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ON SOME ESTIMATES FOR THE LOGARITHMIC MEAN

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ABSTRACT. We show some estimates for the logarithmic mean that are obtained from operator inequalities between the Barbour path and the Heinz means.

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1. INTRODUCTION.

Let $\mathcal{M}^+$ be the set of complex positive definite matrices. For $A, B \in \mathcal{M}^+$, the geometric mean $A\#B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ and the path of the geometric means

$$A\#_x B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^x A^{\frac{1}{2}} \quad (0 \le x \le 1)$$

have been widely discussed in the literature.

On the other hand, a new path of operator means can be defined as

$$\begin{aligned} A\hat{\#}_x B = x(xB^{-1} &+ (1-x)(A\#B)^{-1})^{-1} \\ &+ (1-x)((1-x)A^{-1} + x(A\#B)^{-1})^{-1} \quad (0 \le x \le 1), \end{aligned}$$

and this is called the “Barbour path" [3]. The functions $x \mapsto A\#_x B$ and $x \mapsto A\hat{\#}_x B$ interpolate the points $\{(0, A), (\frac{1}{2}, A\#B), (1, B)\}$ and are convex. Concerning this, an inequality between the integrals of these paths was shown by Nakamura [4], as follows:

$$\int_0^1 A\hat{\#}_x B dx \le \int_0^1 A\#_x B dx.$$

Some properties of the one-parameter family

$$\{\frac{1}{2p}\int_{\frac{1}{2}-p}^{\frac{1}{2}+p} A\hat{\#}_x B dx\}_{0 \le p \le \frac{1}{2}}$$

are also shown [4]. This family is analogous to the family $\{\frac{1}{2p}\int_{\frac{1}{2}-p}^{\frac{1}{2}+p} A\#_x B dx\}_{0 \le p \le \frac{1}{2}}$, which was described in [1].

The purpose of the present paper is to generalize the above inequality by using the above one-parameter family. To see this, we prove the following inequality

$$\frac{1}{2}(A\hat{\#}_x B + A\hat{\#}_{1-x} B) \le \frac{1}{2}(A\#_x B + A\#_{1-x} B) \quad (0 \le x \le 1); \tag{1.1}$$

the method is elementary, but the calculations are a bit complicated. We then discuss some related inequalities.

2. PRELIMINARIES.

Since each side of the inequality (1.1) is an operator mean in the sense of Kubo-Ando [2], (1.1) is equivalent to the inequality between its representation functions. The function which represents the right side is $t \mapsto \frac{1}{2}(t^x + t^{1-x})$, and the one that represents the left side is $t \mapsto \frac{1}{2}(I\hat{\#}_x(tI) + I\hat{\#}_{1-x}(tI))$, where

$$I\hat{\#}_x(tI) = \frac{x}{t^{-1}x + \sqrt{t^{-1}}(1-x)} + \frac{1-x}{\sqrt{t^{-1}}x + (1-x)} = \frac{tx + \sqrt{t}(1-x)}{x + \sqrt{t}(1-x)}.$$

Thus inequality (1.1) is equivalent to

$$\frac{1}{2}\left(\frac{tx + \sqrt{t}(1-x)}{x + \sqrt{t}(1-x)} + \frac{t(1-x) + \sqrt{t}x}{(1-x) + \sqrt{t}x}\right) \le \frac{t^x + t^{1-x}}{2}$$

for all $t > 0, x \in [0, 1]$, and it can be rewritten as

$$\frac{\sqrt{t}\ \left(\sqrt{t}+1\right)^2}{\left(\sqrt{t}^x + \sqrt{t}^{1-x}\right)^2} - \left(\left(1-\sqrt{t}\right)\ (1-x) + \sqrt{t}\right)\ \left(\left(1-\sqrt{t}\right)\ x + \sqrt{t}\right) \le 0.$$

We then let

$$\alpha(s,x) = \frac{s\ (s+1)^2}{(s^x + s^{1-x})^2} - ((1-s)\ (1-x) + s)\ ((1-s)\ x + s)$$

and prove that $\alpha(s,x) \leq 0$ for any $x \in [0,1]$, $s > 0$. Since

$$\alpha(s,x) = s^2 \alpha(\frac{1}{s}, x) = \alpha(s, 1-x)$$

and

$$\alpha(s,1) = \alpha(s,0) = \alpha(s, \frac{1}{2}) = \alpha(1,x) = 0,$$

it is enough to prove it for the case where $s \in (0,1)$ and $0 < x < \frac{1}{2}$.

In the next section, we determine the upper bound of $\alpha(s,x)$ by considering the behavior of the second derivative:

$$\frac{\partial^2}{\partial x^2}\alpha(s,x) = 2\,s\ (s+1)^2\ \log^2 s\ \left(\frac{3\ (s^x - s^{1-x})^2}{(s^x + s^{1-x})^2} - 1\right)\left(s^x + s^{1-x}\right)^{-2} + 2\ (1-s)^2.$$

3. BEHAVIOR OF α.

For $s \in (0,1)$ and $x \in (0, \frac{1}{2})$, put

$$\beta(s,x) = \left(s^x + s^{1-x}\right)^{-2}\left(\frac{3\ (s^x - s^{1-x})^2}{(s^x + s^{1-x})^2} - 1\right).$$

By performing an elementary calculation, we obtain the following: for $s \in (0,1)$, there exists $x_s \in [0, \frac{1}{2})$ such that $\{x \in \left(0, \frac{1}{2}\right] \mid \beta(s,x) < 0\} = \left(x_s, \frac{1}{2}\right]$. Combining this fact and

$$\frac{\partial^2}{\partial x^2}\alpha(s,x) = 2\,s\ (s+1)^2\ (\log s)^2\,\beta(s,x) + 2\ (1-s)^2$$

with

$$\frac{\partial^2}{\partial x^2}\alpha(s,x)\,\Big|_{x=0} > 0, \quad \frac{\partial^2}{\partial x^2}\alpha(s,x)\,\Big|_{x=1/2} < 0,$$

we get the next lemma.

Lemma 3.1. *For $s \in (0,1)$, there exists $x_s \in (0, \frac{1}{2})$ such that*

$$\{x \in \left[0, \tfrac{1}{2}\right] \mid \frac{\partial^2}{\partial x^2}\alpha(s,x) < 0\} = (x_s, \tfrac{1}{2}].$$

Thanks to this lemma and

$$\frac{\partial}{\partial x}\alpha(s,x)\,\Big|_{x=0} < 0, \quad \frac{\partial}{\partial x}\alpha(s,x)\,\Big|_{x=1/2} = 0,$$

the following is obtained.

Lemma 3.2. *For $s \in (0,1)$, there exists $x_s \in (0, \frac{1}{2})$ such that*

$$\{x \in \left[0, \tfrac{1}{2}\right] \mid \frac{\partial}{\partial x}\alpha(s,x) \leq 0\} = [0, x_s],\ \{x \in \left[0, \tfrac{1}{2}\right] \mid \frac{\partial}{\partial x}\alpha(s,x) > 0\} = (x_s, \tfrac{1}{2}].$$

From this lemma, we have $\alpha(s,x) \leq \alpha(s,0) = \alpha(s, \frac{1}{2}) = 0$. Thus the following inequalities are obtained.

Theorem 3.3. *For* $A, B \in \mathcal{M}^+$,

$$A\#B \le \frac{1}{2}(A\hat{\#}_x B + A\hat{\#}_{1-x}B) \le \frac{1}{2}(A\#_x B + A\#_{1-x}B) \quad (0 \le x \le 1),$$

where

$$A\hat{\#}_x B = x(xB^{-1} + (1-x)(A\#B)^{-1})^{-1} + (1-x)((1-x)A^{-1} + x(A\#B)^{-1})^{-1}.$$

Proof. As was stated in [3], the function $x \mapsto A\hat{\#}_x B$ is convex on $[0,1]$, which implies the first inequality. The second one follows from Lemma 3.2. ∎

By integrating both sides of the preceding inequalities, we have the following.

Corollary 3.4. *For* $p \in [0, \frac{1}{2}]$,

$$A\#B \le \frac{1}{2p}\int_{\frac{1}{2}-p}^{\frac{1}{2}+p} A\hat{\#}_x B dx \le \frac{1}{2p}\int_{\frac{1}{2}-p}^{\frac{1}{2}+p} A\#_x B dx.$$

4. RELATED INEQUALITIES.

Several lower bounds for the logarithmic mean have been studied in the literature [5]. Among these, a notably curious one is the inequality

$$\frac{t+t^{\frac{1}{3}}}{1+t^{\frac{1}{3}}} \le \frac{t-1}{\log t}$$

for all $t > 0$. Both sides of this are normalized, positive, operator monotone functions on $(0,\infty)$, and the left-hand side of that is the midpoint of the Barbour path $\frac{tx+t^{\frac{1}{3}}(1-x)}{x+t^{\frac{1}{3}}(1-x)}$.

On the other hand, an interesting lower bound for the logarithmic mean was given by Nakamura [4]:

$$\int_0^1 \frac{tx+t^{\frac{1}{2}}(1-x)}{x+t^{\frac{1}{2}}(1-x)}dx \le \int_0^1 t^x dx = \frac{t-1}{\log t}.$$

In this section, the relationship between these bounds is discussed.

Proposition 4.1.

$$\int_0^1 \frac{tx+t^{\frac{1}{2}}(1-x)}{x+t^{\frac{1}{2}}(1-x)}dx \le \frac{t+t^{\frac{1}{3}}}{1+t^{\frac{1}{3}}}$$

for all $t > 0$.

Proof. Since

$$\int_0^1 \frac{tx+t^{\frac{1}{2}}(1-x)}{x+t^{\frac{1}{2}}(1-x)}dx = \frac{\sqrt{t}\ ((t+1)\log t - 2(t-1)) + 2t\log t}{2(t-1)},$$

we shall show the following :

$$\frac{\sqrt{t}\ ((t+1)\log t - 2(t-1)) + 2t\log t}{2(t-1)} \le \frac{t+t^{\frac{1}{3}}}{1+t^{\frac{1}{3}}}.$$

For $t \ge 1$, the above inequality can be rewritten as

$$\frac{\frac{2(t-1)\left(t+t^{\frac{1}{3}}\right)}{t^{\frac{1}{3}}+1} + 2\ (t-1)\ \sqrt{t}}{\sqrt{t}\ (t+1)+2t} - \log t \ge 0.$$

The first derivative of the left side is

$$\frac{\left(t^{\frac{1}{12}}-1\right)^4\left(t^{\frac{1}{12}}+1\right)^4\left(t^{\frac{1}{6}}-t^{\frac{1}{12}}+1\right)\left(t^{\frac{1}{6}}+t^{\frac{1}{12}}+1\right)}{3\left(t^{\frac{11}{12}}+2t^{\frac{7}{12}}+t^{\frac{1}{4}}\right)t^{\frac{11}{12}}},$$

which is clearly positive. Thanks to

$$\left(\frac{\frac{2(t-1)\left(t+t^{\frac{1}{3}}\right)}{t^{\frac{1}{3}}+1}+2\,(t-1)\,\sqrt{t}}{\sqrt{t}\,(t+1)+2t}-\log t\right)\Bigg|_{t=1}=0,$$

the desired inequality holds for $t \geq 1$.

For $0 < t < 1$, it follows from the above argument that

$$\begin{aligned}\int_0^1 \frac{tx+t^{\frac{1}{2}}(1-x)}{x+t^{\frac{1}{2}}(1-x)}dx &= \int_0^1 \frac{t(1-x)+t^{\frac{1}{2}}x}{1-x+t^{\frac{1}{2}}x}dx\\ &= t\int_0^1 \frac{t^{-1}x+t^{\frac{-1}{2}}(1-x)}{x+t^{\frac{-1}{2}}(1-x)}dx\\ &\leq t\frac{t^{-1}+t^{\frac{-1}{3}}}{1+t^{\frac{-1}{3}}} = \frac{t+t^{\frac{1}{3}}}{1+t^{\frac{1}{3}}}.\end{aligned}$$

∎

Corollary 4.2. *For* $t > 0$,

$$\frac{t+t^{\frac{1}{2}}}{1+t^{\frac{1}{2}}} \leq \int_0^1 \frac{tx+t^{\frac{1}{2}}(1-x)}{x+t^{\frac{1}{2}}(1-x)}dx \leq \frac{t+t^{\frac{1}{3}}}{1+t^{\frac{1}{3}}} \leq \int_0^1 t^x dx.$$

Corollary 4.3. *For* $A, B \in \mathcal{M}^+$,

$$\begin{aligned}A\#B &\leq \int_0^1 A\hat{\#}_x B\,dx\\ &\leq (B^{-1}+(A\#_{\frac{2}{3}}B)^{-1})^{-1}+(A^{-1}+(A\#_{\frac{1}{3}}B)^{-1})^{-1}\\ &\leq \int_0^1 A\#_x B dx.\end{aligned}$$

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