The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 10, Issue 1, Article 7, pp. 1-8, 2013

C^*-ALGEBRAS ASSOCIATED NONCOMMUTATIVE CIRCLE AND THEIR K-THEORY

SALEH OMRAN

Received 24 September, 2012; accepted 11 February, 2013; published 2 May, 2013.

TAIF UNIVERSITY, FACULTY OF SCIENCE, TAIF, KSA

SOUTH VALLEY UNIVERSITY, FACULTY OF SCIENCE, MATH. DEP. QENA, EGYPT
salehomran@yahoo.com

ABSTRACT. In this article we investigate the universal C^*-algebras associated to certain 1- dimensional simplicial flag complexes which describe the noncommutative circle. We denote it by S_1^{nc}. We examine the K-theory of this algebra and the subalgebras $S_1^{nc}/I_k, I_k$. Where I_k, for each k, is the ideal in S_1^{nc} generated by all products of generators h_s containing at least $k+1$ pairwise different generators. Moreover we prove that such algebra divided by the ideal I_2 is commutative.

Key words and phrases: Operator Algebras, C^*-algebra, K-theory for C^*-algebra.

2000 *Mathematics Subject Classification.* 19 K 46.

ISSN (electronic): 1449-5910

1. INTRODUCTION

J. Cuntz in [2] associate to every simplicial complex a universal C^*-algebra with generators and relations. In the following we give some examples and properties of such algebras see also [6].

Definition 1.1. A simplicial complex Σ consists of a set of vertices V_Σ and a set of non-empty subsets of V_Σ, the simplexes in Σ, such that:

- If $s \in V_\Sigma$, then $\{s\} \in \Sigma$.
- If $F \in \Sigma$ and $\emptyset \neq E \subset F$ then $E \in \Sigma$.

Σ is called locally finite if every vertex of Σ is contained in only finitely many simplexes of Σ, and finite-dimensional (of dimension $\leqslant n$) if it contains no simplexes with more than $n+1$-vertices.

For a simplicial complex Σ one can define the topological space $|\Sigma|$ associated to this complex. It is called the "geometric realization" of the complex and can be defined as the space of maps $f : V_\Sigma \longrightarrow [0,1]$ such that $\sum_{s\in V_\Sigma} f(s) = 1$ and $f(s_0).....f(s_i) = 0$ whenever $\{s_0, ..., s_i\} \notin \Sigma$. If Σ is locally finite, then $|\Sigma|$ is locally compact.

- $\mathcal{C}_\Sigma$ is the universal C^*-algebra with positive generators $h_s, s \in V_\Sigma$, satisfying the relations
$$h_{s_0}h_{s_1}...h_{s_n} = 0 \text{ whenever } \{s_0, s_1, ..., s_n\} \notin \Sigma,$$
$$\sum_{s\in V_\Sigma} h_s h_t = h_t \quad \forall\, t \in V_\Sigma.$$
Here the sum is finite, because Σ is locally finite.
- $\mathcal{C}^{ab}_\Sigma$ is the abelian version of the universal C^*-algebra above, i.e. satisfying in addition $h_s h_t = h_t h_s$ forall $s, t \in V_\Sigma$.

Remark 1.1. There exists a canonical surjective map $\mathcal{C}_\Sigma \longrightarrow \mathcal{C}^{ab}_\Sigma$.

A simplicial map between two simplicial complexes Σ and Σ' is a map $\varphi : V_\Sigma \longrightarrow V_{\Sigma'}$ such that, whenever $(t_0, ..., t_n)$ is a simplex in Σ this implies that $(\varphi(t_0), ..., \varphi(t_n))$ is a simplex in Σ'.

Proposition 1.1. *Every simplicial map $\varphi : \Sigma \longrightarrow \Sigma'$ between two simplicial complexes Σ and Σ' induces a $*$- homomorphism $\varphi^* : \mathcal{C}_{\Sigma'} \longrightarrow \mathcal{C}_\Sigma$.*

Proof. Define $\varphi^* : \mathcal{C}_{\Sigma'} \longrightarrow \mathcal{C}_\Sigma$ by $h_s \longmapsto g_s := \sum_{\varphi(t)=s} h_t$ and h_s mapped to 0 if s is not in the image of φ. We verify that the sum of all g_s over s is equal to if one the sum of all h_t over t is equal to one and the products $g_{s_0}.....g_{s_n} = 0$ whenever $h_{s_0}.....h_{s_n} = 0$.

For the first condition, we have
$$\sum_s g_s = \sum_s \Big(\sum_{\varphi(t)=s} h_t\Big) = \sum_t h_t = 1,$$
and for the second condition
$$\begin{aligned} g_{s_0}.....g_{s_n} &= \sum_{\varphi(t_0)=s_0} h_{t_0}.....\sum_{\varphi(t_n)=s_n} h_{t_n} \\ &= \sum_{\varphi(t_0)=s_0}.....\sum_{\varphi(t_n)=s_n} h_{t_n} h_{t_0}......h_{t_n} = 0 \end{aligned}$$
because φ is a simplicial map. ∎

It has been shown in [2] that the K-theory of $\mathcal{C}_\Sigma$ coincides with the K-theory of $\mathcal{C}^{ab}_\Sigma$ (which in turn is isomorphic to $C_0(|\Sigma|)$. In the sequel we will study the K-theory of another C^*-algebra that can be associated with certain complexes.

Definition 1.2. A simplicial complex Σ is called flag or full, if it is determined by its 1-simplexes in the sense that
$\{s_0, ..., s_n\} \in \Sigma \Longleftrightarrow \{s_i, s_j\} \in \Sigma$ for all $0 \leq i < j \leq n$.

Definition 1.3. Let Σ be a locally finite flag complex. Denote by V the set of its vertices. Define $\mathcal{C}_\Sigma^{flag}$ as the universal C^*-algebra with positive generators $h_s, s \in V$, satisfying the relations

$$\sum_{s\in V} h_s h_t = h_t, \quad t \in V$$

and

$$h_s h_t = 0 \quad \text{for} \quad \{s, t\} \notin \Sigma.$$

Denote by I_k the ideal in $\mathcal{C}_\Sigma^{flag}$ generated by products containing at least $n + 1$ different generators. The filtration (I_k) of $\mathcal{C}_\Sigma^{flag}$ is called the skeleton filtration.

For simplicity we denote $\mathcal{C}_\Sigma^{flag}$ by $\mathcal{C}_\Sigma^{f}$. This algebra is an interesting example of a noncommutative C^*- algebra described by a simplicial complex. If we consider the flag complex Σ_{S^1} with vertices $\{0^-, 0^+, 1^-, 1^+\}$ and the condition that exactly the edges $\{i^-, i^+\}$ do not belong to Σ_{S^1}, the geometric realization of Σ_{S^1} is the noncommutative circle S^1. We consider the universal C^*-algebra with 4 positive generators $h_i, i \in V_{\Sigma_S} := \{0^-, 0^+, 1^-, 1^+\}$ and satisfying the relations

$$\sum_i h_{i^+} + \sum_i h_{i^-} = 1, h_{i^+} h_{i^-} = 0 \quad \forall i \in \{0, 1\}.$$

The algebra described above is exactly the algebra $\mathcal{C}^f_{\Sigma_{S^1}}$. We will denote it by S_1^{nc}. The abelianization of this C^*-algebra is isomorphic to the algebra of continuous functions on the circle S^1 as shown in [2]. The K-theory of S_1^{nc} is described by the following theorem.

Theorem 1.2. [2] *The evaluation map $ev : S_1^{nc} \longrightarrow \mathbb{C}$ at the vertex 1^+, which maps the generator h_{1^+} to 1 and all the other generators to 0, induces an isomorphism in K-theory. (The same is true for the evaluation maps, corresponding to the other vertices.)*

Let

$$\Delta := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum_{i=1}^{n} t_i = 1\}$$

be the standard n-simplex. Denote by $\mathcal{C}_\Delta$ the associated universal C^*-algebra with generators h_s, $s \in \{t_0, ..., t_n\}$, such that $h_s \geq 0$ and $\sum_s h_s = 1$. Denote by $\mathcal{I}_\Delta$ the ideal in $\mathcal{C}_\Delta$ generated by products of generators containing all the h_{t_i}, $i = 0, ..., n$. For each k, denote by I_k the ideal in $\mathcal{C}_\Delta$ generated by all products of generators h_s containing at least $k + 1$ pairwise different generators. We also denote by I_k^{ab} the image of I_k in $\mathcal{C}_\Delta^{ab}$. We have the following lemma.

Lemma 1.3. [2] *Let Σ be a locally simplicial complex and I_n be an ideal in $\mathcal{C}_\Sigma$ defined above. Then isomorphism*

$$I_k/I_{k+1} \cong \bigoplus_{\triangle} \mathcal{I}_\triangle,$$

where the sum is taken over all n-simplexes $\triangle$ in Σ.

For any vertex t in Δ there is a natural evaluation map $\mathcal{C}_\triangle \longrightarrow \mathbb{C}$ mapping the generators h_t to 1 and all the other generators to 0.

Proposition 1.4. (i) *The evaluation map* $\mathcal{C}_\triangle \longrightarrow \mathbb{C}$ *defined above induces an isomorphism in* K*-theory.*
(ii) *The surjective map* $\mathcal{I}_\triangle \longrightarrow \mathcal{I}^{ab}_\triangle$ *induces an isomorphism in* K*-theory, where* $\mathcal{I}^{ab}_\triangle$ *is the abelianization of* $\mathcal{I}_\triangle$ *.*

Proof. For (i) it is enough to prove that $\mathcal{C}_\triangle$ is homotopy equivalent to $\mathbb{C}$. Consider the $*$-homomorphisms $\alpha : \mathbb{C} \longrightarrow \mathcal{C}_\triangle,\ \lambda \mapsto c_\lambda := \lambda.1$ and $\beta : \mathcal{C}_\triangle \longrightarrow \mathbb{C},\ h_i \mapsto \frac{1}{n+1},\ i \in \{0, 1, ..., n\}$. It is clear that $\beta \circ \alpha = id_\mathbb{C}$. Define $\varphi_t : \mathcal{C}_\triangle \longrightarrow \mathcal{C}_\triangle$ by $\varphi_t(h_i) = \frac{1-t}{n+1} + th_i,\ t \in [0, 1]$. It is obvious that $\varphi_1 = id_{\mathcal{C}_\triangle}$ and $\varphi_0 = \alpha \circ \beta$. So $\alpha \circ \beta \sim id_{\mathcal{C}_\triangle}$. This implies that $\mathcal{C}_\triangle$ is homotopy equivalent to $\mathbb{C}$.
Using Lemma 1.3 above, one can use induction on the dimension n of $\triangle$ to prove the claim (ii). For the complete proof we refer to [2]. ∎

Remark 1.2. Let Δ and $\mathcal{I}_\triangle \subset \mathcal{C}_\triangle$ as above. Then $K_*(\mathcal{I}_\triangle) \cong K_*(\mathbb{C}),\ * = 0, 1$, if the dimension n of $\triangle$ is even and $K_*(\mathcal{I}_\triangle) \cong K_*(C_0(0, 1)),\ * = 0, 1$, if the dimension n of $\triangle$ is odd.

2. K-THEORY OF NONCOMMUTATIVE CIRCLE

K-theory of noncommutative circle was introduced first by [5]. In this article we introduce this algebras as a 1-dimensional simplicial complexes and and it's skeleton filtration. Basic definitions and facts of C^*-algebras, universal C^*-algebras and their K-theory which we will use in this article can be found in [1],[3] [4], [7]and [8]

Lemma 2.1. $S^{nc}_1/I_1 \cong \mathbb{C}^4$.

Proof. Let $\dot{h_i}$ denote the image of a generator h_i for S^{nc}_1/I_1. One has the following relations :

$$\sum_i \dot{h_i} = 1, \quad \dot{h_i}\dot{h_j} = 0, \quad i \neq j.$$

For every $\dot{h_i}$ in S^{nc}_1/I_1 we have

$$\dot{h_i} = \dot{h_i}(\sum_i \dot{h_i}) = \dot{h}_i^2.$$

Hence S^{nc}_1/I_1 is generated by 4 different orthogonal projections and therefore $S^{nc}_1/I_1 \cong \mathbb{C}^4$. ∎

Lemma 2.2. *In* S^{nc}_1*, we have an isomorphism*

$$I_1/I_2 \cong I^{ab}_1/I^{ab}_2.$$

specially in S^{nc}_1/I_1 *we have* $I_1/I_2 \cong C_0(0, 1)^4$.

Proof. In S^{ab}_1

$$I_1{}^{ab}/I_2{}^{ab} \cong \bigoplus_\sigma \mathcal{I}^{ab}_\sigma.$$

And in S^{nc}_1

$$I_1/I_2 \cong \bigoplus_\sigma \mathcal{I}_\sigma$$

where the direct sum is taken over all the 1-simplexes σ in Σ_{S^1}. $\mathcal{I}_\sigma$ is the ideal generated by products of generators containing h_1 and h_2 in the universal C^*-algebra $\mathcal{C}^f_\sigma$ which is generated by positive elements h_1, h_2, such that $h_1 + h_2 = 1$. This C^*-algebra is commutative. Therefore

$$\mathcal{I}_\sigma \cong C_0(0, 1)$$

and the map $\mathcal{I}_\sigma \longrightarrow \mathcal{I}^{ab}_\sigma$ is an isomorphism. In the algebra S^{nc}_1 , there are four 1-simplexes . So we have $I_1/I_2 \cong C_0(0, 1)^4$. ∎

Lemma 2.3. $\mathcal{C}_{\Sigma_{S^1}}$ *is commutative.*

Proof. An easy computation shows that $\mathcal{C}_{\Sigma_{S^1}}/I_2$ is commutative. Since in the algebra $\mathcal{C}_{\Sigma_{S^1}}$ the the product of any three different generators is zero, so the ideal $I_2 = 0$. Then $\mathcal{C}_{\Sigma_{S^1}}$ is commutative. ∎

Lemma 2.4. *S_1^{nc}/I_2 is isomorphic to $\mathcal{C}_{\Sigma_{S^1}}$.*

Proof. Consider

$$\phi : \mathcal{C}_{\Sigma_{S^1}} \longrightarrow S_1^{nc}/I_2, h_i \longmapsto \dot{h}_i,\ i \in \{0^-, 0^+, 1^-, 1^+\}.$$

The elements h_i in $\mathcal{C}_{\Sigma_{S^1}}$ satisfy the relations of the $\dot{h}_i$ in S_1^{nc}/I_2, so ϕ is a well defined homomorphism. It is evident that ϕ is surjective. It remains to prove that ϕ is injective.
Let

$$\rho : \mathcal{C}_{\Sigma_{S^1}} \longrightarrow \mathcal{B}(\mathcal{H}),\ h_i \longmapsto g_i$$

be a unital representation. So in $\mathcal{B}(\mathcal{H})$, we have

$$\sum_i g_i = \sum_i \rho(h_i) = \rho(\sum_i h_i) = \rho(1) = 1$$

and all g_i commute since $\mathcal{C}_{\Sigma_{S^1}}$ is commutative. Now, define

$$\pi : S_1^{nc} \longrightarrow \mathcal{B}(\mathcal{H}),\ \pi(\dot{h}_i) = g_i.$$

Then π annihilates I_2 and therefore factors as

$$S_1^{nc} \longrightarrow S_1^{nc}/I_2 \xrightarrow{\pi'} \mathcal{B}(\mathcal{H})$$

where π' is a well defined homomorphism such that $\rho = \pi' \circ \phi$.
∎

Proposition 2.5. $S_1^{nc}/I_2 \cong S_1^{ab}$.

Proof. By 2.4, S_1^{nc}/I_2 is isomorphic to the commutative algebra $\mathcal{C}_{\Sigma_{S^1}}$. Thus S_1^{nc}/I_2 is an abelian C^*-algebra. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_1/I_2 & \longrightarrow & S_1^{nc}/I_2 & \longrightarrow & S_1^{nc}/I_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_1^{ab}/I_2^{ab} & \longrightarrow & S_1^{ab}/I_2^{ab} & \longrightarrow & S_1^{ab}/I_1^{ab} & \longrightarrow & 0. \end{array}$$

Since

$$S_1^{nc}/I_1 \cong S_1^{ab}/I_1 \cong \mathbb{C}^4$$

from Lemma 2.1. And

$$I_1/I_2 \cong I_1^{ab}/I_2^{ab}$$

from lemma 2.2. By five-lemma, we get

$$S_1^{nc}/I_2 \cong S_1^{ab}/I_2^{ab}$$

In S_1^{ab}, we have $I_2^{ab} = 0$. So

$$S_1^{nc}/I_2 \cong S_1^{ab} = C(S^1).$$

∎

Remark 2.1. We have that

$$C(|\Sigma_{S^1}|) \cong C(S^1),$$

since $|\Sigma_{S^1}|$ and S^1 are homeomorphic spaces .

We now consider the simplicial flag complex Λ with 3 vertices $\{1, 2, 3\}$ such that $\{1, 3\} \notin \Lambda$.

Lemma 2.6. *The universal C^*-algebra $\mathcal{C}^f_\Lambda$ generated by positive generators h_1, h_2, h_3 with sum equal to one and $h_1h_3 = 0$ is homotopy equivalent to $\mathbb{C}$.*

Proof. Let $\alpha : \mathcal{C}^f_\Lambda \longrightarrow \mathbb{C}$ be the homomorphism which sends h_2 to 1 and h_1, h_3 to 0. And let $\beta : \mathbb{C} \longrightarrow \mathcal{C}^f_\Lambda$ be the natural homomorphism which sends 1 in $\mathbb{C}$ to the the identity element in $\mathcal{C}^f_\Lambda$. It's clear that $\alpha \circ \beta = id_{\mathbb{C}}$. Define

$$\varphi_t : \mathcal{C}^f_\Lambda \longrightarrow \mathcal{C}^f_\Lambda,$$

by mapping h_2 to $h_2 + (1-t)(h_1+h_3)$ and h_i to th_i for $i = 1, 3$. The $\varphi_t(h_i)$ satisfy the following relations :
(i) $\varphi_t(h_i) \geqslant 0\ \forall i \in \{1, 2, 3\}$.
(ii) $\varphi_t(h_1) + \varphi_t(h_2) + \varphi_t(h_3) = th_1 + (h_2 + (1-t)(h_1 + h_3)) + th_3 = h_1 + h_2 + h_3 = 1$.
(iii) $\varphi_t(h_1)\varphi_t(h_3) = th_1th_3 = t^2h_1h_3 = 0$.
Since the elements $\varphi_t(h_i)$ satisfy the relations of the h_i in $\mathcal{C}^f_\Lambda$, φ_t is well defined.
It is obvious that $\varphi_1 = id_{\mathcal{C}^f_\Lambda}$ and $\varphi_0 = \beta \circ \alpha$. This means that $\beta \circ \alpha$ is homotopic to $Id_{\mathcal{C}^f_\Lambda}$. Hence it follows that $\mathcal{C}^f_\Lambda$ is homotopy equivalent to $\mathbb{C}$. ∎

Lemma 2.7. *In the previous lemma, let $\mathcal{I}_\Lambda$ be the ideal in $\mathcal{C}^f_\Lambda$ generated by the products containing all generators h_1, h_2, h_3 . Then $\mathcal{I}_\Lambda$ is homotopy equivalent to zero.*

Proof. We have from the previous lemma that

$$\varphi_t : \mathcal{C}^f_\Lambda \longrightarrow \mathcal{C}^f_\Lambda$$

is well defined .
We show that φ_t maps $\mathcal{I}_\Lambda$ to $\mathcal{I}_\Lambda$ and therefore induces by restriction a homomorphism

$$\varphi_t|_{\mathcal{I}_\Lambda} := \hat{\varphi}_t : \mathcal{I}_\Lambda \longrightarrow \mathcal{I}_\Lambda.$$

Let $x = ...h_1h_2^kh_3...$ be a typical element in $\mathcal{I}_\Lambda$. We have

$$\hat{\varphi}_t(h_1h_2^kh_3) = \varphi_t(h_1)\varphi_t(h_2^k)\varphi_t(h_3)$$
$$= th_1(h_2 + (1-t)(h_1 + h_3)^k)th_3 = h_1P(h_2)h_3$$

where P is polynomial without constant term. So the product is in $\mathcal{I}_\Lambda$. Note that we used in the equations above that $h_1h_3 = 0$.

It is clear that $\hat{\varphi}_0 = 0$ and $\hat{\varphi}_1 = id_{\mathcal{I}_\Lambda}$.
This yields that $\mathcal{I}_\Lambda$ is homotopy equivalent to zero. ∎

Lemma 2.8. *For the skeleton filtration (I_k) in S^{nc}_1, I_2/I_3 has trivial K-theory.*

Proof. Consider the skeleton filtration

$$S^{nc}_1 := I_0 \supset I_1 \supset I_2 \supset I_3.$$

By Lemma 1.1, we have

$$I_2/I_3 \cong \bigoplus\nolimits_{\Lambda_i} \mathcal{I}_{\Lambda_i},$$

where Λ_i is the subcomplex of Σ_{S^1} generated by $\{0^+, 0^-, 1^+, 1^-\} \setminus \{i\}$, and $\mathcal{I}_{\Lambda_i}$ is the ideal generated by products containing all generators h_j , $j \in V_{\Sigma_{S^1}} \setminus \{i\}$.
There are four orthogonal ideals of this form. The orthogonality is clear, since e.g. if $x \in \mathcal{I}_{\Lambda_{0+}}$ and $y \in \mathcal{I}_{\Lambda_{0-}}$, the product

$$xy = (...h_{1^+}h^k_{0^-}h_{1^-}...)(...h_{1^+}h^l_{0^+}h_{1^-}...)$$

contains four different generators, so it is equal to zero in I_2/I_3.

Using Lemma 2.7, we get that $\mathcal{I}_{\Lambda_i}$ is homotopic to zero. This implies that $K_*(I_2/I_3) = 0$. ∎

Proposition 2.9. *In S_1^{nc} we have $K_*(I_2) = K_*(I_3), * = 0, 1$.*

Proof. We construct the short exact sequence

$$0 \longrightarrow I_3 \longrightarrow I_2 \longrightarrow I_2/I_3 \longrightarrow 0.$$

Apply the six term-exact sequence and use the lemma above. We get two isomorphisms in K-theory $K_*(I_2) \cong K_*(I_3), * = 0, 1$. ∎

Proposition 2.10. *We have*

$$K_*(S_1^{nc}/I_3) \cong K_*(S_1^{nc}/I_2) \cong K_*(C(S^1)).$$

Proof. Consider the short exact sequence

$$0 \longrightarrow I_2/I_3 \longrightarrow S_1^{nc}/I_3 \longrightarrow S_1^{nc}/I_2 \longrightarrow 0.$$

Applying the six-term exact sequence, we get

$$\begin{array}{ccccc} K_0(I_2/I_3) & \longrightarrow & K_0(S_1^{nc}/I_3) & \longrightarrow & K_0(S_1^{nc}/I_2) \\ \uparrow & & & & \downarrow \\ K_1(S_1^{nc}/I_2) & \longleftarrow & K_1(S_1^{nc}/I_3) & \longleftarrow & K_1(I_2/I_3). \end{array}$$

From Lemma 2.8 we have $K_*(I_2/I_3) = 0$, so that the above six-term exact sequence reduces to the following two isomorphisms

$$K_0(S_1^{nc}/I_3) \cong K_0(S_1^{nc}/I_2)$$

and

$$K_1(S_1^{nc}/I_3) \cong K_1(S_1^{nc}/I_2).$$

Note that by Lemmas 2.5 and 2.1 the C^*-algebras S_1^{nc}/I_2 and $C(S^1)$ have the same K-theory. This proves the proposition. ∎

Proposition 2.11. *In the algebra S_1^{nc}, we have $K_0(I_2) = K_0(I_3) = \mathbb{Z}$ and $K_1(I_2) = K_1(I_3) = 0$.*

Proof. I_2 is a closed two sided ideal in S_1^{nc}. We have the following short exact sequence

$$0 \longrightarrow I_2 \xrightarrow{i} S_1^{nc} \xrightarrow{\pi} S_1^{nc}/I_2 \longrightarrow 0.$$

During the rest of this section, denote $K_*(i)$ by i_* and $K_*(\pi)$ by π_* for $* = 0, 1$. From the above exact sequence we obtain the following six-term exact sequence .

$$\begin{array}{ccccc} K_0(I_2) & \xrightarrow{i_0} & K_0(S_1^{nc}) & \xrightarrow{\pi_0} & K_0(S_1^{nc}/I_2) \\ \uparrow & & & & \downarrow \\ K_1(S_1^{nc}/I_2) & \longleftarrow & K_1(S_1^{nc}) & \longleftarrow & K_1(I_2). \end{array}$$

We have from Theorem 1.2

$$K_*(S_1^{nc}) \cong K_*(\mathbb{C}),$$

which is generated by $[1_{S_1^{nc}}]$, where $1_{S_1^{nc}}$ denotes the identity element in S_1^{nc}.
And from the above lemma we have

$$K_*(S_1^{nc}/I_2) \cong K_*(C(S^1)).$$

It's well known that $K_*(C(S^1)) \cong \mathbb{Z}$, for $* = 0, 1$ So, the above six-term exact sequence reads as

$$\begin{array}{ccccc} K_0(I_2) & \xrightarrow{i_0} & \mathbb{Z} & \xrightarrow{\pi_0} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & K_1(I_2). \end{array}$$

With respect to the isomorphism $K_0(S_1^{nc}/I_2) \cong K_0(C(S^1))$, the image $\pi_0([1_{S_1^{nc}}])$ of the generator of $K_0(S_1^{nc})$ corresponds to the generator $[1_{C(S^1)}]$ of $K_0(C(S^1))$.

So π_0 is bijective. Then i_0 is zero, and we have $K_0(I_2) = \mathbb{Z}$ and $K_1(I_2) = 0$. By proposition 2.9, we have also $K_0(I_3) = \mathbb{Z}$ and $K_1(I_3) = 0$.
∎

Proposition 2.12. *Consider the skeleton filtration*

$$S_1^{nc} = I_0 \supset I_1 \supset I_2 \supset I_3.$$

The short exact sequence

$$0 \longrightarrow I_k \xrightarrow{i} I_{k-1} \xrightarrow{\pi} I_{k-1}/I_k \longrightarrow 0$$

induces $i_* : K_*(I_k) \longrightarrow K_*(I_{k-1})$ *which is zero for* $1 \leq k \leq 2$, *and* $* = 0, 1$.

Proof. For $k = 1$, we have the following six-term exact sequence

$$\begin{array}{ccccc} K_0(I_1) & \xrightarrow{i_0} & K_0(S_1^{nc}) & \xrightarrow{\pi_0} & K_0(S_1^{nc}/I_1) \\ \uparrow & & & & \downarrow \\ K_1(S_1^{nc}/I_1) & \xleftarrow{\pi_1} & K_1(S_1^{nc}) & \xleftarrow{i_1} & K_1(I_1). \end{array}$$

From Theorem 1.2 $K_*(S_1^{nc}) \cong K_*(\mathbb{C})$ and by Lemma 2.1 $K_0(S_1^{nc}/I_1) \cong \mathbb{Z}^4$ and $K_1(S_n^{nc}/I_1) = 0$. So there is an embedding $\mathbb{Z} \overset{\pi_0}{\hookrightarrow} K_0(S_n^{nc}/I_1)$, therefore $i_0 = 0$. It is already $i_1 = 0$. Moreover, it is also clear that $K_0(I_1) = 0$. For $k = 2$, we get the six-term exact sequence

$$\begin{array}{ccccc} K_0(I_2) & \xrightarrow{i_0} & K_0(I_1) & \xrightarrow{\pi_0} & K_0(I_1/I_2) \\ \uparrow & & & & \downarrow \\ K_1(I_1/I_2) & \xleftarrow{\pi_1} & K_1(I_1) & \xleftarrow{i_1} & K_1(I_2). \end{array}$$

From above $K_0(I_1) = 0$, and from proposition 2.11 $K_1(I_2) = 0$, so $i_* = 0$. ∎

For $k = 3$, proposition 2.9 gives a counterexample, since i_* is an isomorphism between $K_*(I_2)$ and $K_*(I_3)$, and therefore $i_* \neq 0$ for $k = 3$.

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