ON SOME ESTIMATES FOR THE LOGARITHMIC MEAN

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ABSTRACT. We show some estimates for the logarithmic mean that are obtained from operator inequalities between the Barbour path and the Heinz means.

Key words and phrases: Operator mean; Barbour path; Logarithmic mean.

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1. Introduction.

Let \( \mathcal{M}^+ \) be the set of complex positive definite matrices. For \( A, B \in \mathcal{M}^+ \), the geometric mean \( A\#B = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} \) and the path of the geometric means
\[
A\#_xB = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^xA^{\frac{1}{2}} \quad (0 \leq x \leq 1)
\]
have been widely discussed in the literature.

On the other hand, a new path of operator means can be defined as
\[
A\#_xB = x(xB^{-1} + (1 - x)(A\#B)^{-1})^{-1} + (1 - x)((1 - x)A^{-1} + x(A\#B)^{-1})^{-1} \quad (0 \leq x \leq 1),
\]
and this is called the “Barbour path” [4]. The functions \( x \mapsto A\#_xB \) and \( x \mapsto A\#_xB \) interpolate the points \( \{(0, A), (\frac{1}{2}, A\#B), (1, B)\} \) and are convex. Concerning this, an inequality between the integrals of these paths was shown by Nakamura [4], as follows:
\[
\int_0^1 A\#_xBdx \leq \int_0^1 A\#_Bdx.
\]
Some properties of the one-parameter family
\[
\left\{ \frac{1}{2p} \int_{\frac{1}{2}-p}^{\frac{1}{2}+p} A\#_xBdx \right\}_{0 \leq p \leq \frac{1}{2}}
\]
are also shown [4]. This family is analogous to the family \( \left\{ \frac{1}{2p} \int_{\frac{1}{2}-p}^{\frac{1}{2}+p} A\#_xBdx \right\}_{0 \leq p \leq \frac{1}{2}} \), which was described in [1].

The purpose of the present paper is to generalize the above inequality by using the above one-parameter family. To see this, we prove the following inequality
\[
(1.1) \quad \frac{1}{2}(A\#_xB + A\#_{1-x}B) \leq \frac{1}{2}(A\#_xB + A\#_{1-x}B) \quad (0 \leq x \leq 1);
\]
the method is elementary, but the calculations are a bit complicated. We then discuss some related inequalities.

2. Preliminaries.

Since each side of the inequality (1.1) is an operator mean in the sense of Kubo-Ando [2], (1.1) is equivalent to the inequality between its representation functions. The function which represents the right side is \( t \mapsto \frac{1}{2}(t^x + t^{1-x}) \), and the one that represents the left side is \( t \mapsto \frac{1}{2}(I\#_x(tI) + I\#_{1-x}(tI)) \), where
\[
I\#_x(tI) = \frac{x}{t^{-1}x + \sqrt{t^{-1}(1 - x)}} + \frac{1 - x}{\sqrt{t^{-1}x} + (1 - x)} = \frac{tx + \sqrt{t}(1 - x)}{x + \sqrt{t}(1 - x)}.
\]
Thus inequality (1.1) is equivalent to
\[
\frac{1}{2} \left( \frac{tx + \sqrt{t}(1 - x)}{x + \sqrt{t}(1 - x)} + \frac{t(1 - x) + \sqrt{t}x}{(1 - x) + \sqrt{t}x} \right) \leq \frac{t^x + t^{1-x}}{2}
\]
for all \( t > 0, x \in [0, 1] \), and it can be rewritten as
\[
\frac{\sqrt{t} \left( \sqrt{t} + 1 \right)^2}{\left( \sqrt{t^x} + \sqrt{t^{1-x}} \right)^2} - \left( (1 - \sqrt{t}) (1 - x) + \sqrt{t} \right) \left( (1 - \sqrt{t}) x + \sqrt{t} \right) \leq 0.
\]
We then let
\[ \alpha(s, x) = \frac{s (s + 1)^2}{(s^x + s^{1-x})^2} - ((1 - s) (1 - x) + s) ((1 - s) x + s) \]
and prove that \( \alpha(s, x) \leq 0 \) for any \( x \in [0, 1], s > 0. \) Since
\[ \alpha(s, x) = s^2 \alpha\left(\frac{1}{s}, x\right) = \alpha(s, 1 - x) \]
and
\[ \alpha(s, 1) = \alpha(s, 0) = \alpha(s, \frac{1}{2}) = \alpha(1, x) = 0, \]
it is enough to prove it for the case where \( s \in (0, 1) \) and \( 0 < x < \frac{1}{2}. \)

In the next section, we determine the upper bound of \( \alpha(s, x) \) by considering the behavior of the second derivative:
\[ \frac{\partial^2}{\partial x^2} \alpha(s, x) = 2 s (s + 1)^2 \log^2 s \left( \frac{3 (s^x - s^{1-x})}{(s^x + s^{1-x})^2} - 1 \right) (s^x + s^{1-x})^{-2} + 2 (1 - s)^2. \]

3. Behavior of \( \alpha. \)

For \( s \in (0, 1) \) and \( x \in (0, \frac{1}{2}), \) put
\[ \beta(s, x) = (s^x + s^{1-x})^{-2} \left( \frac{3 (s^x - s^{1-x})}{(s^x + s^{1-x})^2} - 1 \right). \]

By performing an elementary calculation, we obtain the following: for \( s \in (0, 1), \) there exists \( x_s \in [0, \frac{1}{2}) \) such that \( \{ x \in (0, \frac{1}{2}] \mid \beta(s, x) < 0 \} = (x_s, \frac{1}{2}). \) Combining this fact and
\[ \frac{\partial^2}{\partial x^2} \alpha(s, x) = 2 s (s + 1)^2 (\log s)^2 \beta(s, x) + 2 (1 - s)^2 \]
with
\[ \frac{\partial^2}{\partial x^2} \alpha(s, x) \bigg|_{x=0} > 0, \quad \frac{\partial^2}{\partial x^2} \alpha(s, x) \bigg|_{x=\frac{1}{2}} < 0, \]
we get the next lemma.

Lemma 3.1. For \( s \in (0, 1), \) there exists \( x_s \in (0, \frac{1}{2}) \) such that
\[ \{ x \in [0, \frac{1}{2}] \mid \frac{\partial^2}{\partial x^2} \alpha(s, x) < 0 \} = (x_s, \frac{1}{2}). \]

Thanks to this lemma and
\[ \frac{\partial}{\partial x} \alpha(s, x) \bigg|_{x=0} < 0, \quad \frac{\partial}{\partial x} \alpha(s, x) \bigg|_{x=\frac{1}{2}} = 0, \]
the following is obtained.

Lemma 3.2. For \( s \in (0, 1), \) there exists \( x_s \in (0, \frac{1}{2}) \) such that
\[ \{ x \in [0, \frac{1}{2}] \mid \frac{\partial}{\partial x} \alpha(s, x) \leq 0 \} = [0, x_s], \quad \{ x \in [0, \frac{1}{2}] \mid \frac{\partial}{\partial x} \alpha(s, x) > 0 \} = (x_s, \frac{1}{2}). \]

From this lemma, we have \( \alpha(s, x) \leq \alpha(s, 0) = \alpha(s, \frac{1}{2}) = 0. \) Thus the following inequalities are obtained.
Theorem 3.3. For $A, B \in M^+$,

$$A \# B \leq \frac{1}{2} (A \#_x B + A \#_{1-x} B) \leq \frac{1}{2} (A \# B + A \#_{1} B) \quad (0 \leq x \leq 1),$$

where

$$A \#_x B = x(xB^{-1} + (1-x)(A(B)^{-1})^{-1} + (1-x)((1-x)A^{-1} + x(A#B)^{-1})^{-1}.$$  

Proof. As was stated in [3], the function $x \mapsto A \#_x B$ is convex on $[0, 1]$, which implies the first inequality. The second one follows from Lemma 3.2.

By integrating both sides of the preceding inequalities, we have the following.

Corollary 3.4. For $p \in [0, \frac{1}{2}]$,

$$A \# B \leq \frac{1}{2p} \int_{\frac{1}{2}-p}^{\frac{1}{2}+p} A \#_x B dx \leq \frac{1}{2p} \int_{\frac{1}{2}-p}^{\frac{1}{2}+p} A \#_x B dx.$$  

4. RELATED INEQUALITIES.

Several lower bounds for the logarithmic mean have been studied in the literature [5]. Among these, a notably curious one is the inequality

$$\frac{t + t^{\frac{1}{3}}}{1 + t^{\frac{1}{3}}} \leq \frac{t - 1}{\log t}$$

for all $t > 0$. Both sides of this are normalized, positive, operator monotone functions on $(0, \infty)$, and the left-hand side of that is the midpoint of the Barbour path $\frac{t+1}{x+t^{\frac{1}{3}}(1-x)}$.

On the other hand, an interesting lower bound for the logarithmic mean was given by Nakamura [4]:

$$\int_0^1 \frac{tx + t^{\frac{1}{3}}(1-x)}{x + t^{\frac{1}{3}}(1-x)} dx \leq \int_0^1 t^2 dx = \frac{t - 1}{\log t}.$$  

In this section, the relationship between these bounds is discussed.

Proposition 4.1.

$$\int_0^1 \frac{tx + t^{\frac{1}{3}}(1-x)}{x + t^{\frac{1}{3}}(1-x)} dx \leq \frac{t + t^{\frac{1}{3}}}{1 + t^{\frac{1}{3}}}$$

for all $t > 0$.

Proof. Since

$$\int_0^1 \frac{tx + t^{\frac{1}{3}}(1-x)}{x + t^{\frac{1}{3}}(1-x)} dx = \frac{\sqrt{t} \ ((t+1) \log t - 2(t - 1)) + 2 t \log t}{2(t - 1)},$$

we shall show the following:

$$\frac{\sqrt{t} \ ((t+1) \log t - 2(t - 1)) + 2 t \log t}{2(t - 1)} \leq \frac{t + t^{\frac{1}{3}}}{1 + t^{\frac{1}{3}}}.$$  

For $t \geq 1$, the above inequality can be rewritten as

$$\frac{2 (t-1) (t+1)}{t^{\frac{1}{3}} + 1} + 2 (t - 1) \sqrt{t} \quad \frac{\sqrt{t} \ (t + 1) + 2 t}{(t + 1) + 2 t} - \log t \geq 0.$$
The first derivative of the left side is
\[
\left( \frac{t^{1/2} - 1}{t^{1/2} + 1} \right)^4 \left( \frac{t^{1/2} + 1}{t^{1/2} - 1} \right)^4 \left( \frac{t^{1/2} - t^{1/2} + 1}{t^{1/2} + t^{1/2} + 1} \right) \frac{3}{\left( t^{1/2} + 2 t^{1/2} + t^{1/2} \right)^2} t^{1/2},
\]
which is clearly positive. Thanks to
\[
\left( \frac{2 (t-1) (t+t^{1/2})}{t^{1/2} + 1} + 2 (t-1) \sqrt{t} \frac{- \log t}{\sqrt{t} (t+1) + 2 t} \right) \bigg|_{t=1} = 0,
\]
the desired inequality holds for \( t \geq 1 \).

For \( 0 < t < 1 \), it follows from the above argument that
\[
\int_0^1 \frac{tx + t^{1/2} (1-x)}{x + t^{1/2} (1-x)} \, dx = \int_0^1 \frac{t(1-x) + t^{1/2} x}{1-x + t^{1/2} x} \, dx
\]
\[
= t \int_0^1 \frac{t^{-1} x + t^{1/2} (1-x)}{x + t^{1/2} (1-x)} \, dx
\]
\[
\leq \frac{t^{-1} + t^{1/2}}{1 + t^{1/2}} \leq \frac{t + t^{1/2}}{1 + t^{1/2}}.
\]

**Corollary 4.2.** For \( t > 0 \),
\[
\frac{t + t^{1/2}}{1 + t^{1/2}} \leq \int_0^1 \frac{tx + t^{1/2} (1-x)}{x + t^{1/2} (1-x)} \, dx \leq \frac{t + t^{1/2}}{1 + t^{1/2}} \leq \int_0^1 t^x \, dx.
\]

**Corollary 4.3.** For \( A, B \in \mathcal{M}^+ \),
\[
A \# B \leq \int_0^1 A \#_x B \, dx
\]
\[
\leq (B^{-1} + (A \#_{1/3} B)^{-1} - 1) + (A^{-1} + (A \#_{1/3} B)^{-1} - 1)
\]
\[
\leq \int_0^1 A \#_x B \, dx.
\]

**References**


