+d}{2}\right)\mathrm{d}x + \frac{1}{d-c}\int_c^d \varphi\left(\frac{a+b}{2}, y\right)\mathrm{d}y\right] \\ \le\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \varphi(x,y)\,\mathrm{d}x\mathrm{d}y \\ \le\ & \frac{1}{8}\left[\frac{1}{b-a}\int_a^b \left(\varphi(x,c) + \varphi(x,d) + 2\varphi\left(x, \frac{c+d}{2}\right)\right)\mathrm{d}x + \right. \\ & \left. \frac{1}{d-c}\int_c^d \left(\varphi(a,y) + \varphi(b,y) + 2\varphi\left(\frac{a+b}{2}, y\right)\right)\mathrm{d}y\right] \\ \le\ & \frac{\varphi(a,c) + \varphi(a,d) + \varphi(b,c) + \varphi(b,d)}{16} + \frac{1}{4}\varphi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \\ & \frac{\varphi\left(\frac{a+b}{2}, c\right) + \varphi\left(\frac{a+b}{2}, d\right) + \varphi\left(a, \frac{c+d}{2}\right) + \varphi\left(b, \frac{c+d}{2}\right)}{8} \end{aligned} \tag{2.4}$$

These inequalities are sharp.

Proof. The idea of the proof is the same as in [1] except that we use (2.3). Since φ is convex on the coordinates we know that $\varphi_y : [a,b] \to \mathbb{R}$ defined by $\varphi_y(u) = \varphi(u,y)$, and $\varphi_x : [c,d] \to \mathbb{R}$ defined by $\varphi_x(v) = \varphi(x,v)$, are convex for all $y \in [c,d]$ and $x \in [a,b]$. By (2.3) we have

$$\begin{aligned} \varphi_y\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \varphi_y(x)\mathrm{d}x \\ &\leq \frac{\varphi_y(a)+\varphi_y(b)}{4} + \frac{1}{2}\varphi_y\left(\frac{a+b}{2}\right) \end{aligned}$$

for all $y \in [c,d]$, that is

$$\begin{aligned} \varphi\left(\frac{a+b}{2},y\right) &\leq \frac{1}{b-a}\int_a^b \varphi(x,y)\mathrm{d}x \\ &\leq \frac{\varphi(a,y)+\varphi(b,y)}{4} + \frac{1}{2}\varphi\left(\frac{a+b}{2},y\right). \end{aligned}$$

Integrating this on $[c,d]$ and dividing by $(d-c)$ we obtain

$$\begin{aligned} &\frac{1}{d-c}\int_c^d \varphi\left(\frac{a+b}{2},y\right)\mathrm{d}y \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \varphi(x,y)\mathrm{d}x\mathrm{d}y \\ &\leq \frac{1}{4}\left[\frac{1}{d-c}\int_c^d \left(\varphi(a,y)+\varphi(b,y)+2\varphi\left(\frac{a+b}{2},y\right)\right)\right]\mathrm{d}y. \end{aligned} \tag{2.5}$$

In a similar way we get

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \varphi\left(x,\frac{c+d}{2}\right)\mathrm{d}x \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \varphi(x,y)\mathrm{d}x\mathrm{d}y \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b \left(\varphi(x,c)+\varphi(x,d)+2\varphi\left(x,\frac{c+d}{2}\right)\right)\right]\mathrm{d}x. \end{aligned} \tag{2.6}$$

Summing (2.5) and (2.6) we obtain

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{b-a}\int_a^b \varphi\left(x,\frac{c+d}{2}\right)\mathrm{d}x + \frac{1}{d-c}\int_c^d \varphi\left(\frac{a+b}{2},y\right)\mathrm{d}y\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \varphi(x,y)\mathrm{d}x\mathrm{d}y \\ &\leq \frac{1}{8}\left[\frac{1}{b-a}\int_a^b \left(\varphi(x,c)+\varphi(x,d)+2\varphi\left(x,\frac{c+d}{2}\right)\right)\mathrm{d}x + \right. \\ &\qquad \left.\frac{1}{d-c}\int_c^d \left(\varphi(a,y)+\varphi(b,y)+2\varphi\left(\frac{a+b}{2},y\right)\right)\mathrm{d}y\right] \end{aligned}$$

which are the second and the third inequality in (2.4).
By the Hermite-Hadamard inequality (or the left hand side of (2.3)) we also have

$$\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(x,\frac{c+d}{2})\mathrm{d}x$$

and

$$\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d\varphi\left(\frac{a+b}{2},y\right)\mathrm{d}y,$$

hence

$$\begin{aligned}&\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ \leq\;&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\varphi\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d\varphi\left(\frac{a+b}{2},y\right)\mathrm{d}y\right],\end{aligned}$$

which is the first inequality in (2.4). In the same way, by (2.3), we know that

$$\begin{aligned}\frac{1}{b-a}\int_a^b\varphi(x,c)\mathrm{d}x&\leq\frac{\varphi(a,c)+\varphi(b,c)}{4}+\frac{1}{2}\varphi\left(\frac{a+b}{2},c\right)\\ \frac{1}{b-a}\int_a^b\varphi(x,d)\mathrm{d}x&\leq\frac{\varphi(a,d)+\varphi(b,d)}{4}+\frac{1}{2}\varphi\left(\frac{a+b}{2},d\right)\\ \frac{1}{d-c}\int_c^d\varphi(a,y)\,\mathrm{d}y&\leq\frac{\varphi(a,c)+\varphi(a,d)}{4}+\frac{1}{2}\varphi\left(a,\frac{c+d}{2}\right)\\ \frac{1}{d-c}\int_c^d\varphi(b,y)\,\mathrm{d}y&\leq\frac{\varphi(b,c)+\varphi(b,d)}{4}+\frac{1}{2}\varphi\left(b,\frac{c+d}{2}\right)\\ \frac{1}{b-a}\int_a^b\varphi(x,\frac{c+d}{2})\mathrm{d}x&\leq\frac{\varphi\left(a,\frac{c+d}{2}\right)+\varphi\left(b,\frac{c+d}{2}\right)}{4}+\frac{1}{2}\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ \frac{1}{d-c}\int_c^d\varphi(\frac{a+b}{2},y)\mathrm{d}y&\leq\frac{\varphi\left(\frac{a+b}{2},c\right)+\varphi\left(\frac{a+b}{2},d\right)}{4}+\frac{1}{2}\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\end{aligned}$$

which gives, by addition,

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\varphi(x,c)\mathrm{d}x+\frac{1}{b-a}\int_a^b\varphi(x,d)\mathrm{d}x+\\ &\frac{1}{d-c}\int_c^d\varphi(a,y)\,\mathrm{d}y+\frac{1}{d-c}\int_c^d\varphi(b,y)\,\mathrm{d}y+\\ &\frac{2}{b-a}\int_a^b\varphi(x,\frac{c+d}{2})\mathrm{d}x+\frac{2}{d-c}\int_c^d\varphi(\frac{a+b}{2},y)\mathrm{d}y\\ \leq\;&\frac{\varphi(a,c)+\varphi(a,d)+\varphi(b,c)+\varphi(b,d)}{2}+\\ &\frac{\varphi\left(\frac{a+b}{2},c\right)+\varphi\left(\frac{a+b}{2},d\right)+\varphi\left(a,\frac{c+d}{2}\right)+\varphi\left(b,\frac{c+d}{2}\right)}{2}+\\ &\frac{\varphi\left(a,\frac{c+d}{2}\right)+\varphi\left(b,\frac{c+d}{2}\right)}{2}+\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\\ &\frac{\varphi\left(\frac{a+b}{2},c\right)+\varphi\left(\frac{a+b}{2},d\right)}{2}+\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ =\;&\frac{\varphi(a,c)+\varphi(a,d)+\varphi(b,c)+\varphi(b,d)}{2}+2\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\varphi\left(\frac{a+b}{2},c\right)+\varphi\left(\frac{a+b}{2},d\right)+\varphi\left(a,\frac{c+d}{2}\right)+\varphi\left(b,\frac{c+d}{2}\right).\end{aligned}$$

Taking all this into consideration we obtain the last inequality in (2.4).

If we choose φ defined by $\varphi(x,y) = xy$ then inequalities (2.4) become equalities which shows that they are sharp. ∎

Remark 2.1. It is easy to prove (in several steps) using the second inequality in (2.3) that

$$\begin{aligned}
&\frac{1}{8}\left[\frac{1}{b-a}\int_a^b\left(\varphi(x,c)+\varphi(x,d)+2\varphi\left(x,\frac{c+d}{2}\right)\right)\mathrm{d}x+\right.\\
&\left.\frac{1}{d-c}\int_c^d\left(\varphi(a,y)+\varphi(b,y)+2\varphi\left(\frac{a+b}{2},y\right)\right)\mathrm{d}y\right]\\
\leq\;&\frac{1}{4}\left[\frac{1}{b-a}\int_a^b\varphi(x,c)\,\mathrm{d}x+\frac{1}{b-a}\int_a^b\varphi(x,d)\,\mathrm{d}x+\right.\\
&\left.\frac{1}{d-c}\int_c^d\varphi(a,y)\,\mathrm{d}y+\frac{1}{d-c}\int_c^d\varphi(b,y)\,\mathrm{d}y\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\varphi(a,c)+\varphi(a,d)+\varphi(b,c)+\varphi(b,d)}{16}+\frac{1}{4}\varphi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\\
&\frac{\varphi\left(\frac{a+b}{2},c\right)+\varphi\left(\frac{a+b}{2},d\right)+\varphi\left(a,\frac{c+d}{2}\right)+\varphi\left(b,\frac{c+d}{2}\right)}{8}\\
\leq\;&\frac{\varphi(a,c)+\varphi(a,d)+\varphi(b,c)+\varphi(b,d)}{4}
\end{aligned}$$

meaning that (2.4) is in fact an improvement of (1.3).

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