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# ON REFORMATIONS OF 2–HILBERT SPACES

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ABSTRACT. In this paper, first, we introduce the new concept of (complex) 2–Hilbert spaces, that is, we define the concept of 2–inner product spaces with a complex valued 2–inner product by using the 2–norm. Next, we prove some theorems on Schwartz's inequality, the polarization identity, the parallelogram laws and related important properties. Finally, we give some open problems related to 2–Hilbert spaces.

*Key words and phrases:* 2–inner product, 2–norm, $x$–linear function, linear 2–function, $x$–orthogonal, 2–Hilbert space.

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## 1. INTRODUCTION

The concepts of $2$-metric spaces and linear $2$–normed spaces were first introduced by Gähler in 1963 and 1964 [11, 12]. Since then, many authors, Freese et al., Gähler, Cho et al., Diminnie et al., Gunawan et al. have developed extensively topological and geometric structures of $2$–normed spaces, $2$–metric spaces, semi–$2$–normed spaces, semi–$2$–metric spaces, ultra–$2$–normed spaces, non–Archimedean $2$–normed spaces and the relations among $2$–norms, bi–vectors and usual norms, and others (see [1, 2, 3, 4, 7, 8, 9, 13, 14, 15, 18, 19, 20, 21, 22, 23]). Particulary, in [8], Freese and Cho showed that a $2$–metric space in which unique "lines" exist and expressed entirely in terms of the $2$–metric is a linear $2$–normed space.

A linear $2$–normed space is a pair $(H, \|\cdot,\cdot\|)$, where $H$ is a a complex vector space of a dimension greater than one and $\|\cdot,\cdot\|$ is a real-valued mapping on $H \times H$ satisfying the following conditions: for all $a, b \in H$ and $\alpha \in \mathbb{C}$,

(2N1) $\|\alpha a, b\| = |\alpha| \|a, b\|$;
(2N2) $\|b, a\| = \|a, b\|$;
(2N3) $\|a, b\| = 0$ if and only if $a$ and $b$ are linearly dependent;
(2N4) $\|a_1 + a_2, b\| \leq \|a_1, b\| + \|a_2, b\|$.

We can consider a $2$–norm on $H$ defined by an inner product $\langle\cdot,\cdot\rangle$ on $H$ as follows:

$$\|a, b\| := \begin{vmatrix} \langle a, a\rangle & \langle a, b\rangle \\ \langle b, a\rangle & \langle b, b\rangle \end{vmatrix}^{\frac{1}{2}}$$

for all $a, b \in H$. Geometrically, a $2$–norm function generalizes the concept of the parallelogram law spanned by the vectors $a$ and $b$, which may be given explicitly by the formula $\|a, b\| = |a_1 b_2 - a_2 b_1|$, where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ (see [14]). Observe that, in a linear $2$–normed space $(H, \|\cdot,\cdot\|)$,

$$\|a, b\| = \|a, b + \alpha a\|$$

for all $a, b \in H$ and $\alpha \in \mathbb{C}$.

The properties of linear $2$–normed spaces have been extensively studied by many authors, but many properties of them are almost same to the properties of usual normed spaces. The same properties also occur in $2$–inner product spaces, which were introduced by Diminnie et al. [6].

A pair $(H, (\cdot,\cdot|\cdot)$ is a $2$–inner product space, where $(\cdot,\cdot|\cdot)$ is a real-valued function defined on $H \times H \times H$ satisfying the following conditions:

(2I1) $(a, a|b) = (b, b|a)$ and $(a, b|d) = \overline{(b, a|d)}$;
(2I2) $(a, a|b) \geq 0$ and $(a, a|b) = 0$ if and only if $a$ and $b$ are linearly dependent;
(2I3) $(\alpha a_1 + a_2, b|d) = \alpha(a_1, b|d) + (a_2, b|d)$ for all $a, b, d \in H$ and $\alpha \in \mathbb{C}$.

Many authors have obtained important results on orthogonality in linear $2$–normed spaces and $2$–inner product spaces (see, for example, [1, 14, 15, 16, 17, 24]). But it seems that there exists a fault in the definition of a $2$–inner product space. The most important defect is the following:

*We can not define $2$–Hilbert spaces by the above definition.*

As we know, theory of Hilbert spaces plays a central role in many areas of mathematics, not only in analysis, but also including (differential) geometry, group theory, stochastic and even number theory. The earliest Hilbert spaces were studied from this point of view in the first decade of the 20th century by Hilbert, Schmidt and Riesz. They are indispensable tools in the theories of partial differential equations, quantum mechanics, Fourier analysis (which includes applications to signal processing and heat transfer) and Ergodic Theory which forms the mathematical underpinning of the study of thermodynamics. Today, the most spectacular new applications of theory of Hilbert spaces are given by Noncommutative Geometry [5], where the motivation from pure mathematics is merged with the physical input from quantum mechanics. Consequently, this is an important field of research in pure mathematics as well as in mathematical physics.

In this paper, we introduce the best generalized definition of a $2$–inner product space which includes all the previous definitions and the geometric approach to the concept of a more appropriate. Then we define newly the concept of $2$–Hilbert spaces and give several important properties.

In the near future, we hope that the notion of $2$–Hilbert spaces provides the new mathematical foundation of physics (specially in quantum theory) and engineering, typically as infinite–dimensional function spaces.

## 2. New definition of $2$–inner product spaces

**Definition 2.1.** A complex vector space $H$ is called a $2$–inner product space if there exists a complex-valued function $\langle(\cdot,\cdot),(\cdot,\cdot)\rangle$ on $H^2 \times H^2$ such that, for all $a, b, c, d \in H$ and $\alpha \in \mathbb{C}$,
(I1) $\langle(a,b),(c,d)\rangle = \overline{\langle(c,d),(a,b)\rangle}$;
(I2) If $a$ and $b$ are linearly independent in $H$, then $\langle(a,b),(a,b)\rangle > 0$;
(I3) $\langle(a,b),(c,d)\rangle = -\langle(b,a),(c,d)\rangle$;
(I4) $\langle(\alpha a+\tilde{a},b),(c,d)\rangle = \alpha\langle(a,b),(c,d)\rangle + \langle(\tilde{a},b),(c,d)\rangle$.

Some immediate consequences of these axioms are as follows: for all $a, b, c, d, \tilde{b}, \tilde{c}, \tilde{d} \in H$ and $\alpha, \beta \in \mathbb{C}$,
(P1) $\langle(0,b),(c,d)\rangle = \langle(b,\alpha b),(c,d)\rangle 0$;
(P1) $\langle(a,\alpha b+\tilde{b}),(c,d)\rangle = \alpha\langle(a,b),(c,d)\rangle + \langle(a,\tilde{b}),(c,d)\rangle$;
(P1) $\langle(a,b),(\beta c+\tilde{c},d)\rangle = \overline{\beta}\langle(a,b),(c,d)\rangle + \langle(a,b),(\tilde{c},d)\rangle$;
(P1) $\langle(a,b),(c,\beta d+\tilde{d})\rangle = \overline{\beta}\langle(a,b),(c,d)\rangle + \langle(a,b),(c,\tilde{d})\rangle$;
(P1) $\langle(a,b),(c,d)\rangle = \langle(b,a),(d,c)\rangle$;
(P1) $\langle(a,b),(a,b)\rangle = 0 \iff a, b$ are linearly dependent;
(P1) $\langle(a,b),(a,b)\rangle \geq 0$.

**Theorem 2.2.** (The Schwartz Inequality) Let $H$ be a $2$–inner product space. Then we have the following:

$$|\langle(a,b),(c,b)\rangle|^2 \leq \langle(a,b),(a,b)\rangle\langle(c,b),(c,b)\rangle$$

for all $a, b, c \in H$.

*Proof.* For any complex number $\lambda$, we have

$$\begin{aligned} 0 &\leq \langle(\lambda a+c,b),(\lambda a+c,b)\rangle \\ &= \langle(\lambda a,b),(\lambda a+c,b)\rangle + \langle(c,b),(\lambda a+c,b)\rangle \\ &= \lambda\bar{\lambda}\langle(a,b),(a,b)\rangle + \lambda\langle(a,b),(c,b)\rangle + \bar{\lambda}\langle(c,b),(a,b)\rangle + \langle(c,b),(c,b)\rangle. \end{aligned}$$

Let $a$ and $b$ are linearly independent. Then, by putting

$$\lambda = -\frac{\langle(c,b),(a,b)\rangle}{\langle(a,b),(a,b)\rangle}$$

in the above inequality, we have

$$\frac{|\langle (a,b),(c,b)\rangle|^2}{\langle (a,b),(a,b)\rangle} - \frac{\langle (c,b),(a,b)\rangle\langle (a,b),(c,b)\rangle}{\langle (a,b),(a,b)\rangle} - \frac{\langle (a,b),(c,b)\rangle\ \langle (c,b),(a,b)\rangle}{\langle (a,b),(a,b)\rangle} + \langle (c,b),(c,b)\rangle \geq 0.$$

Therefore, we have

$$|\langle (a,b),(c,b)\rangle|^2 \leq \langle (a,b),(a,b)\rangle\ \langle (c,b),(c,b)\rangle.$$

This completes the proof. ∎

**Theorem 2.3.** Let $H$ be a 2–inner product space. Then the real-valued function $\|\cdot,\cdot\| : H \times H \to \mathbb{R}$ defined by

$$\|a,b\| := \langle (a,b),(a,b)\rangle^{\frac{1}{2}}$$

is a 2–norm on $H$.

*Proof.* Now, we verify that $\|\cdot,\cdot\|$ satisfies the following properties of 2–norms:

(1) (I4) and (P3) show that $\|\alpha a,b\| = \langle (\alpha a,b),(\alpha a,b)\rangle^{\frac{1}{2}} = |\alpha|\|a,b\|$ for all $a,b \in H$ and all $\alpha \in \mathbb{C}$.

(2) It follows from (P5) that $\|b,a\| = \langle (b,a),(b,a)\rangle^{\frac{1}{2}} = \|a,b\|$ for all $a,b \in H$.

(3) (P6) follows that $\|a,b\| = 0$ if and only if $a$ and $b$ are linearly dependent for all $a,b \in H$.

(4) By the Schwartz inequality, we have

$$\begin{aligned}
\|a+\tilde{a},b\|^2 &= \langle (a+\tilde{a},b),(a+\tilde{a},b)\rangle \\
&= \langle (a,b),(a,b)\rangle + \langle (\tilde{a},b),(\tilde{a},b)\rangle + \langle (a,b),(\tilde{a},b)\rangle + \langle (\tilde{a},b),(a,b)\rangle \\
&\leq \langle (a,b),(a,b)\rangle + \langle (\tilde{a},b),(\tilde{a},b)\rangle +\ 2|\langle (a,b),(\tilde{a},b)\rangle| \\
&\leq \|a,b\| + \|\tilde{a},b\| + 2\|a,b\|\|\tilde{a},b\| \\
&\leq (\|a,b\| + \|\tilde{a},b\|)^2
\end{aligned}$$

for all $a,\tilde{a},b \in H$. This shows that $H$ is a linear 2–normed space. This completes the proof. ∎

**Example 2.4.** Let $H$ be a complex vector space with inner product $\langle\cdot,\cdot\rangle$. We define

$$\langle (a,b),(c,d)\rangle := \begin{vmatrix} \langle a,c\rangle & \langle a,d\rangle \\ \langle b,c\rangle & \langle b,d\rangle \end{vmatrix}$$

for all $a,b,c,d \in H$. Then we have the following:

(1) $\langle (a,b),(c,d)\rangle = \begin{vmatrix} \langle a,c\rangle & \langle a,d\rangle \\ \langle b,c\rangle & \langle b,d\rangle \end{vmatrix} = \begin{vmatrix} \overline{\langle c,a\rangle} & \overline{\langle d,a\rangle} \\ \overline{\langle c,b\rangle} & \overline{\langle d,b\rangle} \end{vmatrix} = \overline{\langle (c,d),(a,b))\rangle}.$

(2) If $a$ and $b$ be linearly independent, then $\|a\|\|b\| > |\langle a,b\rangle|$ and so we have

$$\langle (a,b),(a,b)\rangle = \begin{vmatrix} \langle a,a\rangle & \langle a,b\rangle \\ \langle b,a\rangle & \langle b,b\rangle \end{vmatrix} > 0.$$

(3) $-\langle (b,a),(c,d)\rangle = -\begin{vmatrix} \langle b,c\rangle & \langle b,d\rangle \\ \langle a,c\rangle & \langle a,d\rangle \end{vmatrix} = \langle (a,b),(c,d)\rangle.$

(4) Similarly, we have

$$\langle (\alpha a,b),(c,d)\rangle = \alpha\langle (a,b),(c,d)\rangle$$

and

$$\langle (a+\tilde{a},b),(c,d)\rangle = \langle (a,b),(c,d)\rangle + \langle (\tilde{a},b),(c,d)\rangle$$

for all $a,b,c,d,\tilde{a}, \in H$ and $\alpha \in \mathbb{C}$. Therefore, it follows that $H$ is a 2–inner product space.

Consider a space **F** of well–defined complex–valued functions over an interval $[a, b]$. Then a 2–inner product is given by

$$\langle (f,g),(h,k)\rangle := \frac{1}{(b-a)^2}\begin{vmatrix} \int_a^b f\bar{h}\,dt & \int_a^b f\bar{k}\,dt \\ \int_a^b g\bar{h}\,dt & \int_a^b g\bar{k}\,dt \end{vmatrix}$$

for all $f, g, h, k \in \mathbf{F}$.

## 3. **Main results**

We know that every 2–inner product is a linear 2–normed space with respect to the 2–norm defined in Theorem 2.3. Is the converse true? That is, when a 2–norm is derived from a 2–inner product in this way? The answer is given by the following theorems:

**Lemma 3.1.** (Polarization identity) Let $H$ be a 2–inner product space. Then we have

$$\begin{aligned} &\langle (a,b),(c,d)\rangle \\ &= \frac{1}{8}[\|a+c,b+d\|^2 + \|a-c,b-d\|^2 - \|a+ic,b+id\|^2 - \|a-ic,b-id\|^2] \\ &\quad + \frac{1}{8}\, i[\|a+ic,b+d\|^2 + \|a-ic,b+d\|^2 - \|a+c,b-id\|^2 - \|a-c,b+id\|^2] \end{aligned}$$

for all $a, c, b, d \in H$.

*Proof.* We have

$$\begin{aligned} &\|a+c,b+d\|^2 + \|a-c,b-d\|^2 \\ &= 2[\|a,b\|^2 + \|c,b\|^2 + \|a,d\|^2 + \|c,d\|^2] + \langle (a,b),(c,d)\rangle + \langle (c,d),(a,b)\rangle \\ &\quad + \langle (c,b),(a,d)\rangle + \langle (a,d),(c,b)\rangle, \end{aligned}$$

$$\begin{aligned} &\|a+ic,b+id\|^2 + \|a-ic,b-id\|^2 \\ &= 2[\|a,b\|^2 + \|c,b\|^2 + \|a,d\|^2 + \|c,d\|^2] - \langle (a,b),(c,d)\rangle - \langle (c,d),(a,b)\rangle \\ &\quad + \langle (c,b),(a,d)\rangle + \langle (a,d),(c,b)\rangle \end{aligned}$$

for all $a, c, b, d \in H$ and so

$$Re(\langle (a,b),(c,d)\rangle) = \frac{1}{8}[\|a+c,b+d\|^2 + \|a-c,b-d\|^2 - \|a+ic,b+id\|^2 - \|a-ic,b-id\|^2]$$

for all $a, c, b, d \in H$.

On the other hand, we have

$$\langle (a,b),(c,d)\rangle = Re(\langle (a,b),(c,d)\rangle) + iRe(-i\langle (a,b),(c,d)\rangle)$$

for all $a, c, b, d \in H$. This completes the proof. ∎

**Theorem 3.2.** (Parallelogram laws) A linear 2–norm space $H$ is a 2–inner product space with the 2—norm derived from the 2–inner product defined in Theorem 2.3 if and only if the following two equations hold:

$$\begin{aligned} &\|a+c,b+d\|^2 + \|a-c,b-d\|^2 + \|c+b,d+a\|^2 + \|c-b,d-a\|^2 \\ &= \|a+ic,b+id\|^2 + \|a-ic,b-id\|^2 + \|c+ib,d+ia\|^2 + \|c-ib,d-ia\|^2, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\|a+c, b+id\|^2 + \|a+c, b-id\|^2 + \|a-c, b+id\|^2 + \|a-c, b-id\|^2 \\ &= \|b+d, a+ic\|^2 + \|b+d, a-ic\|^2 + \|b-d, a+ic\|^2 + \|b-d, a-ic\|^2 \end{aligned} \tag{3.2}$$

for all $a, c, b, d \in H$.

*Proof.* Let $H$ be a 2–inner product space. Then, by the above lemma, one can prove that the equations (3.1) and (3.2) hold.

Conversely, let $H$ be a linear 2–normed space satisfying (3.1) and (3.2). Then we can show that

$$\begin{aligned} &\langle (a,b),(c,d)\rangle \\ &= \frac{1}{8}[\|a+c, b+d\|^2 + \|a-c, b-d\|^2 - \|a+ic, b+id\|^2 - \|a-ic, b-id\|^2] \\ &\quad + \frac{1}{8}\, i[\|a+ic, b+d\|^2 + \|a-ic, b+d\|^2 - \|a+c, b-id\|^2 - \|a-c, b+id\|^2] \end{aligned}$$

for all $a, c, b, d \in H$ defines a 2–inner product on $H$. To this end, we have the following:

(A) It follows from (3.1) that

$$\begin{aligned} &\overline{\langle (c,d),(a,b))\rangle} \\ &= \frac{1}{8}[\|a+c, b+d\|^2 + \|a-c, b-d\|^2 - \|a-ic, b-id\|^2 - \|a+ic, b+id\|^2] \\ &\quad - \frac{1}{8}\, i[\|c+ia, b+d\|^2 + \|a+ic, b+d\|^2 - \|c-a, d+ib\|^2 - \|a+c, d-ib\|^2] \\ &= \langle (a,b),(c,d)\rangle \end{aligned}$$

for all $a, c, b, d \in H$.

(B) If $a$ and $b$ are linearly independent, then $\|a, b\| > 0$ and so

$$\langle (a,b),(a,b)\rangle = \|a,b\|^2 > 0.$$

(C) (3.2) shows that $\langle (a,b),(c,d)\rangle = -\langle (b,a),(c,d)\rangle$ for all $a, c, b, d \in H$.

(D) It is easy to show that

$$\langle (\alpha a + \tilde{a}, b),(c,d)\rangle = \alpha\langle (a,b),(c,d)\rangle + \langle (\tilde{a},b),(c,d)\rangle$$

for all $a, c, b, d \in H$. This completes the proof. ∎

**Theorem 3.3.** Let $H$ be a 2–inner product space. Then, for any fixed $x, c \in H$, the mappings

$$a \mapsto \langle (a,x),(c,x)\rangle, \quad a \mapsto \langle (c,x),(a,x)\rangle, \quad a \mapsto \|a,x\|$$

are continuous on $H$.

*Proof.* The Schwartz inequality implies that

$$|\langle (a_1,x),(c,x)\rangle - \langle (a_2,x),(c,x)\rangle| = |\langle (a_1-a_2,x),(c,x)\rangle| \le \|a_1-a_2,x\|\|c,x\|,$$

which shows that $a \mapsto \langle (a,x),(c,x)\rangle$ is uniformly continuous and so continuous.

In the same way, the mapping $a \mapsto \langle (c,x),(a,x)\rangle$ is continuous.

Now, by the triangle inequality $\|a_1,x\| \le \|a_1-a_2,x\| + \|a_2,x\|$, we have

$$\|a_1,x\| - \|a_2,x\| \le \|a_1-a_2,x\|.$$

If we interchange $a_1$ and $a_2$ in the above inequality, then we have

$$|\|a_1,x\| - \|a_2,x\|| \le \|a_1-a_2,x\|.$$

Thus $a \mapsto \|a,x\|$ is also uniformly continuous and so continuous. This completes the proof. ∎

There are well–known different definitions of an orthogonality, namely, pythagorean orthogonality, isosceles orthogonality and Brushoff–James orthogonality in linear 2–normed spaces and 2–inner product spaces [1, 2, 17] and others. Recently, Mazaheri et al. [16, 23, 24] introduced $b$–orthogonality by a different method and Gozali and Gunawan [14] observed that this definition is too loose and proved that every two linearly independent vectors are $b$–orthogonal.

Let $(H, \|\cdot,\cdot\|)$ be a linear 2–normed space and $a, c \in H$. Then we have the following definitions of an orthogonality:

**Pythagorean orthogonality:** $a \perp^P c \iff \|a,k\|^2 + \|c,k\|^2 = \|a+c,k\|^2$ for all $k \in H$.
**Isosceles orthogonality:** $a \perp^I c \iff \|a-c,k\| = \|a+c,k\|$ for all $k \in H$.
**Brushoff–James orthogonality:** $a \perp^{BJ} c \iff \|a,k\| \leq \|a+\alpha c,k\|$ for all $\alpha \in R$ and $k \in H$.
$b$**–Orthogonality:** $a \perp^b c \iff$ there exists $b \in H$ with $\|a,b\| \neq 0$ such that $\|a,b\| \leq \|a+\alpha c,b\|$ for all $\alpha \in R$.

Now, we define newly the orthogonality in 2–inner product spaces as follows:

**Definition 3.4.** Let $(H, \langle(\cdot,\cdot),(\cdot,\cdot)\rangle)$ be a 2–inner product space, $0 \neq x \in H$ and $a, c \in H - \langle x\rangle$ (where $\langle x\rangle$ is the subspace of $H$ generated by $x$).
(1) We say that $a$ is $x$–orthogonal to $c$ if

$$\langle(a,x),(c,x)\rangle = 0,$$

which is denoted by $a \perp^x c$.

Since $\langle(a,x),(c,x)\rangle = 0$ implies $\langle(c,x),(a,x)\rangle = 0$, the relation $\perp^x$ is symmetric.

(2) We say that $a$ is orthogonal to $c$ (we write $a \perp c$) if, for all $x \in H$, $a$ is $x$–orthogonal to $c$.

Let $a \in H$ and $S$ be a subset of $H$. The $x$–orthogonal complements $a_x^\perp$, $a^\perp$ and $S^\perp$ are defined by

$$a_x^\perp = \{a' \in H : a \perp^x a'\}, \quad a^\perp = \bigcap_{x\in H} a_x^\perp, \quad S^\perp = \bigcap_{a\in S} a^\perp.$$

Since $a \perp^x c$ and $a \perp^x \tilde{c}$ imply $a \perp^x \alpha c + \tilde{c}$ and $a_x^\perp$ is precisely the set of points where the function $c \mapsto \langle(a,x),(c,x)\rangle$ is continuous (Theorem 3.3),it follows that $a_x^\perp$ is a closed subspace of $H$. Therefore, $a^\perp$ and $S^\perp$ are closed subspaces of $H$. It is easy to show that, for all $a, c \in H$ and $S, S_1, S_2 \subset H$,
(A) $a \perp^a c$, $0 \perp^a c$ and $a \perp^0 c$;
(B) $a \perp^c a \iff a$ and $c$ are linearly dependent;
(C) $S_1 \supseteq S_2 \implies S_1^\perp \subseteq S_2^\perp$;
(D) $S \subseteq S^{\perp\perp}$;
(E) $S \cap S^\perp = \{0\}$.

**Theorem 3.5.** Let $(H, \langle(\cdot,\cdot),(\cdot,\cdot)\rangle)$ be a 2–inner product space. Then we have

$$a \perp c \iff a \perp^P c$$

for all $a, c \in H$.

*Proof.* We can easily establish the Pythagorean law for 2–inner product space and

$$\langle(a,x),(c,x)\rangle = 0 \iff \|a+c,x\|^2 = \|a,x\|^2 + \|c,x\|^2$$

for all $a, x, c \in H$. Now, for all $x \in H$, we have

$$\begin{aligned} a \perp^P c &\iff \|a,x\|^2 + \|c,x\|^2 = \|a+c,x\|^2 \\ &\iff \langle (a,x),(c,x)\rangle = 0 \\ &\iff a \perp^x c \\ &\iff a \perp c. \end{aligned}$$

This completes the proof. ∎

**Definition 3.6.** Let $(H, \langle (\cdot,\cdot),(\cdot,\cdot)\rangle)$ be a 2–inner product space and $0 \neq x \in H$.

(1) A sequence $\{x_n\}$ in $H$ is a $x$–Cauchy sequence if, for any $\epsilon > 0$, there exists a positive integer $N$ such that

$$0 < \|x_m - x_n, x\| < \epsilon$$

for all $m, n \geq N$.

(2) $H$ is $x$–Hilbert space if every $x$–Cauchy sequence converges a point in a semi–2–normed space $(H, \|\cdot, x\|)$. Also, $H$ is called a 2–Hilbert space if $H$ is a $x$–Hilbert space for any $x \in H$.

(3) A subset $A$ of 2–Hilbert space $H$ is said to be closed if $A$ is closed in a semi–2–normed space $(H, \|\cdot, x\|)$ for all $x \in H$.

**Lemma 3.7.** If $N$ is a closed subspace of a $x$–Hilbert space $H$ and $h \in H$, then

$$d_x(h,N) = \|h - n_o, x\| \iff \langle (h-n_o, x),(n,x)\rangle = 0$$

for all $n \in N$.

*Proof.* Suppose that $d_x(h,N) = \|h - n_o, x\|$ and write $z = h - n_o$. Then, for all non–zero $n \in N$, we have

$$\begin{aligned} \|z,x\|^2 &\leq \|z - \frac{\langle (z,x),(n,x)\rangle n}{\|n\|^2}, x\|^2 \\ &= \|z,x\|^2 - \frac{2|\langle (z,x),(n,x)\rangle|^2}{\|n\|^2} + \frac{|\langle (z,x),(n,x)\rangle|^2}{\|n\|^2} \\ &= \|z,x\|^2 - \frac{|\langle (z,x),(n,x)\rangle|^2}{\|n\|^2}. \end{aligned}$$

Thus it follows that $\langle (z,x),(n,x)\rangle = 0$.

Conversely, by the Pythagorean law, we have

$$\|h-n,x\|^2 = \|h-n_o+n_o-n,x\|^2 = \|h-n_o,x\|^2 + \|n_o-n,x\|^2 \geq \|h-n_o,x\|^2$$

for all $n \in N$. Hence $\inf_{n\in N} \|h-n,x\|$ is attained at $n_o$. ∎

**Corollary 3.8.** If $N$ is a closed subspace of a 2–Hilbert space $H$ and $h \in H$, then we have

$$d_x(h,N) = \|h-n_o,x\|,\ \forall x \in H,\ \iff\ h - n_o \perp n,\ \forall n \in N.$$

**Theorem 3.9.** If $N$ is a closed subspace of a 2–Hilbert space $H$, then we have

$$N^\perp = \{0\} \iff N = H.$$

*Proof.* Clearly, if N = H, then $N^\perp = \{0\}$.

Conversely, if $N \neq H$ and $h \notin N$, then there exists $n_o \in N$ such that $d_x(h,N) = \|h-n_o,x\|$ for all $x \in H$. It follows from Corollary 3.8 that $0 \neq h - n_o \perp N$ and so $N^\perp \neq \{0\}$. This completes the proof. ∎

**Theorem 3.10.** If $N$ is a closed subspace of a 2–Hilbert space $H$, then we have
$$N \oplus N^{\perp} = H.$$

*Proof.* First, we show that $N \oplus N^{\perp}$ is a closed subspace of $H$. Let $c_i \in N \oplus N^{\perp}$ such that $c_i \to c$. Then $c \in N \oplus N^{\perp}$. In fact, suppose that $c_i = a_i + b_i$ with $a_i \in N$ and $b_i \in N^{\perp}$ for all $i \in \mathbb{N}$. Therefore, using the Pythagoras law, since $N \perp N^{\perp}$, we have
$$\|c_{i+p} - c_i, x\|^2 = \|a_{i+p} - a_i, x\|^2 + \|b_{i+p} - b_i, x\|^2$$
for all $x \in H$. Since $\{c_n\}$ is convergent, it is a $x$-cauchy sequence. It follows easily from the above that both $\{a_i\}$ and $\{b_i\}$ are $x$–Cauchy sequences and so, since $H$ is a 2–Hilbert space, there exist $a, b \in H$ such that $a_i \to a$ and $b_i \to b$.

On the other hand, $N$ and $N^{\perp}$ are closed and so $a \in N$ and $b \in N^{\perp}$. Hence
$$c = \lim_{i\to\infty}(a_i + b_i) = a + b \in N \oplus N^{\perp}.$$

Now, if $n' \in [N \oplus N^{\perp}]^{\perp}$, then $n' \in N^{\perp} \cap N^{\perp\perp}$. Therefore, $n' \perp n'$, that is, $n' \perp^x n'$ for all $x \in H$. Thus $n'$ and $x$ are linearly dependent for all $x \in H$. Also, $n'$ and $0$ are linearly dependent. Thus it follows that $n' = 0$ and $N^{\perp} \cap N^{\perp\perp} = \{0\}$. Therefore, it follows form Theorem 3.9 that $N \oplus N^{\perp} = H$. This completes the proof. ∎

**Definition 3.11.** Let $H$ be a $x$- Hilbert space and $\langle x\rangle$ be a vector space generated by $x \in H$. A function $F : H \times \langle x\rangle \to \mathbb{C}$ is called a $x$–linear function if it satisfy the following conditions: for all $a, \tilde{a} \in H$ and $x \in \langle x\rangle$,

(1) $F(a + \tilde{a}, x) = F(a, x) + F(\tilde{a}, x)$;

(2) $F(\alpha a, x) = \alpha F(a, x)$.

Moreover, a $x$–linear function $F$ is said to be bounded if there exists a positive number $\|F\|_x$ such that
$$\|F\|_x = \inf\{K > 0 : |F(a, x)| \leq K\|a, x\|\}.$$

**Example 3.12.** Let $H$ is a $x$–Hilbert space and $y \in H$. A function $L_{(x,y)} : H \times \langle x\rangle \to \mathbb{C}$ defined by the formula
$$L_{(x,y)}(a, x) := \langle (a, x), (y, x)\rangle$$
is $x$–linear and bounded. Moreover, $\|L_{(x,y)}\| = \|x, y\|$). In fact, for all $a, \tilde{a} \in H$ and $\alpha \in \mathbb{C}$, we have

(1) $L_{(x,y)}(a + \tilde{a}, x) = \langle (a + \tilde{a}, x), (y, x)\rangle = L_{(x,y)}(a, x) + L_{(x,y)}(\tilde{a}, x)$;

(2) $L_{(x,y)}(\alpha\, a, x) = \langle (\alpha a, x), (y, x)\rangle = \alpha L_{(x,y)}(a, x)$;

(3) $|L_{(x,y)}(a, x)| = |\langle (a, x), (y, x)\rangle| \leq \|y, x\|\|a, x\|$ (by the Schwartz inequality).

**Theorem 3.13.** Let $H$ be a $x$–Hilbert space. If $F$ is continuous and $x$–linear function on $H \times \langle x\rangle$, then there exists $y \in H$ such that
$$F(a, x) = L_{(x,y)}(a, x)$$
for all $a \in H$.

*Proof.* If $F(a, x) = 0$ for all $a \in H$, then take $y = 0$. Otherwise, define
$$N := \{a : F(a, x) = 0\}.$$
Since $F$ is a continuous and $x$–linear function, it follows that $N$ is a closed subspace of $H$. Since $F(a, x) \neq 0$ for some $a \in H$, it follows that from Theorem 3.10 that there exists $z \in N^{\perp}$ such that $\|z, x\| = 1$. Put
$$u := F(a, x)z - F(z, x)a.$$

It is easy to see that $u \in N$ and $\langle (u,x),(z,x)\rangle = 0$, that is,

$$\langle (F(a,x)z - F(z,x)a, x),(z,x)\rangle = F(a,x)\|z,x\|^2 - F(z,x)\langle (a,x),(z,x)\rangle = 0.$$

Thus it follows that

$$F(a,x) = F(a,x)\|z,x\|^2 = \langle (a,x),(\overline{F(z,x)}z,x)\rangle.$$

Take $y = \overline{F(z,x)}z$. Then we have

$$F(a,x) = L_{(x,y)}(a,x)$$

for all $a \in H$. This completes the proof. ∎

## 4. **Problems**

We know that the Riesz representation theorem in Hilbert spaces is one of the most important tools in applications of Hilbert spaces in pure and applied mathematics and natural sciences. In this paper, by using our results, we can not prove a version of the Riesz representation theorem in 2–Hilbert spaces. We need to define the dual space of a $2$–Hilbert space and a $x$–Hilbert space. In the other word, we have to define the bounded linear operators between $2$–Hilbert spaces or linear $2$–normed spaces. In fact, there are some works to define the bounded linear mappings between linear 2–normed spaces (see, for example, [18, 20]), but we need some more information in this problem to prove our ideas.

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