The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 10, Issue 1, Article 9, pp. 1-11, 2013

SOME APPLICATIONS OF FEJÉR'S INEQUALITY FOR CONVEX FUNCTIONS (I)

S.S. DRAGOMIR[1,2] AND I. GOMM[1]

Received 5 January, 2013; accepted 10 April, 2013; published 2 May, 2013.

[1] MATHEMATICS, SCHOOL OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA.
sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2] SCHOOL OF COMPUTATIONAL & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA.

ABSTRACT. Some applications of Fejér's inequality for convex functions are explored. Upper and lower bounds for the weighted integral

$$\int_a^b (b-x)(x-a) f(x)\, dx$$

under various assumptions for f with applications to the trapezoidal quadrature rule are given. Some inequalities for special means are also provided.

Key words and phrases: Convex functions, Hermite-Hadamard inequality, Fejér's Inequality, Special means.

1991 *Mathematics Subject Classification.* 26D15; 25D10.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The *Hermite-Hadamard* integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for special means.

For related results, see for instance the research papers [1], [8], [9], [10], [12], [11], [13], [14], [15], the monograph online [7] and the references therein.

In 1906, Fejér, while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1.1. *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval (a,b) and w is a positive function in the same interval such that*

$$w(a+t)=w(b-t),\quad 0\leq t\leq\frac{1}{2}(a+b),$$

i.e., $y=w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the $x-$axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx\leq\int_a^b h(x)\,w(x)\,dx\leq\frac{h(a)+h(b)}{2}\int_a^b w(x)\,dx. \tag{1.1}$$

If h is concave on (a,b), then the inequalities reverse in (1.1).

Clearly, for $w(x)\equiv 1$ on $[a,b]$ we get HH.

We observe that, if we take $w(x)=(b-x)(x-a)$, $x\in[a,b]$, then w satisfies the conditions in Theorem 1.1,

$$\int_a^b (b-x)(x-a)\,dx=\frac{1}{6}(b-a)^3$$

and by 1.1 we have the following inequality

$$\begin{aligned}\frac{1}{6}h\left(\frac{a+b}{2}\right)(b-a)^3&\leq\int_a^b(b-x)(x-a)\,h(x)\,dx\\&\leq\frac{h(a)+h(b)}{12}(b-a)^3,\end{aligned} \tag{1.2}$$

for any convex function $h:[a,b]\to\mathbb{R}$. If the function h is concave the inequalities in (1.2) reverse.

In this paper we establish amongst other some better bounds for the weighted integral

$$\int_a^b (b-x)(x-a)\,h(x)\,dx$$

in the case of convex functions $h:[a,b]\to\mathbb{R}$. We also investigate the connection with the trapezoid rule and apply some of the obtained results for special means.

2. THE RESULTS

The following result holds.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and such that the second derivative f'' is convex on (a,b). Then*

$$\frac{1}{12}f''\left(\frac{a+b}{2}\right)(b-a)^2 \le \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f''(a)+f''(b)}{24}(b-a)^2. \tag{2.1}$$

Proof. We know, see for instance [7, Lemma 4, p. 38], that

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{2(b-a)}\int_a^b (x-a)(b-x)f''(x)\,dx. \tag{2.2}$$

Since f'' is convex on (a,b), then by (1.2) we have

$$\frac{1}{6}f''\left(\frac{a+b}{2}\right)(b-a)^3 \le \int_a^b (b-x)(x-a)f''(x)\,dx \le \frac{f''(a)+f''(b)}{12}(b-a)^3. \tag{2.3}$$

Utilising (2.2) and (2.3) we deduce the desired result (2.1). ∎

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b).*

If there exists a real number m such that $f''(x)\ge m$ for any $x\in(a,b)$, then

$$\begin{aligned}(2.4)\qquad &\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{240}m(b-a)^5\\ &\le \int_a^b (b-x)(x-a)f(x)\,dx\\ &\le \frac{f(a)+f(b)}{12}(b-a)^3-\frac{1}{60}m(b-a)^5,\end{aligned}$$

If there exists a real number M such that $f''(x)\le M$ for any $x\in(a,b)$, then

$$\begin{aligned}(2.5)\qquad &\frac{f(a)+f(b)}{12}(b-a)^3-\frac{1}{60}M(b-a)^5\\ &\le \int_a^b (b-x)(x-a)f(x)\,dx\\ &\le \frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{240}M(b-a)^5.\end{aligned}$$

Proof. Define the function $h_m:[a,b]\to\mathbb{R}$ by

$$h_m(x):=f(x)+\frac{1}{2}m(x-a)(b-x).$$

This function is twice differentiable and the second derivative is

$$h_m''(x)=f''(x)-m\ge 0,\ x\in(a,b)$$

showing that h_m is convex on $[a,b]$.

If we apply the inequality (1.2) for h_m, then we have

$$\begin{aligned}\frac{1}{6}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{8}m\left(b-a\right)^2\right]\left(b-a\right)^3 \\ \leq \int_a^b \left(b-x\right)\left(x-a\right)f\left(x\right)dx+\frac{1}{2}m\int_a^b \left(b-x\right)^2\left(x-a\right)^2 dx \\ \leq \frac{f\left(a\right)+f\left(b\right)}{12}\left(b-a\right)^3 .\end{aligned} \tag{2.6}$$

Observe that

$$\begin{aligned}&\frac{1}{6}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{8}m\left(b-a\right)^2\right]\left(b-a\right)^3 \\ &=\frac{1}{6}f\left(\frac{a+b}{2}\right)\left(b-a\right)^3+\frac{1}{48}m\left(b-a\right)^5 .\end{aligned}$$

We also have

$$\begin{aligned}\int_a^b \left(b-x\right)^2\left(x-a\right)^2 dx &= \frac{1}{3}\left(x-a\right)^3\left(b-x\right)^2\Big|_a^b+\frac{2}{3}\int_a^b \left(b-x\right)\left(x-a\right)^3 dx \\ &=\frac{2}{3}\left[\frac{1}{4}\left(b-x\right)\left(x-a\right)^4\Big|_a^b+\frac{1}{4}\int_a^b \left(x-a\right)^4 dx\right] \\ &=\frac{1}{30}\left(b-a\right)^5 .\end{aligned}$$

Then (2.6) becomes

$$\begin{aligned}&\frac{1}{6}f\left(\frac{a+b}{2}\right)\left(b-a\right)^3+\frac{1}{48}m\left(b-a\right)^5 \\ &\leq \int_a^b \left(b-x\right)\left(x-a\right)f\left(x\right)dx+\frac{1}{60}m\left(b-a\right)^5 \\ &\leq \frac{f\left(a\right)+f\left(b\right)}{12}\left(b-a\right)^3\end{aligned}$$

which is equivalent with (2.4).

Now define the function $h_M : [a,b] \to \mathbb{R}$ by

$$h_M\left(x\right) := -f\left(x\right)-\frac{1}{2}M\left(x-a\right)\left(b-x\right).$$

This function is twice differentiable and

$$h_M''\left(x\right) := M-f''\left(x\right)\geq 0,\ x\in\left(a,b\right)$$

showing that h_M is convex on $[a,b]$.

If we apply the inequality (1.2) for h_M, then we have

$$\begin{aligned}&\frac{1}{6}\left[-f\left(\frac{a+b}{2}\right)-\frac{1}{8}M\left(b-a\right)^2\right]\left(b-a\right)^3 \\ &\leq \int_a^b \left(b-x\right)\left(x-a\right)\left[-f\left(x\right)-\frac{1}{2}M\left(x-a\right)\left(b-x\right)\right]dx \\ &\leq \frac{-f\left(a\right)-f\left(b\right)}{12}\left(b-a\right)^3 ,\end{aligned}$$

which, by multiplication with -1, produces

$$\begin{aligned}
&\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{48}M(b-a)^5\\
&\geq\int_a^b(b-x)(x-a)f(x)\,dx+\frac{1}{2}M\int_a^b(x-a)^2(b-x)^2\,dx\\
&\geq\frac{f(a)+f(b)}{12}(b-a)^3
\end{aligned}$$

that is equivalent with

$$\begin{aligned}
&\frac{f(a)+f(b)}{12}(b-a)^3-\frac{1}{60}M(b-a)^5\\
&\leq\int_a^b(b-x)(x-a)f(x)\,dx\\
&\leq\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{240}M(b-a)^5
\end{aligned}$$

and the inequality (2.5) is proved. ∎

Corollary 2.3. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable function on* (a,b). *If there exists a* $K>0$ *such that* $|f''(x)|\leq K$ *for any* $x\in(a,b)$, *then*

$$\begin{aligned}
&\left|\int_a^b(b-x)(x-a)f(x)\,dx-\frac{1}{12}(b-a)^3\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right|\\
&\leq\frac{1}{96}K(b-a)^5.
\end{aligned}\tag{2.7}$$

Proof. If we write the inequality (2.4) for $m=-K$ and the inequality (2.5) for $M=K$ we have

$$\begin{aligned}
&\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3-\frac{1}{240}K(b-a)^5\\
&\leq\int_a^b(b-x)(x-a)f(x)\,dx\\
&\leq\frac{f(a)+f(b)}{12}(b-a)^3+\frac{1}{60}K(b-a)^5,
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
&\frac{f(a)+f(b)}{12}(b-a)^3-\frac{1}{60}K(b-a)^5\\
&\leq\int_a^b(b-x)(x-a)f(x)\,dx\\
&\leq\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{240}K(b-a)^5.
\end{aligned}\tag{2.9}$$

If we add the inequality (2.8) with (2.8) and divide the sum by 2 we get

$$\begin{aligned}
&\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{f(a)+f(b)}{24}(b-a)^3-\frac{1}{96}K(b-a)^5\\
&\le\int_a^b(b-x)(x-a)f(x)\,dx\\
&\le\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{f(a)+f(b)}{24}(b-a)^3+\frac{1}{96}K(b-a)^5,
\end{aligned}$$

which is equivalent with the desired result (2.7). ∎

Remark 2.1. We observe that the case $m>0$ in the inequality (2.4) produces a better result than (1.2).

For twice differentiable functions we can provide the following *perturbed trapezoid quadrature rule*

$$\begin{aligned}
\int_a^b f(x)\,dx &\simeq \frac{f(a)+f(b)}{2}(b-a)\\
&\quad-\frac{1}{24}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right].
\end{aligned}\tag{2.10}$$

Denote $R_{P,T}(f;a,b)$ the error in approximating the integral as in (2.10), namely

$$\begin{aligned}
R_{P,T}(f;a,b)&:=\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}(b-a)\\
&\quad+\frac{1}{24}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right].
\end{aligned}$$

The following result that provides an *a priory* error bound for functions whose forth derivatives are bounded, holds.

Proposition 2.4. *Let $f:[a,b]\to\mathbb{R}$ be a four time differentiable function on (a,b). If there exists a $K>0$ such that $\left|f^{(4)}(x)\right|\le K$ for any $x\in(a,b)$, then*

$$|R_{P,T}(f;a,b)|\le\frac{1}{192}K(b-a)^5.\tag{2.11}$$

Proof. Writing the inequality (2.7) for the second derivative f'' we have

$$\begin{aligned}
&\left|\int_a^b(b-x)(x-a)f''(x)\,dx\right.\\
&\left.-\frac{1}{12}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right]\right|\\
&\le\frac{1}{96}K(b-a)^5.
\end{aligned}$$

Dividing this inequality by 2 and utilizing the representation (2.2) we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right.\\
&\left.-\frac{1}{24}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right]\right|\\
&\le\frac{1}{192}K(b-a)^5,
\end{aligned}$$

and the inequality (2.11) is proved. ∎

The following result that improves the inequality (1.2) also holds.

Theorem 2.5. *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function. Then*

$$\begin{aligned}
\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 &\le 2\int_a^b\left(x-\frac{a+b}{2}\right)^2 f\left(\frac{x+\frac{a+b}{2}}{2}\right)dx \\
&\le \int_a^b (b-x)(x-a)\,f(x)\,dx \\
&\le \int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx+\frac{(b-a)^3}{12}f\left(\frac{a+b}{2}\right) \\
&\le \frac{(b-a)^3}{12}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \\
&\le \frac{f(a)+f(b)}{12}(b-a)^3.
\end{aligned}\tag{2.12}$$

Proof. Denote, as usual, $F(x):=\int_a^x f(t)\,dt$, $x\in[a,b]$. By the Hermite-Hadamard inequality we have for any $x\in[a,b]$, $x\neq\frac{a+b}{2}$ that

$$f\left(\frac{x+\frac{a+b}{2}}{2}\right)\le\frac{F(x)-F\left(\frac{a+b}{2}\right)}{x-\frac{a+b}{2}}\le\frac{1}{2}\left[f(x)+f\left(\frac{a+b}{2}\right)\right],$$

which, by multiplication with $\left(x-\frac{a+b}{2}\right)^2\ge 0$ implies

$$\begin{aligned}
&f\left(\frac{x+\frac{a+b}{2}}{2}\right)\left(x-\frac{a+b}{2}\right)^2 \\
&\le\left[F(x)-F\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right) \\
&\le\frac{1}{2}\left[f(x)+f\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right)^2,
\end{aligned}\tag{2.13}$$

that holds for any $x\in[a,b]$.

Integrating the inequality (2.13) on the interval $[a,b]$ we get

$$\begin{aligned}
&\int_a^b\left(x-\frac{a+b}{2}\right)^2 f\left(\frac{x+\frac{a+b}{2}}{2}\right)dx \\
&\le\int_a^b\left[F(x)-F\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right)dx \\
&\le\frac{1}{2}\int_a^b\left[f(x)+f\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right)^2dx \\
&=\frac{1}{2}\left[\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx+f\left(\frac{a+b}{2}\right)\frac{(b-a)^3}{12}\right].
\end{aligned}\tag{2.14}$$

Now, observe that

$$\begin{aligned}
&\int_a^b \left[F(x) - F\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right)dx \\
&= \int_a^b F(x)\left(x - \frac{a+b}{2}\right)dx = \frac{1}{2}\int_a^b F(x)\, d\left(x - \frac{a+b}{2}\right)^2 \\
&= \frac{1}{2}\left[F(x)\left(x - \frac{a+b}{2}\right)^2\Big|_a^b - \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx\right] \\
&= \frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 \int_a^b f(x) - \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx\right] \\
&= \frac{1}{2}\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2\right] f(x)\,dx \\
&= \frac{1}{2}\int_a^b (b-x)(x-a) f(x)\,dx
\end{aligned}$$

and by (2.14) we have

$$\begin{aligned}
&\int_a^b \left(x - \frac{a+b}{2}\right)^2 f\left(\frac{x + \frac{a+b}{2}}{2}\right)dx \\
&\le \frac{1}{2}\int_a^b (b-x)(x-a) f(x)\,dx \\
&= \frac{1}{2}\left[\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx + f\left(\frac{a+b}{2}\right)\frac{(b-a)^3}{12}\right],
\end{aligned}$$

which proves the second and the third inequality in (2.12).

The function $g(x) := f\left(\frac{x+\frac{a+b}{2}}{2}\right)$ is convex on $[a,b]$ and $w(x) := \left(x - \frac{a+b}{2}\right)^2$ is nonnegative and symmetric on $[a,b]$. Applying Fejér's first inequality we have

$$f\left(\frac{\frac{a+b}{2} + \frac{a+b}{2}}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \le \int_a^b f\left(\frac{x + \frac{a+b}{2}}{2}\right)\left(x - \frac{a+b}{2}\right)^2 dx$$

i.e.

$$\frac{(b-a)^3}{12} f\left(\frac{a+b}{2}\right) \le \int_a^b \left(x - \frac{a+b}{2}\right)^2 f\left(\frac{x + \frac{a+b}{2}}{2}\right)dx,$$

which proves the first inequality in (2.12).

From the Fejér's second inequality for the convex function f function and the weight $w(x) := \left(x - \frac{a+b}{2}\right)^2$ we also have

$$\begin{aligned}
\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx &\le \frac{f(a)+f(b)}{2}\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \\
&= \frac{f(a)+f(b)}{24}(b-a)^3,
\end{aligned}$$

which proves the fourth inequality in (2.12).

The last inequality is obvious. ∎

Corollary 2.6. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and such that the second derivative f'' is convex on (a,b). Then*

$$\begin{aligned}(2.15)\quad \frac{1}{12}f''\left(\frac{a+b}{2}\right)(b-a)^2 &\le \int_a^b\left(x-\frac{a+b}{2}\right)^2 f''\left(\frac{x+\frac{a+b}{2}}{2}\right)dx\\ &\le \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\\ &\le \frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 f''(x)\,dx+\frac{(b-a)^3}{24}f''\left(\frac{a+b}{2}\right)\\ &\le \frac{(b-a)^3}{24}\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right]\\ &\le \frac{f''(a)+f''(b)}{24}(b-a)^3.\end{aligned}$$

We observe that the inequality (2.15) is a better result than (2.1).

3. APPLICATIONS FOR SPECIAL MEANS

Let us recall the following means for two positive numbers.

(1) *The Arithmetic mean*

$$A=A(a,b):=\frac{a+b}{2},\ a,b>0;$$

(2) *The Geometric mean*

$$G=G(a,b):=\sqrt{ab},\ a,b>0;$$

(3) *The Harmonic mean*

$$H=H(a,b):=\frac{2ab}{a+b},\ a,b>0;$$

(4) *The Logarithmic mean*

$$L=L(a,b):=\begin{cases} a & \text{if}\quad a=b\\ \frac{b-a}{\ln b-\ln a} & \text{if}\quad a\neq b;\end{cases},\ a,b>0,$$

(5) *The Identric mean*

$$I=I(a,b):=\begin{cases} a & \text{if}\quad a=b\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if}\quad a\neq b\end{cases},\ a,b>0;$$

(6) *The p-Logarithmic mean*

$$L_p=L_p(a,b):=\begin{cases} a & \text{if}\quad a=b\\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if}\quad a\neq b\end{cases},\quad a,b>0.$$

The following inequality is well known in the literature:

$$(3.1)\qquad H\le G\le L\le I\le A.$$

It is also known that L_p is monotonically increasing over $p\in\mathbb{R}$, denoting $L_0=I$ and $L_{-1}=L$.

Consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=x^{p}$ for $p\geq 3$. We have the fourth derivative of the function given by

$$f^{(4)}(x)=p(p-1)(p-2)(p-3)x^{p-4}$$

which shows that the second derivative f'' is convex on $[a,b]$. Applying the inequality (2.1) we have

$$\begin{aligned}\frac{1}{12}p(p-1)\left(\frac{a+b}{2}\right)^{p-2}(b-a)^{2} &\leq \frac{a^{p}+b^{p}}{2}-\frac{1}{b-a}\int_{a}^{b}x^{p}dx\\ &\leq p(p-1)\frac{a^{p-2}+b^{p-2}}{24}(b-a)^{2},\end{aligned}$$

which in terms of the special means define above can be written as

$$\begin{aligned}\frac{1}{12}p(p-1)A^{p-2}(a,b)(b-a)^{2} &\leq A(a^{p},b^{p})-L_{p}^{p}(a,b)\\ &\leq \frac{1}{12}p(p-1)A\left(a^{p-2},b^{p-2}\right)(b-a)^{2},\end{aligned}\tag{3.2}$$

that holds for any $a,b>0$ and $p\geq 3$.

Consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=\frac{1}{x}$. Then $f''(x)=\frac{2}{x^{3}}$ and $f^{(4)}(x)=\frac{24}{x^{5}}$ showing that the second derivative is convex on $[a,b]$. Applying the inequality (2.1) we have

$$\begin{aligned}\frac{1}{6}\frac{(b-a)^{2}}{A^{3}(a,b)} &\leq \frac{\frac{1}{a}+\frac{1}{b}}{2}-\frac{\ln b-\ln a}{b-a}\\ &\leq \frac{\frac{2}{a^{3}}+\frac{2}{b^{3}}}{24}(b-a)^{2}.\end{aligned}$$

which is equivalent with

$$\frac{1}{6}\frac{(b-a)^{2}}{A^{3}(a,b)}\leq\frac{L(a,b)-H(a,b)}{L(a,b)H(a,b)}\leq\frac{1}{6}\frac{(b-a)^{2}}{H(a^{3},b^{3})}\tag{3.3}$$

that holds for any $a,b>0$.

Consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=-\ln x$. Then $f''(x)=\frac{1}{x^{2}}$ and $f^{(4)}(x)=\frac{6}{x^{4}}$ showing that the second derivative is convex on $[a,b]$. Applying the inequality (2.1) we have

$$\begin{aligned}\frac{1}{12}\frac{(b-a)^{2}}{A^{2}(a,b)} &\leq \frac{-\ln a-\ln b}{2}+\frac{1}{b-a}\int_{a}^{b}\ln x dx\\ &\leq \frac{\frac{1}{a^{2}}+\frac{1}{b^{2}}}{24}(b-a)^{2}.\end{aligned}$$

Observe that

$$\begin{aligned}\frac{1}{b-a}\int_{a}^{b}\ln x dx &= \frac{1}{b-a}\left[x\ln x|_{a}^{b}-(b-a)\right]=\\ &= \left[\ln\left(\frac{b^{b}}{a^{a}}\right)^{1/(b-a)}-1\right]=\ln I(a,b),\end{aligned}$$

and

$$\frac{-\ln a-\ln b}{2}=\ln\frac{1}{G(a,b)}.$$

Then we get

$$\frac{1}{12}\frac{(b-a)^2}{A^2(a,b)} \le \ln\left(\frac{I(a,b)}{G(a,b)}\right) \le \frac{1}{12}\frac{(b-a)^2}{H(a^2,b^2)} \tag{3.4}$$

that holds for any $a, b > 0$.

The interested reader may apply the inequality (2.11) or (2.15) to obtain other similar results. However, the details are omitted here.

REFERENCES

[1] A.G. AZPEITIA, Convex functions and the Hadamard inequality. *Rev. Colombiana Mat.* **28** (1994), no. 1, pp. 7–12.

[2] S.S. DRAGOMIR, A mapping in connection to Hadamard's inequalities, *An. Öster. Akad. Wiss. Math.-Natur.,* (Wien), **128**(1991), pp. 17-20. MR 934:26032. ZBL No. 747:26015.

[3] S.S. DRAGOMIR, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.,* **167**(1992), pp. 49-56. MR:934:26038, ZBL No. 758:26014.

[4] S.S. DRAGOMIR, On Hadamard's inequalities for convex functions, *Mat. Balkanica,* **6**(1992), pp. 215-222. MR: 934: 26033.

[5] S.S. DRAGOMIR, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), pp. 471-476.

[6] S.S. DRAGOMIR, D.S. MILOŚEVIĆ and J. SÁNDOR, On some refinements of Hadamard's inequalities and applications, *Univ. Belgrad, Publ. Elek. Fak. Sci. Math.,* **4**(1993), pp. 21-24.

[7] S.S. DRAGOMIR and C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online `http://rgmia.org/monographs/hermite_hadamard.html`].

[8] A. GUESSAB and G. SCHMEISSER, Sharp integral inequalities of the Hermite-Hadamard type. *J. Approx. Theory* **115** (2002), no. 2, 260–288.

[9] E. KILIANTY and S.S. DRAGOMIR, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space. *Math. Inequal. Appl.* **13** (2010), no. 1, pp. 1–32.

[10] M. MERKLE, Remarks on Ostrowski's and Hadamard's inequality. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **10** (1999), pp. 113–117.

[11] C. E. M. PEARCE and A. M. RUBINOV, P-functions, quasi-convex functions, and Hadamard type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, pp. 92–104.

[12] J. PEČARIĆ and A. VUKELIĆ, Hadamard and Dragomir-Agarwal inequalities, the Euler formulae and convex functions. Functional equations, inequalities and applications, pp. 105–137, Kluwer Acad. Publ., Dordrecht, 2003.

[13] G. TOADER, Superadditivity and Hermite-Hadamard's inequalities. *Studia Univ. Babeş-Bolyai Math.* **39** (1994), no. 2, pp. 27–32.

[14] G.-S. YANG and M.-C. HONG, A note on Hadamard's inequality. *Tamkang J. Math.* **28** (1997), no. 1, pp. 33–37.

[15] G.-S. YANG and K.-L. TSENG, On certain integral inequalities related to Hermite-Hadamard inequalities. *J. Math. Anal. Appl.* **239** (1999), no. 1, pp. 180–187.