The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 11, Issue 1, Article 10, pp. 1-13, 2014

L^{∞}- ERROR ESTIMATE OF SCHWARZ ALGORITHM FOR ELLIPTIC QUASI-VARIATIONAL INEQUALITIES RELATED TO IMPULSE CONTROL PROBLEM

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Received 16 May, 2012; accepted 2 November, 2012; published 24 September, 2014.

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ABSTRACT. In this work, we study Schwarz method for a class of elliptic quasi-variational inequalities. The principal result of this investigation is to prove the error estimate in L^{∞}-norm for two domains with overlapping nonmatching grids, using the geometrical convergence, and the uniform convergence of Cortey Dumont.

Key words and phrases: Schwarz Algorithm, Quasi-variational Inequalities, $L^{\infty}-$Error Estimates.

2000 *Mathematics Subject Classification.* 05C38,15A15, 05A15, 15A18.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Schwarz algorithms were proposed more than one hundred years ago for proving the solvability of PDEs on a complicated domain. With parallel calculators, this rediscovery of these methods as algorithms of calculations, based on a modern variational approach. Pierre-Louis Lions, was the starting point of an intense research activity to develop this tool of calculation, see, e.g., [10, 11], and the references therein [1, 5, 9, 12, 13].
In this work, we are interested in the error analysis in the maximum norm of obstacle type problems, quasi-variational inequalities (Q.V.I) related to stochastic inventory problems with impulse control problem governed by partial differential inequations [2, 3, 4, 7] in the context of overlapping nonmatching grids: we consider a domain Ω which is the union of two overlapping subdomains where each subdomain has its own triangulation.
In section 2, we state the second order elliptic problem with Dirichlet condition, study some qualitative properties, define the continuous Schwarz sequences for Q.V.I and give their respective finite element counterparts in the context of overlapping grids. In section 3, we prove a fundamental lemma for two auxiliary sequences and establish a main result concerning the error estimate of solution in the L^∞-norm, taking into account the combination of geometrical convergence, and the error estimate in L^∞-norm of Cortey Dumont [8]. It worth mentioning that the method presented in this paper is simpler than that introduced in [9]. Finally, In Section 4, to validate our theory, we present some numerical results for a quasi-variational inequality.

2. SCHWARZ ALGORITHM FOR QUASI-VARIATIONAL INEQUALITIES

2.1. **Some Preliminary Results on the Q.V.I..** Let Ω be a convex domain in IR^N with sufficiently smooth boundary $\partial\Omega$. We define the variational form, for any $u, v \in H^1(\Omega)$,

$$a(u,v) = \int_\Omega \left(\sum_{1\le i,j\le N} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{1\le j\le N} a_j \frac{\partial u}{\partial x_j} v + a_0 uv \right) dx, \tag{2.1}$$

and the differential operator associated with the bilinear form $a(\cdot,\cdot)$

$$A = -\sum_{1\le i,j\le N} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + \sum_{1\le j\le N} a_j \frac{\partial}{\partial x_j} + a_0,$$

the coefficients $a_{ij}(x), a_j(x), a_0(x)$ are supposed to be sufficiently smooth and $a_0(x)$ satisfied

$$a_0(x) \ge \beta > 0, \forall x \in \Omega. \tag{2.2}$$

We also suppose that the bilinear form is continuous and strongly coercive,

$$\exists \alpha > 0 : a(v,v) \ge \alpha \|v\|^2_{H^1(\Omega)}, \tag{2.3}$$

we are also given right-hand side

$$f \in L^\infty(\Omega), f \ge 0, \tag{2.4}$$

the obstacle Mu of impulse control defined by

$$Mu(x) = k + \inf u(x+\xi), x \in \Omega, \xi \ge 0, x+\xi \in \Omega, k > 0, \tag{2.5}$$

the operator M maps $L^\infty(\Omega)$ into itself and possesses the following properties [2]

$$\|Mu - Mv\|_{L^\infty(\Omega)} \le \|u - v\|_{L^\infty(\Omega)}, \forall u, v \in L^\infty(\Omega), \tag{2.6}$$

$$Mu \le M\widetilde{u}, \text{whenever } u \le \widetilde{u} \tag{2.7}$$

and the nonempty convex set

$$K_g(u) = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega, 0 \le v \le Mu \text{ in } \Omega\}, \tag{2.8}$$

where g is a regular function defined on $\partial\Omega$,

$$g \in W^{2,p}(\Omega), 2 \leq p < \infty, g \geq 0. \tag{2.9}$$

We consider the following elliptic quasi variational inequality (Q.V.I)

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ satisfying} \\ a(u, v-u) \geq (f, v-u), \forall v \in K_g(u) \end{cases} \tag{2.10}$$

$(.,.)$ denotes the usual inner product in $L^2(\Omega)$.

Remark 2.1. In problem (2.10), the condition $u \geq 0$ is added because this appears naturally in the theory of impulse control [2].

Theorem 2.1. [2] *Under the preceding notations and conditions* (2.2) *to* (2.9)*, the problem* (2.10) *has a unique solution, moreover* u *satisfies the regularity property* $u \in W^{2,p}(\Omega), 2 \leq\ p <\ \infty$.

Let V_h denote the standard piecewise linear finite element space. The discrete counterpart of (2.10) consists of finding $u_h \in K_{gh}(u_h)$ solution of

$$a(u_h, v-u_h) \geq (f, v-u_h), \forall v \in K_{gh}(u_h) \tag{2.11}$$

where

$$K_{gh}(u_h) = \left\{ v \in V_h : v = \pi_h g \text{ on } \partial\Omega, 0 \leq v \leq r_h M u_h \text{ in } \Omega \right\}, \tag{2.12}$$

π_h is an interpolation operator on $\partial\Omega$ and r_h is the usual finite element restriction operator in Ω.

Let u_h be the discrete solution of Q.V.I

$$\begin{cases} a(u_h, v-u_h) \geq (f, v-u_h), \forall v \in V_h \\ u_h = \pi_h g \text{ on } \partial\Omega, u_h \leq r_h M u_h \text{ in } \Omega \\ v = \pi_h g \text{ on } \partial\Omega, v \leq r_h M u_h \text{ in } \Omega \end{cases} \tag{2.13}$$

in a similar way let $\widetilde{u}_h$ be the discrete solution of V.I

$$\begin{cases} a(\widetilde{u}_h, v-\widetilde{u}_h) \geq (f, v-\widetilde{u}_h), \forall v \in V_h \\ \widetilde{u}_h = \pi_h \widetilde{g} \text{ on } \partial\Omega, \widetilde{u}_h \leq r_h M u_h \text{ in } \Omega \\ v = \pi_h \widetilde{g} \text{ on } \partial\Omega, v \leq r_h M u_h \text{ in } \Omega \end{cases} \tag{2.14}$$

where $\widetilde{g}$ is a regular function defined on $\partial\Omega$.
Let us write $\sigma_h(g, Mu_h)$ the solution of the problem (2.13), where σ_h is a mapping from $L^\infty(\Omega)$ into itself. We establish the monotonicity and stability properties of solution.

Lemma 2.2. *Let* $g, \widetilde{g}$ *be two functions given and* $u_h = \sigma_h(g, Mu_h), \widetilde{u}_h = \sigma_h(\widetilde{g}, Mu_h)$ *corresponding discrete solutions of* (2.13) $(resp.(2.14))$*. If* $g \geq \widetilde{g}$ *, then* $\sigma_h(g, Mu_h) \geq \sigma_h(\widetilde{g}, Mu_h)$.

Proof. Let

$$\varphi \in H^1(\Omega) \cap L^\infty(\Omega), \varphi^- = max(-\varphi, 0),$$

we denote by $\varphi \to \varphi^-$ a continuous mapping on $H^1(\Omega) \cap L^\infty(\Omega)$ into itself, we have

$$(u_h - \widetilde{u}_h)^- = max(\widetilde{u}_h - u_h, 0) \text{ on } \partial\Omega,$$

we can take $v = u_h + (u_h - \widetilde{u}_h)^-$ as a trial function in (2.13), this gives

$$a(u_h, (u_h - \widetilde{u}_h)^-) \geq (f, (u_h - \widetilde{u}_h)^-), \tag{2.15}$$

also, for $v = \widetilde{u}_h - (u_h - \widetilde{u}_h)^-$ in (2.14), we get

$$a(\widetilde{u}_h, -(u_h - \widetilde{u}_h)^-) \geq (f, -(u_h - \widetilde{u}_h)^-), \tag{2.16}$$

so, by addition of (2.15) and (2.16), we obtain

$$a(u_h - \widetilde{u}_h, (u_h - \widetilde{u}_h)^-) \geq 0,$$

and this implies

$$a((u_h - \widetilde{u}_h)^-, (u_h - \widetilde{u}_h)^-) \leq 0,$$

as $a(.,.)$ is coercive this also implies

$$(u_h - \widetilde{u}_h)^- = 0,$$

thus

$$u_h = \widetilde{u}_h.$$

∎

Remark 2.2. The proof is similar in continuous case.

Proposition 2.3. *Under the notations and conditions of preceding lemma, we have*

$$\|u_h - \widetilde{u}_h\|_{L^\infty(\Omega)} \leq \|g - \widetilde{g}\|_{L^\infty(\partial\Omega)}. \tag{2.17}$$

Proof. Let us pose

$$\phi = \|g - \widetilde{g}\|_{L^\infty(\partial\Omega)},$$

we have

$$g - \widetilde{g} \leq \|g - \widetilde{g}\|_{L^\infty(\partial\Omega)},$$

thus

$$g \leq \widetilde{g} + \phi,$$

by lemma 2.2, it follows that

$$\sigma_h(g, Mu_h) = \sigma_h(\widetilde{g} + \phi, Mu_h + \phi),$$

however

$$\sigma_h(\widetilde{g} + \phi, Mu_h + \phi) = \sigma_h(\widetilde{g}, Mu_h) + \phi,$$

from where

$$\sigma_h(g, Mu_h) - \sigma_h(\widetilde{g}, Mu_h) \leq \phi.$$

Similarly by interchanging the roles of g and $\widetilde{g}$, we also get

$$\sigma_h(\widetilde{g}, Mu_h) - \sigma_h(g, Mu_h) \leq \phi.$$

This completes the demonstration. ∎

Remark 2.3. The preceding proposition remains valid in continuous case.

Theorem 2.4. [8] *Under the preceding notations and conditions, and discrete maximum principle assumption, there exists a constant c independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq ch^2 |\log h|^2. \tag{2.18}$$

2.2. **The Continuous Schwarz Sequences.** Consider the model obstacle problem: find $u \in K_0(u)(g=0)$ such that

$$a(u, v-u) \geq (f, v-u), \forall v \in K_0(u). \tag{2.19}$$

We decompose Ω into two overlapping polygonal subdomains Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2, \Omega_1 \cap \Omega_2 \neq \emptyset,$$

and u satisfies the local regularity condition

$$u_{|\Omega_i} \in W^{2,p}(\Omega_i), 2 \leq p < \infty.$$

We denote by $\partial\Omega_i$ the boundary of Ω_i, and $\Gamma_i = \partial\Omega_i \cap \Omega_j$. The intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j, i \neq j$ is assumed to be empty.
Let

$$V_i(\Omega_i) = \left\{ v \in H^1(\Omega_i)/v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \right\}, i = 1, 2.$$

For $w \in C(\overline{\Gamma}_i)$, we define

$$V_i^{(w)} = \left\{ v \in V_i(\Omega_i)/v = w \text{ on } \Gamma_i \right\}, i = 1, 2.$$

Let $u^0 \in C^0(\overline{\Omega})$ be the initial value such that

$$a(u^0, v) = (f, v), v \in H_0^1(\Omega) \tag{2.20}$$

we respectively define the Schwarz sequence (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in V_1^{(u_2^n)}$ solves

$$\begin{cases} a_1(u_1^{n+1}, v - u_1^{n+1}) \geq (f_1, v - u_1^{n+1}), \forall v \in V_1^{(u_2^n)} \\ u_1^{n+1} \leq Mu_1^n \text{ in } \Omega_1, v \leq Mu_1^n \text{ in } \Omega_1, \end{cases} \tag{2.21}$$

and (u_2^{n+1}) on Ω_2 such that $u_2^{n+1} \in V_2^{(u_1^{n+1})}$ solution of

$$\begin{cases} a_2(u_2^{n+1}, v - u_2^{n+1}) \geq (f_2, v - u_2^{n+1}), \forall v \in V_2^{(u_1^{n+1})} \\ u_2^{n+1} \leq Mu_2^n \text{ in } \Omega_2, v \leq Mu_2^n \text{ in } \Omega_2, \end{cases} \tag{2.22}$$

where

$$a_i(u, v) = \int_{\Omega_i} \left(\sum_{1 \leq i,j \leq N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{1 \leq j \leq N} a_j \frac{\partial u}{\partial x_j} v + a_0 uv \right) dx, i = 1, 2$$

$$u_1^0 = u^0 \text{ in } \Omega_1, u_2^0 = u^0 \text{ in } \Omega_2, u_1^{n+1} = 0 \text{ in } \overline{\Omega} - \overline{\Omega}_1 \text{ and } u_2^{n+1} = 0 \text{ in } \overline{\Omega} - \overline{\Omega}_2.$$

Now, consider a function $w_i \in L^\infty(\Omega_i)$ be continuous on $\overline{\Omega}_i - \overline{\partial\Omega_i \cap \Omega} \cap \partial\Omega$ such that

$$\begin{cases} \Delta w_i = 0 \text{ in } \Omega_i, i = 1, 2 \\ w_i = \begin{cases} 0 \text{ on } \partial\Omega_i - \overline{\partial\Omega_i \cap \Omega} \\ 1 \text{ on } \partial\Omega_i \cap \Omega \end{cases} \end{cases} \tag{2.23}$$

with $w_i = 0$ in $\overline{\Omega} - \overline{\Omega}_i$. From maximum principle, we have $0 < w_i < 1$ in Ω_i (see [11]).
We prove a geometrical convergence result.

Theorem 2.5. *Under maximum principle assumption, the sequences* $(u_1^{n+1}), (u_2^{n+1}), n \geq 0$ *produced by the Schwarz algorithm converge geometrically to the solution u of the obstacle problem (2.19). More precisely, there exists two constants* $k_1, k_2 \in]0, 1[$ *such that for all* $n \geq 0$,

$$\|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} \leq k_1^n k_2^n \|u - u^0\|_{L^\infty(\Gamma_1)}, \tag{2.24}$$

$$\|u_2 - u_2^{n+1}\|_{L^\infty(\Omega_2)} \leq k_1^{n+1} k_2^n \|u - u^0\|_{L^\infty(\Gamma_2)}, \tag{2.25}$$

Proof. From the maximum principle, we have

$$\|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} \leq \|u_1 - u_1^{n+1}\|_{L^\infty(\Gamma_1)},$$

and

$$\begin{aligned}
\|u_1 - u_1^{n+1}\|_{L^\infty(\Gamma_1)} &\leq \|u_2 - u_2^{n}\|_{L^\infty(\Gamma_1)} \leq \|w_1 u_2 - w_1 u_2^{n}\|_{L^\infty(\Gamma_1)} \\
&\leq \|w_1 u_2 - w_1 u_2^{n}\|_{L^\infty(\Omega_2)} \leq \|w_1 u_2 - w_1 u_2^{n}\|_{L^\infty(\Gamma_2)} \\
&\leq \|w_1\|_{L^\infty(\Gamma_2)}\|u_2 - u_2^{n}\|_{L^\infty(\Gamma_2)} \leq \|w_1\|_{L^\infty(\Gamma_2)}\|w_2 u_2 - w_2 u_2^{n}\|_{L^\infty(\Gamma_2)} \\
&\leq \|w_1\|_{L^\infty(\Gamma_2)}\|w_2 u_1 - w_2 u_1^{n}\|_{L^\infty(\Gamma_2)} \leq \|w_1\|_{L^\infty(\Gamma_2)}\|w_2 u_1 - w_2 u_1^{n}\|_{L^\infty(\Omega_1)} \\
&\leq \|w_1\|_{L^\infty(\Gamma_2)}\|w_2 u_1 - w_2 u_1^{n}\|_{L^\infty(\Gamma_1)} \leq \|w_2\|_{L^\infty(\Gamma_1)}\|w_1\|_{L^\infty(\Gamma_2)}\|u_1 - u_1^{n}\|_{L^\infty(\Gamma_1)},
\end{aligned}$$

putting

$$k_i = \sup\{w_j(x)/x \in \partial\Omega_i \cap \Omega, i \neq j\} \in]0,1[, \forall i,j = 1,2,$$

then

$$\|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} \leq k_1 k_2 \|u_1 - u_1^{n}\|_{L^\infty(\Gamma_1)},$$

by induction, we get

$$\|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} \leq k_1^n k_2^n \|u_1 - u_1^{1}\|_{L^\infty(\Gamma_1)}$$

$$\leq k_1^n k_2^n \|u - u^{0}\|_{L^\infty(\Gamma_1)},$$

where $u_1^1 = u^0$ on Γ_1 and $u_1^1 = 0$ on $\partial\Omega_1 \cap \partial\Omega$.
Similarly

$$\begin{aligned}
\|u_2 - u_2^{n+1}\|_{L^\infty(\Omega_2)} &\leq \|u_2 - u_2^{n+1}\|_{L^\infty(\Gamma_2)} \\
&\leq \|w_2 u_1 - w_2 u_1^{n+1}\|_{L^\infty(\Gamma_2)} \leq \|w_2 u_1 - w_2 u_1^{n+1}\|_{L^\infty(\Omega_1)} \\
&\leq \|w_2 u_1 - w_2 u_1^{n+1}\|_{L^\infty(\Gamma_1)} \leq \|w_2\|_{L^\infty(\Gamma_1)}\|u_1 - u_1^{n+1}\|_{L^\infty(\Gamma_1)} \\
&\leq \|w_2\|_{L^\infty(\Gamma_1)}\|w_1 u_1 - w_1 u_1^{n+1}\|_{L^\infty(\Gamma_1)} \leq \|w_2\|_{L^\infty(\Gamma_1)}\|w_1 u_2 - w_1 u_2^{n}\|_{L^\infty(\Gamma_1)} \\
&\leq \|w_2\|_{L^\infty(\Gamma_1)}\|w_1 u_2 - w_1 u_2^{n}\|_{L^\infty(\Omega_2)} \leq \|w_2\|_{L^\infty(\Gamma_1)}\|w_1 u_2 - w_1 u_2^{n}\|_{L^\infty(\Gamma_2)} \\
&\leq \|w_2\|_{L^\infty(\Gamma_1)}\|w_1\|_{L^\infty(\Gamma_2)}\|u_2 - u_2^{n}\|_{L^\infty(\Gamma_2)} \leq k_1 k_2 \|u_2 - u_2^{n}\|_{L^\infty(\Gamma_2)},
\end{aligned}$$

by induction, we obtain

$$\begin{aligned}
\|u_2 - u_2^{n+1}\|_{L^\infty(\Omega_2)} &\leq k_1^n k_2^n \|u_2 - u_2^{1}\|_{L^\infty(\Gamma_2)} \leq k_1^n k_2^n \|w_2 u_1 - w_2 u_1^{1}\|_{L^\infty(\Gamma_2)} \\
&\leq k_1^n k_2^n \|w_2 u_1 - w_2 u_1^{1}\|_{L^\infty(\Omega_1)} \leq k_1^n k_2^n \|w_2 u_1 - w_2 u_1^{1}\|_{L^\infty(\Gamma_1)} \\
&\leq k_1^{n+1} k_2^n \|u_1 - u_1^{1}\|_{L^\infty(\Gamma_1)} \leq k_1^{n+1} k_2^n \|u_2 - u_2^{0}\|_{L^\infty(\Gamma_1)} \\
&\leq k_1^{n+1} k_2^n \|u_2 - u_2^{0}\|_{L^\infty(\Omega_2)} \leq k_1^{n+1} k_2^n \|u_2 - u_2^{0}\|_{L^\infty(\Gamma_2)} \\
&\leq k_1^{n+1} k_2^n \|u - u^{0}\|_{L^\infty(\Gamma_2)},
\end{aligned}$$

which completes the proof. ∎

Remark 2.4. The demonstration of theorem 2.5 is an adaptation of the one in [11] given for the problem of variational equation. This theorem remains true for the problem introduced in this paper.

2.3. **The Discretization.** For $i = 1, 2$, let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i, h_i being the meshsize. We assume that the two triangulations are mutually independent on $\Omega_1 \cap \Omega_2$, in the sense that a triangle belonging to one triangulation does not necessarily belong to the other.
Let $V_{h_i}(\Omega_i)$ be the space of continuous linear functions piecewise on τ^{h_i} which vanish on $\partial\Omega \cap \partial\Omega_i$. For $w \in C(\overline{\Gamma}_i)$ we define

$$V_{h_i}^{(w)} = \left\{ v \in V_{h_i}(\Omega_i)/v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i, v = \pi_{h_i}(w) \text{ on } \Gamma_i \right\}, i = 1, 2,$$

where π_{h_i} denotes a suitable interpolation operator on Γ_i. We now define the discrete analogous of Schwarz sequences.
We also assume that the respective matrices resulting from the discretizations of problems (2.21) and (2.22), are M-matrices [6].
Let $u_{h_i}^0 = r_{h_i} u^0$ be given, find the sequence $u_{1h_1}^{n+1} \in V_{h_1}^{(u_{2h_2}^n)}$ such that

$$\left\{ \begin{array}{l} a_1(u_{1h_1}^{n+1}, v - u_{1h_1}^{n+1}) \geq (f_1, v - u_{1h_1}^{n+1}), \forall v \in V_{h_1}^{(u_{2h_2}^n)} \\ u_{1h_1}^{n+1} \leq r_{h_1} M u_{1h_1}^n \text{ in } \tau^{h_1}, v \leq r_{h_1} M u_{1h_1}^n \text{ in } \tau^{h_1}, \end{array} \right. \tag{2.26}$$

respectively, find the sequence $u_{2h_2}^{n+1} \in V_{h_2}^{(u_{1h_1}^{n+1})}$ solution of

$$\left\{ \begin{array}{l} a_2(u_{2h_2}^{n+1}, v - u_{2h_2}^{n+1}) \geq (f_2, v - u_{2h_2}^{n+1}), \forall v \in V_{h_2}^{(u_{1h_1}^{n+1})} \\ u_{2h_2}^{n+1} \leq r_{h_2} M u_{2h_2}^n \text{ in } \tau^{h_2}, v \leq r_{h_2} M u_{2h_2}^n \text{ in } \tau^{h_2}, \end{array} \right. \tag{2.27}$$

where r_{h_i} is a usual restriction operator in Ω_i and $u_{1h_1}^0 = u_{h_1}^0$ in $\Omega_1, u_{2h_2}^0 = u_{h_2}^0$ in Ω_2.

3. L^∞-ERROR ANALYSIS

This section is devoted to demonstration of main result of this work. For that we start by introducing two discrete auxiliary sequences and prove a fundamental lemma.

3.1. **Definition of two Auxiliary Sequences.** For $\widetilde{w}_{h_i}^0 = u_{h_i}^0$, we define the sequence $\widetilde{w}_{1h_1}^{n+1} \in V_{h_1}^{(u_2^n)}$, discrete solution of V.I

$$\left\{ \begin{array}{l} a_1(\widetilde{w}_{1h_1}^{n+1}, v - \widetilde{w}_{1h_1}^{n+1}) \geq (f_1, v - \widetilde{w}_{1h_1}^{n+1}), \forall v \in V_{h_1}^{(u_2^n)} \\ \widetilde{w}_{1h_1}^{n+1} \leq r_{h_1} M u_{1h_1}^n \text{ in } \tau^{h_1}, v \leq r_{h_1} M u_{1h_1}^n \text{ in } \tau^{h_1}, \end{array} \right. \tag{3.1}$$

respectively the sequence $\widetilde{w}_{2h_2}^{n+1} \in V_{h_2}^{(u_1^{n+1})}$ satisfied

$$\left\{ \begin{array}{l} a_2(\widetilde{w}_{2h_2}^{n+1}, v - \widetilde{w}_{2h_2}^{n+1}) \geq (f_2, v - \widetilde{w}_{2h_2}^{n+1}), \forall v \in V_{h_2}^{(u_1^{n+1})} \\ \widetilde{w}_{2h_2}^{n+1} \leq r_{h_2} M u_{2h_2}^n \text{ in } \tau^{h_2}, v \leq r_{h_2} M u_{2h_2}^n \text{ in } \tau^{h_2}, \end{array} \right. \tag{3.2}$$

Notation: In the continuation of this work, we will adopt the following notations.

$$|.|_1 = \|.\|_{L^\infty(\Gamma_1)}, |.|_2 = \|.\|_{L^\infty(\Gamma_2)},$$

$$\|.\|_1 = \|.\|_{L^\infty(\Omega_1)}, \|.\|_2 = \|.\|_{L^\infty(\Omega_2)},$$

$$h_1 = h_2 = h, r_{h_1} = r_{h_2} = r_h, \pi_{h_1} = \pi_{h_2} = \pi_h.$$

The following lemma will play a crucial role in demonstration of the main result.

Lemma 3.1. *Let* $(u_i^{n+1}), (u_{ih}^{n+1}), i=1,2$ *be the sequences defined in* $(2.21), (2.22), (2.26), (2.27)$ *respectively. Then, we have*

$$\|u_1^{n+1}-u_{1h}^{n+1}\|_1 \leq \sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n}\|u_2^p-\widetilde{w}_{2h}^p\|_2, \tag{3.3}$$

$$\|u_2^{n+1}-u_{2h}^{n+1}\|_2 \leq \sum_{p=0}^{n+1}\|u_2^p-\widetilde{w}_{2h}^p\|_2+\sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1, \tag{3.4}$$

Proof. Let us reason by recurrence, for $n=1$: according to proposition 2.3, we have

$$\|\widetilde{w}_{1h}^1-u_{1h}^1\|_1 \leq |\pi_h u_2^0-\pi_h u_{2h}^0|_1 \leq \|u_2^0-u_{2h}^0\|_2 \leq \|u_2^0-\widetilde{w}_{2h}^0\|_2,$$

thus

$$\|u_1^1-u_{1h}^1\|_1 \leq \|u_1^1-\widetilde{w}_{1h}^1\|_1+\|u_2^0-\widetilde{w}_{2h}^0\|_2,$$

in the same way

$$\begin{aligned}
\|u_2^1-u_{2h}^1\|_2 &\leq \|u_2^1-\widetilde{w}_{2h}^1\|_2+\|\widetilde{w}_{2h}^1-u_{2h}^1\|_2\\
&\leq \|u_2^1-\widetilde{w}_{2h}^1\|_2+|\pi_h u_1^1-\pi_h u_{1h}^1|_2\\
&\leq \|u_2^1-\widetilde{w}_{2h}^1\|_2+\|u_1^1-u_{1h}^1\|_1\\
&\leq \sum_{p=0}^{1}\|u_2^p-\widetilde{w}_{2h}^p\|_2+\sum_{p=1}^{1}\|u_1^1-\widetilde{w}_{1h}^1\|_1,
\end{aligned}$$

for $n=2$: using proposition 2.3, we have

$$\begin{aligned}
\|u_1^2-u_{1h}^2\|_1 &\leq \|u_1^2-\widetilde{w}_{1h}^2\|_1+\|\widetilde{w}_{1h}^2-u_{1h}^2\|_1\\
&\leq \|u_1^2-\widetilde{w}_{1h}^2\|_1+|\pi_h u_2^1-\pi_h u_{2h}^1|_1\\
&\leq \|u_1^2-\widetilde{w}_{1h}^2\|_1+\|u_2^1-u_{2h}^1\|_2\\
&\leq \|u_1^2-\widetilde{w}_{1h}^2\|_1+\|u_1^1-\widetilde{w}_{1h}^1\|_1+\sum_{p=0}^{1}\|u_2^p-\widetilde{w}_{2h}^p\|_2\\
&\leq \sum_{p=1}^{2}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{1}\|u_2^p-\widetilde{w}_{2h}^p\|_2,
\end{aligned}$$

and

$$\begin{aligned}
\|u_2^2-u_{2h}^2\|_2 &\leq \|u_2^2-\widetilde{w}_{2h}^2\|_2+\|\widetilde{w}_{2h}^2-u_{2h}^2\|_2\\
&\leq \|u_2^2-\widetilde{w}_{2h}^2\|_2+|\pi_h u_1^2-\pi_h u_{1h}^2|_2\\
&\leq \|u_2^2-\widetilde{w}_{2h}^2\|_2+\|u_1^2-u_{1h}^2\|_1\\
&\leq \|u_2^2-\widetilde{w}_{2h}^2\|_2+\sum_{p=1}^{2}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{1}\|u_2^p-\widetilde{w}_{2h}^p\|_2\\
&\leq \sum_{p=1}^{2}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{2}\|u_2^p-\widetilde{w}_{2h}^p\|_2.
\end{aligned}$$

Let us now suppose that

$$\|u_2^n-u_{2h}^n\|_2 \leq \sum_{p=1}^{n}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n}\|u_2^p-\widetilde{w}_{2h}^p\|_2,$$

then

$$\begin{aligned}
\|u_1^{n+1}-u_{1h}^{n+1}\|_1 &\le \|u_1^{n+1}-\widetilde{w}_{1h}^{n+1}\|_1+\|\widetilde{w}_{1h}^{n+1}-u_{1h}^{n+1}\|_1\\
&\le \|u_1^{n+1}-\widetilde{w}_{1h}^{n+1}\|_1+|\pi_h u_2^n-\pi_h u_{2h}^n|_1\\
&\le \|u_1^{n+1}-\widetilde{w}_{1h}^{n+1}\|_1+\|u_2^n-u_{2h}^n\|_2\\
&\le \|u_1^{n+1}-\widetilde{w}_{1h}^{n+1}\|_1+\sum_{p=1}^{n}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n}\|u_2^p-\widetilde{w}_{2h}^p\|_2,
\end{aligned}$$

and consequently

$$\|u_1^{n+1}-u_{1h}^{n+1}\|_1 \le \sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n}\|u_2^p-\widetilde{w}_{2h}^p\|_2,$$

likewise, using the above estimate, we get

$$\begin{aligned}
\|u_2^{n+1}-u_{2h}^{n+1}\|_2 &\le \|u_2^{n+1}-\widetilde{w}_{2h}^{n+1}\|_2+\|\widetilde{w}_{2h}^{n+1}-u_{2h}^{n+1}\|_2\\
&\le \|u_2^{n+1}-\widetilde{w}_{2h}^{n+}\|_2+|\pi_h u_1^{n+1}-\pi_h u_{1h}^{n+1}|_2\\
&\le \|u_2^{n+1}-\widetilde{w}_{2h}^{n+1}\|_2+\|u_1^{n+1}-u_{1h}^{n+1}\|_1\\
&\le \sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n+1}\|u_2^p-\widetilde{w}_{2h}^p\|_2.
\end{aligned}$$

Hence,

$$\|u_2^{n+1}-u_{2h}^{n+1}\|_2 \le \sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n+1}\|u_2^p-\widetilde{w}_{2h}^p\|_2.$$

∎

3.2. L^∞-Error Estimate.

Theorem 3.2. *There exists a constant c independent of n and h such that*

$$\|u_i-u_{ih}^{n+1}\|_i \le ch^2|\log h|^3, i=1,2. \tag{3.5}$$

Proof. Let us give the proof for $i=1$.The case $i=2$ is similar.
Indeed, let $k=max(k_1,k_2)$

$$\|u_1-u_{1h}^{n+1}\|_1 \le \|u_1-u_1^{n+1}\|_1+\|u_1^{n+1}-u_{1h}^{n+1}\|_1,$$

we use theorem 2.5 and lemma 3.1

$$\begin{aligned}
&\le k^{2n}|u^0-u|_1+\sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n}\|u_2^p-\widetilde{w}_{2h}^p\|_2\\
&\le |u^0-u|_1+\sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n}\|u_2^p-\widetilde{w}_{2h}^p\|_2\\
&\le \|u^0-u\|_2+\sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n}\|u_2^p-\widetilde{w}_{2h}^p\|_2\\
&\le (\|u^0-u_h\|_2+\|u_h-u\|_2)+\sum_{p=1}^{n+1}\|u_1^p-\widetilde{w}_{1h}^p\|_1+\sum_{p=0}^{n}\|u_2^p-\widetilde{w}_{2h}^p\|_2,
\end{aligned}$$

according to theorem 2.4

$$\begin{aligned}\|u_1 - u_{1h}^{n+1}\|_1 &\leq (ch^2|\log h|^2 + ch^2|\log h|^2) + (n+1)ch^2|\log h|^2 + (n+1)ch^2|\log h|^2 \\ &\leq ch^2|\log h|^2 + 2(n+1)ch^2|\log h|^2.\end{aligned}$$

Now setting

$$k^n = h,$$

we obtain

$$\|u_1 - u_{1h}^{n+1}\|_1 \leq ch^2|\log h|^3,$$

which is the desired error estimate. ∎

Conclusion 1. *We have established a convergence order of Schwarz algorithm for two subdomains. However, it is worth noting that the approach developed in this paper relies on the geometrical convergence for the systems (2.21), (2.22), where the constant* $k_i \in]0,1[$ *depends on* Ω_j *and* Γ_i*. The method relies also on the discrete maximum principle. This assumption imposes a restriction on bilinear form* $a_i(.,.)$ *as well as the triangulation. In 2-D, for example, angles of triangles must be acute* [6]. *The error estimate obtained contains a logarithmic factor with an extra power of* $|\log h|$ *than expected.*

4. NUMERICAL EXPERIMENTATION

In this section, we present some results of a sample numerical test involving nonmatching overlapping grids. We put

$$\begin{aligned}\Omega &= [0,1]\times[0,1], V = H_0^1(\Omega) \\ Au &= -\Delta u + u \\ f(x,y) &= (2\pi^2+1)\sin(\pi x)\sin(\pi y) \\ Mu(x) &= 1 + \inf\{u(x+\xi), \xi \geq 0, x+\xi \in \Omega\}, \forall x \in \Omega \\ \Omega_1 &= [0,x_1]\times[0,1], \Omega_2 = [x_2,1]\times[0,1] \\ d &= x_1 - x_2, 0 < x_2 < x_1 < 1\end{aligned}$$

we stop the iteration when

$$\|u_{1h_1}^n - u_{1h_1}^{n-1}\|_1 < 10^{-6} \text{ and } \|u_{2h_2}^n - u_{2h_2}^{n-1}\|_2 < 10^{-6}.$$

Let $u_{ih}^0 = 0$ be the initial value. For each macro-iteration (Schwarz iteration) we use on each subdomain the relaxation iterative method by projection, where the relaxation parameter is $w = 1.5$

	l	error in Ω_1	error in Ω_2	it_3	it_2	it_1
$d = 0.5$	1	/	/	6	22	23
$x_2 = 0.25$	2	$3.4460e-02(/)$	$1.4128e-02(/)$	6	65	23
$h_1 = \frac{1}{2^{l+1}}$	3	$8.6500e-03(3.98)$	$3.2638e-03(4.32)$	6	250	67
$h_2 = \frac{1}{2^{l+2}}$	4	$2.2461e-03(3.85)$	$8.0498e-04(4.05)$	6	885	254
$d = 0.25$	1	/	/	15	21	23
$x_2 = 0.5$	2	$3.3886e-02(/)$	$1.2371e-02(/)$	16	34	23
$h_1 = \frac{1}{2^{l+1}}$	3	$8.4832e-03(3.99)$	$2.9105e-03(4.25)$	17	143	67
$h_2 = \frac{1}{2^{l+2}}$	4	$2.1911e-03(3.87)$	$7.3943e-04(3.93)$	17	516	254
$d = 1/3$	1	/	/	9	22	21
$x_2 = 1/3$	2	$4.6320e-02(/)$	$2.0774e-02(/)$	10	26	23
$h_1 = \frac{1}{3\times 2^{l-1}}$	3	$1.2975e-02(3.57)$	$5.2879e-03(3.93)$	10	125	28
$h_2 = \frac{1}{3\times 2^{l}}$	4	$3.1613e-03(4.10)$	$1.2342e-03(4.28)$	10	456	129

Table 4.1: Errors and iterations number

it_3, indicates the macro-iteration of Schwarz (external iteration),
it_1, it_2, indicates the micro-iterations of relaxation method by projection for each subdomain (interior iterations).
l, indicates the refinement level.
In the row l, the number between bracket $(.)$ indicates the quotient of the value in the row $(l-1)$ over the value in the row l. The quotient shows the good precision of the discretization, thus the convergence order equals 2. We notice that the iterations number of Schwarz it_3 is bounded independently of the meshsizes.

$k = it_3$	1	2	3	4	5	6
$u^k(0.25, 0.25)$	0.42204973	0.49844705	0.50031850	0.50036433	0.50036546	0.50036548
$u^k(0.5, 0.5)$	0.69966404	0.99328516	1.00047814	1.00065431	1.00065863	1.00065873
$u^k(0.75, 0.75)$	0.48780615	0.49975636	0.50004905	0.50005622	0.50005639	0.50005640

Table 4.2: iterations of solution for $h_1 = \frac{1}{2^5}, h_2 = \frac{1}{2^6}, d = 0.5$

$d \backslash l$	1	2	3	4
1/2	6	6	6	6
1/3	9	10	10	10
1/4	15	16	17	17

Table 4.3: iterations number of Schwarz

We notice that if the geometrical distance d decreases, the iterations number of Schwarz it_3 increases, this proves that the iterations number of Schwarz depends imperatively of the geometrical interface $\Omega_1 \cap \Omega_2$.

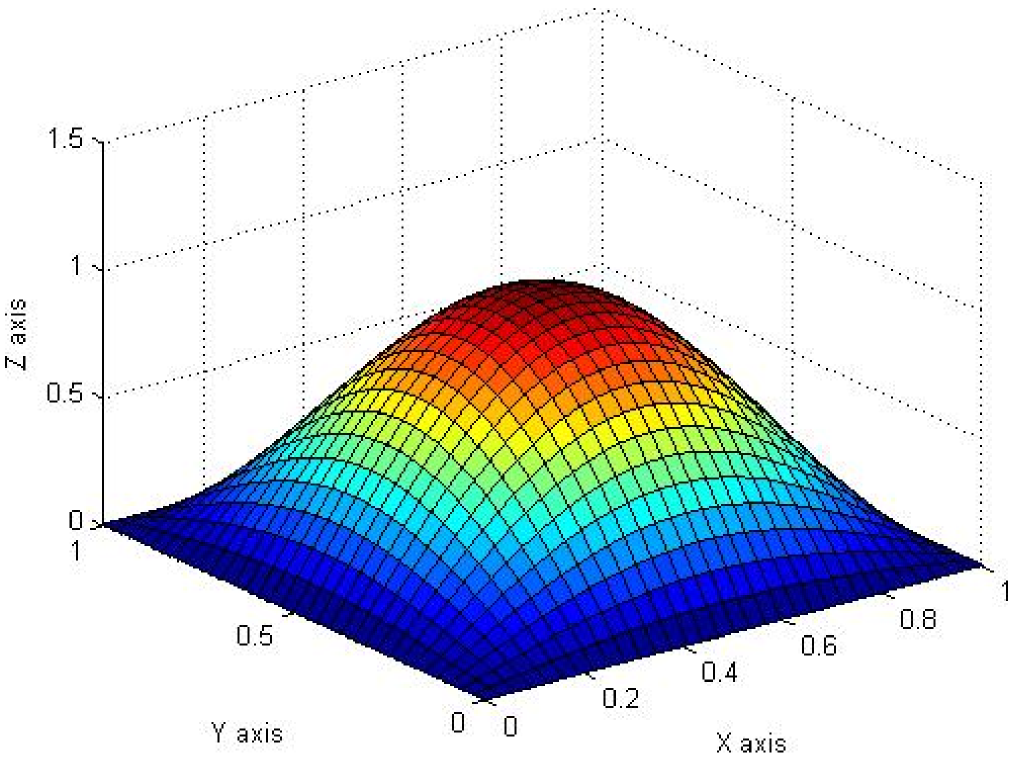

Figure 1: Surface of numerical solution

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