CERTAIN COMPACT GENERALIZATIONS OF WELL-KNOWN POLYNOMIAL INEQUALITIES

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ABSTRACT. In this paper, certain sharp compact generalizations of well-known Bernstein-type inequalities for polynomials, from which a variety of interesting results follow as special cases, are obtained.

Key words and phrases: Polynomials; Inequalities in the complex domain; Bernstein’s inequality.

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1. Introduction and Statement of Results

Let \( \mathcal{P}_n \) denote the space of all complex polynomials \( P(z) = \sum_{j=0}^{n} a_j z^j \) of degree \( n \). A famous result known as Bernstein’s inequality (for reference, see \[8\, p.531\], \[10\, p.508\] or \[11\] states that if \( P \in \mathcal{P}_n \), then

\[
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|,
\]

whereas concerning the maximum modulus of \( P(z) \) on the circle \( |z| = R \geq 1 \), we have

\[
\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|, \quad R \geq 1.
\]

(For reference, see \[8\, p.442\] or \[9\, vol.I, p.137\].)

If we restrict ourselves to the class of polynomials \( P \in \mathcal{P}_n \) having no zero in \( |z| < 1 \), then inequalities (1.1) and (1.2) can be respectively replaced by

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|,
\]

and

\[
\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|, \quad R \geq 1.
\]

Inequality (1.3) was conjectured by Erdős and later verified by Lax \[7\], whereas inequality (1.4) is due to Ankey and Ravilin \[1\]. Aziz and Dawood \[2\] further improved inequalities (1.3) and (1.4) under the same hypothesis and proved that,

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|,
\]

and

\[
\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \frac{R^n - 1}{2} \min_{|z|=1} |P(z)|, \quad R \geq 1.
\]

Jain \[5\] generalized both the inequalities (1.3) and (1.4) and proved that if \( P \in \mathcal{P}_n \) and \( P(z) \neq 0 \) in \( |z| < 1 \), then for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( |z| = 1 \) and \( R \geq 1 \),

\[
|zP'(z) + \frac{n\beta}{2} P(z)| \leq \frac{n}{2} \left\{ \left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right\} \max_{|z|=1} |P(z)|,
\]

and

\[
|P(Rz) + \beta \left( \frac{R + 1}{2} \right)^n P(z)| \leq \frac{1}{2} \left[ \left| R^n + \beta \left( \frac{R + 1}{2} \right)^n \right| + \left| 1 + \beta \left( \frac{R + 1}{2} \right)^n \right| \right] \max_{|z|=1} |P(z)|.
\]

Jain \[6\] obtained a result concerning minimum modulus of polynomials and proved the following result.

**Theorem 1.1.** If \( P \in \mathcal{P}_n \) and \( P(z) \) has all its zeros in \( |z| \leq 1 \), then for every real of complex \( \beta \) with \( |\beta| \leq 1 \),

\[
\min_{|z|=1} |zP'(z) + \frac{n\beta}{2} P(z)| \geq n \left| 1 + \frac{\beta}{2} \right| \min_{|z|=1} |P(z)|.
\]

As a refinement of inequalities (1.7) and (1.8), Jain \[6\] proved:
Theorem 1.2. If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish in \( |z| < 1 \), then for every real of complex \( \beta \) with \( |\beta| \leq 1 \) and \( R \geq 1 \),

\[
|zP'(z) + \frac{n\beta}{2} P(z)| \leq \frac{n}{2} \left\{ \left| 1 + \frac{\beta}{2} \right| + \frac{|\beta|}{|z|=1} \max |P(z)| \right\} |z|=1
\]

(1.10)

and

\[
\max_{|z|=1} \left| P(Rz) + \beta \left( \frac{R+1}{2} \right)^n P(z) \right| \leq \frac{1}{2} \left\{ \left| R^n + \beta \left( \frac{R+1}{2} \right)^n \right| + \left| 1 + \beta \left( \frac{R+1}{2} \right)^n \right| \right\} \max_{|z|=1} |P(z)| |z|=1
\]

(1.11)

Inequalities (1.9) and (1.10) have recently appeared in [4] also.

In this paper, we first present the following interesting result which yields a number of well-known polynomial inequalities as special cases.

Theorem 1.3. If \( P \in \mathcal{P}_n \), then for \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \), \( R > r \geq 1 \) and \( |z| \geq 1 \),

\[
k^n |P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n \right\} P(rz)| \leq |z|^n \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n \right\} r^n \right| \max_{|z|=k} |P(z)|.
\]

(1.12)

The result is best possible and equality in (1.12) holds for \( P(z) = az^n, a \neq 0 \).

If we choose \( \alpha = 0 \) in Theorem 1.3, we get the following result.

Corollary 1.4. If \( P \in \mathcal{P}_n \), then for \( \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \), \( k \leq 1 \), \( R > r \geq 1 \) and \( |z| \geq 1 \),

\[
k^n |P(Rz) + \beta \left( \frac{R+k}{k+r} \right)^n P(rz)| \leq |z|^n \left| R^n + \beta r^n \left( \frac{R+k}{k+r} \right)^n \right| \max_{|z|=k} |P(z)|
\]

(1.13)

Equality in (1.13) holds for \( P(z) = az^n, a \neq 0 \).

Dividing the two sides of (1.12) by \( R - r \) with \( \alpha = 1 \) and then letting \( R \to r \), we get.

Corollary 1.5. If \( P \in \mathcal{P}_n \), then for \( \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \), \( k \leq 1 \), \( R > r \geq 1 \) and \( |z| \geq 1 \),

\[
k^n |zP'(r) + \frac{n\beta}{k+r} P(rz)| \leq n^r |z|^n \left| \frac{1}{r} + \frac{\beta}{k+r} \right| \max_{|z|=k} |P(z)|.
\]

(1.14)

The result is best possible and equality in (1.14) holds for \( P(z) = az^n, a \neq 0 \).

The following compact generalization of inequalities (1.11) and (1.12) immediately follows from Theorem 1.3 by taking \( k = 1 \) and \( \beta = 0 \) in (1.12).

Corollary 1.6. If \( P \in \mathcal{P}_n \), then for \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( R > r \geq 1 \) and \( |z| \geq 1 \),

\[
|P(Rz) - \alpha P(rz)| \leq |z|^n \left| R^n - \alpha r^n \right| \max_{|z|=1} |P(z)|
\]

(1.15)
The result is best possible as shown by $P(z) = az^n, a \neq 0$.

**Remark 1.1.** For $\alpha = 0$, (1.15) reduces to (1.2). For $\alpha = r = 1$, if we divide the two sides of (1.15) by $R - 1$ and make $R \to 1$, we get inequality (1.1).

Next, we present the following result which includes Theorem 1.1 as a special case.

**Theorem 1.7.** If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$,

$$\min_{|z|=1} \left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right| k^n \geq \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \min_{|z|=k} |P(z)|. \tag{1.16}$$

The result is best possible as shown by $P(z) = az^n, a \neq 0$.

If we divide the two sides of inequality (1.16) by $R - r$, with $\alpha = 1$ and then making $R \to r$ we get

**Corollary 1.8.** If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$ then for $|\beta| \leq 1$ and $r \geq 1$,

$$\min_{|z|=1} \left| z P'(rz) + \frac{n\beta}{k+r} P(rz) \right| k^n \geq \frac{1}{r} + \frac{\beta}{k+r} \min_{|z|=k} |P(z)|. \tag{1.17}$$

The result is sharp.

**Remark 1.2.** For $k = r = 1$, inequality (1.16) reduces to Theorem (1.1).

Setting $\beta = 0$ in Theorem 1.7 we obtain:

**Corollary 1.9.** If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > r \geq 1$,

$$\min_{|z|=1} |P(Rz) - \alpha P(rz)| k^n \geq |R^n - \alpha r^n| \min_{|z|=k} |P(z)|. \tag{1.18}$$

For polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < k$, we also establish the following result which leads to a compact generalization of inequalities (1.7) and (1.8).

**Theorem 1.10.** If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in the disk $|z| < k$ where $k \leq 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,

$$\left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right| \leq \frac{1}{2} \left[ 1 - \alpha + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right]$$

$$+ \frac{|z|^n}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \max_{|z|=k} |P(z)|. \tag{1.18}$$

**Remark 1.3.** If we take $\alpha = k = r = 1$ in Theorem 1.10 and divide the two sides of inequality (1.18) by $R - 1$ and then make $R \to 1$, we get inequality (1.7), whereas inequality (1.8) follows from Theorem 1.10 when $\alpha = 0$ and $k = 1$.

For $k = 1$, Theorem 1.10 reduces to the result due to Aziz and Rather [3].
As a refinement of Theorem 1.10, we finally prove the following result, which provides a compact generalization of inequalities (1.7), (1.8) and (1.10) as well.

**Theorem 1.11.** If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish in the disk \( |z| < k, k \leq 1 \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( R > r \geq 1 \),

\[
\begin{align*}
\max_{|z|=1} |P(Rz) - \alpha P(rz) + \beta \left( \frac{R + k}{k + r} \right)^n - |\alpha| \} P(rz)| \\
\leq \frac{1}{2} \left\{ \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right| r^n \right. \\
\left. + \left| 1 - \alpha + \beta \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right| \right\} \max_{|z|=k} |P(z)| \\
- \left\{ \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right| r^n \right. \\
\left. - \left| 1 - \alpha + \beta \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right| \right\} \min_{|z|=k} |P(z)| \\
\end{align*}
\] (1.19)

If we take \( \alpha = 1 \) and divide the two sides of inequality (1.19) by \( R - r \) then letting \( R \to r \), we get:

**Corollary 1.12.** If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish in \( |z| \leq k \) where \( k \leq 1 \), then for \( |\beta| \leq 1 \) and \( r \geq 1 \),

\[
\begin{align*}
\max_{|z|=1} \left| zP'(rz) + \frac{n\beta}{k + r} P(rz) \right| \\
\leq \frac{n}{2} \left\{ \frac{r^n}{k^n} \left| 1 + \frac{\beta}{k + r} \right| + \frac{\beta}{k + r} \right\} \max_{|z|=k} |P(z)| \\
- \left\{ \frac{r^n}{k^n} \left| 1 + \frac{\beta}{k + r} \right| - \frac{\beta}{k + r} \right\} \min_{|z|=k} |P(z)| \\
\end{align*}
\] (1.20)

**Remark 1.4.** For \( k = r = 1 \), inequality (1.20) reduces to Theorem 1.12

For \( \alpha = 0 \), Theorem 1.11 reduces following result.

**Corollary 1.13.** If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish in \( |z| < k, k \leq 1 \) then for all \( \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \) and \( R > r \geq 1 \),

\[
\begin{align*}
\max_{|z|=1} |P(Rz) + \beta \left( \frac{R + k}{k + r} \right)^n P(rz)| \\
\leq \frac{1}{2} \left\{ \frac{1}{k^n} \left| R^n + \beta r^n \left( \frac{R + k}{k + r} \right)^n \right| + |1 + \beta \left( \frac{R + k}{k + r} \right)^n | \right\} \max_{|z|=k} |P(z)| \\
- \left\{ \frac{1}{k^n} \left| R^n + \beta r^n \left( \frac{R + k}{k + r} \right)^n \right| - |1 + \beta \left( \frac{R + k}{k + r} \right)^n | \right\} \min_{|z|=k} |P(z)| \\
\end{align*}
\] (1.21)

Corollary 1.13 leads to a refinement of inequality (1.11) for \( k = 1 \). The following result immediately follows from Theorem 1.11 by taking \( \beta = 0 \) and \( k = 1 \).
Corollary 1.14. If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish in \( |z| < 1 \), then for all \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \) and \( R > r \geq 1 \),

\[
\begin{align*}
\max_{|z|=1} |P(Rz) - \alpha P(rz)| & \leq \left( \frac{|R^n - \alpha r^n| + |1 - \alpha|}{2} \right) \max_{|z|=1} |P(z)| \\
& \quad - \left( \frac{|R^n - \alpha r^n| - |1 - \alpha|}{2} \right) \min_{|z|=1} |P(z)|.
\end{align*}
\]
(1.22)

The result is sharp and extremal polynomial is \( P(z) = az^n + b, |a| = |b| \neq 0 \).

Remark 1.5. For \( \alpha = 0 \), inequality (1.22) reduces to inequality (1.6). Also if we take \( r = 1 \) and divide the two sides if inequality (1.22) by \( R - 1 \) with \( \alpha = 1 \) and let \( R \to 1 \), we get inequality (1.5).

2. LEMMAS

For the proof of these theorems, we need the following Lemmas.

Lemma 2.1. If \( P \in \mathcal{P}_n \) and \( P(z) \) have all its zeros in \( |z| \leq k \) where \( k \leq 1 \), then for every \( R \geq r \geq 1 \) and \( |z| = 1 \),

\[ |P(Rz)| \geq \left( \frac{R + k}{r + k} \right)^n |P(rz)|. \]

Proof. Since all the zeros of \( P(z) \) lie in \( |z| \leq k \), \( k \leq 1 \) we write

\[ P(z) = C \prod_{j=1}^{n} (z - r_j e^{i\theta_j}) , \]

where \( r_j \leq k \leq 1 \). Now for \( 0 \leq \theta < 2\pi \), \( R > r \geq 1 \), we have

\[
\left| \frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}} \right| = \left\{ \frac{R^2 + r_j^2 - 2Rr_j \cos(\theta - \theta_j)}{r^2 + r_j^2 - 2rr_j \cos(\theta - \theta_j)} \right\}^{1/2} , \]

\[
\geq \left\{ \frac{R + r_j}{r + r_j} \right\} , \]

\[
\geq \left\{ \frac{R + k}{r + k} \right\}, \text{for } j = 1, 2, \ldots, n .
\]

Hence

\[
\left| \frac{P(Re^{i\theta})}{P(re^{i\theta})} \right| = \prod_{j=1}^{n} \left| \frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}} \right| ,
\]

\[
\geq \prod_{j=1}^{n} \left( \frac{R + k}{r + k} \right) ,
\]

\[
= \left( \frac{R + k}{r + k} \right)^n ,
\]

for \( 0 \leq \theta < 2\pi \). This implies for \( |z| = 1 \) and \( R > r \geq 1 \),

\[ |P(Rz)| \geq \left( \frac{R + k}{r + k} \right)^n |P(rz)| ,
\]

which completes the proof of Lemma 2.1.
Lemma 2.2. If $F \in \mathcal{P}_n$ and $F(z)$ has all its zeros in the disk $|z| \leq k$ where $k \leq 1$ and $P(z)$ is a polynomial of degree at most $n$ such that

$$|P(z)| \leq |F(z)| \quad \text{for} \quad |z| = k,$$

then for arbitrary real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1,$

$$|P(Rz) - \alpha P(rz) + \beta \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right) |P(rz)| \leq |F(Rz) - \alpha F(rz) + \beta \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right) F(rz)|.$$ (2.1)

Proof. Since polynomial $F(z)$ of degree $n$ has all its zeros in $|z| \leq k$ and $P(z)$ is a polynomial of degree at most $n$ such that

$$|P(z)| \leq |F(z)| \quad \text{for} \quad |z| = k,$$

therefore, if $F(z)$ has a zero of multiplicity $s$ at $z = ke^{i\theta}$, then $P(z)$ has a zero of multiplicity at least $s$ at $z = ke^{i\theta}$. If $P(z)/F(z)$ is a constant, then inequality (2.1) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \quad \text{for} \quad |z| > k.$$ (2.2)

Suppose $F(z)$ has $m$ zeros on $|z| = k$ where $0 \leq m < n$, so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree $m$ whose all zeros lie on $|z| = k$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < k$. This implies with the help of inequality (2.2) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Again, from inequality (2.2), we have

$$|P_1(z)| \leq |F_2(z)| \quad \text{for} \quad |z| = k$$

where $F_2(z) \neq 0$ for $|z| = k$. Therefore for every real or complex number $\lambda$ with $|\lambda| > 1$, a direct application of Rouche’s Theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < k$. Hence the polynomial

$$G(z) = F_1(z) \left( P_1(z) - \lambda F_2(z) \right) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq k$ with at least one zero in $|z| < k$, so that we can write

$$f(z) = (z - te^{i\delta})H(z)$$

where $t < k$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \leq k$. Applying Lemma 2.1 to the polynomial $H(z)$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$|G(Re^{i\theta})| = |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})|$$

$$\geq \left| Re^{i\theta} - te^{i\delta} \right| \left( \frac{R+k}{k+r} \right)^{n-1} |H(Re^{i\theta})|,$$

$$= \left( \frac{R+k}{k+r} \right)^{n-1} \left| \frac{Re^{i\theta} - te^{i\delta}}{re^{i\theta} - te^{i\delta}} \right| (re^{i\theta} - te^{i\delta}) |H(Re^{i\theta})|,$$

$$\geq \left( \frac{R+k}{k+r} \right)^{n-1} \left( \frac{R+t}{r+t} \right) |G(Re^{i\theta})|. $$
This implies for \( R > r \geq 1 \) and \( 0 \leq \theta < 2\pi \),

\[
(2.3) \quad \left( \frac{r + t}{R + t} \right) |G(Re^{i\theta})| \geq \left( \frac{R + k}{k + r} \right)^{n-1} |G(re^{i\theta})|.
\]

Since \( R > r \geq 1 > t \) so that \( G(Re^{i\theta}) \neq 0 \) for \( 0 \leq \theta < 2\pi \) and \( \frac{r + k}{k + r} > \frac{r + t}{R + t} \), from inequality (2.3), we obtain

\[
(2.4) \quad |G(Re^{i\theta})| > \left( \frac{R + k}{k + r} \right)^n |G(re^{i\theta})|, \quad R > 1 \text{ and } 0 \leq \theta < 2\pi.
\]

Equivalently,

\[
|G(Rz)| > \left( \frac{R + k}{k + r} \right)^n |G(rz)|
\]

for \( |z| = 1 \) and \( R > r \geq 1 \). Hence for every real or complex number \( \alpha \) with \( |\alpha| \leq 1 \) and \( R > r \geq 1 \), we have

\[
(2.5) \quad |G(Rz) - \alpha G(rz)| \geq |G(Rz)| - |\alpha| |G(rz)| > \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} |G(rz)|, \quad \text{for } |z| = 1.
\]

Also, inequality (2.4) can be written in the form

\[
(2.6) \quad |G(re^{i\theta})| < \left( \frac{k + r}{R + k} \right)^n |G(Re^{i\theta})|
\]

for every \( R > r \geq 1 \) and \( 0 \leq \theta < 2\pi \). Since \( G(Re^{i\theta}) \neq 0 \) and \( \left( \frac{k + r}{R + k} \right)^n < 1 \), from inequality (2.6), we obtain for \( 0 \leq \theta < 2\pi \) and \( R > r \geq 1 \),

\[
|G(re^{i\theta})| < |G(Re^{i\theta})|.
\]

That is,

\[
|G(rz)| < |G(Rz)| \quad \text{for } |z| = 1.
\]

Since all the zeros of \( G(Rz) \) lie in \( |z| \leq (k/R) < 1 \), a direct application of Rouche’s Theorem shows that the polynomial \( G(Rz) - \alpha G(rz) \) has all its zeros in \( |z| < 1 \) for every real or complex number \( \alpha \) with \( |\alpha| \leq 1 \). Applying Rouche’s Theorem again, it follows from (2.5) that for arbitrary real or complex numbers \( \alpha, \beta \) with \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( R > r \geq 1 \), all the zeros of the polynomial

\[
T(z) = G(Rz) - \alpha G(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} G(rz)
\]

lie in \( |z| < 1 \). This implies

\[
(2.7) \quad \left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz) \right| \\
\leq |F(Rz) - \alpha F(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} F(rz)|
\]
for $|z| \geq 1$ and $R > r \geq 1$. If inequality (2.7) is not true, then there a point $z = z_0$ with $|z_0| \geq 1$ such that

$$P(Rz_0) - \alpha P(rz_0) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz_0)$$

$$> \left| F(Rz_0) - \alpha F(rz_0) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} F(rz_0) \right|$$

But all the zeros of $F(Rz)$ lie in $|z| < (k/R) < 1$, therefore, it follows (as in case of $G(z)$) that all the zeros of $F(Rz) - \alpha F(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} F(rz)$ lie in $|z| < 1$. Hence

$$F(Rz_0) - \alpha F(rz_0) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} F(rz_0) \neq 0$$

with $|z_0| \geq 1$. We take

$$\lambda = \frac{P(Rz_0) - \alpha P(rz_0) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz_0)}{F(Rz_0) - \alpha F(rz_0) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} F(rz_0)},$$

then $\lambda$ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of $\lambda$, we obtain $T(z_0) = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus (2.7) holds for $|\alpha| \leq 1$, $|\beta| \leq 1$, $|z| \geq 1$, and $R > r \geq 1$.

**Lemma 2.3.** If $P \in \mathcal{P}_n$ and $P(z)$ have no zero in $|z| < k, k \leq 1$, then for $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$

$$|P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz)|$$

$$\leq k^n \left| Q \left( Rz/k^2 \right) - \alpha Q \left( rz/k^2 \right) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q \left( rz/k^2 \right) \right|$$

(2.8)

where $Q(z) = z^nP(1/z)$.

**Proof.** By hypothesis, the polynomial $P(z) \neq 0$ in $|z| < k$, where $k \leq 1$. Therefore, all the zeros of polynomial $Q(z/k^2)$ lie in $|z| < k \leq 1$. As

$$|k^n Q(z/k^2)| = |P(z)| \text{ for } |z| = k,$$

Applying Lemma 2.2 with $F(z)$ replaced by $k^n Q(z/k^2)$, we get for arbitrary real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,

$$|P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz)|$$

$$\leq k^n \left| Q \left( Rz/k^2 \right) - \alpha Q \left( rz/k^2 \right) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q \left( rz/k^2 \right) \right|$$

This proves the Lemma 2.3.
Lemma 2.4. If \( P \in \mathcal{S}_n \) and \( Q(z) = z^n P(1/z) \) then for \( \alpha, \beta \in \mathbb{C} \), with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1, k \leq 1 \) and \( |z| \geq 1 \),

\[
\left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz) \right.
\]
\[+ k^n \left| Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q(rz/k^2) \right|
\leq \left[ \frac{|z|^n}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right| \right] \max_{|z|=k} |P(z)|.
\]

(2.9)

\[ + \left| 1 - \alpha + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} \right| \left\| \text{Max}_{|z|=k} |P(z)| \right\|.
\]

Proof. Let \( M = \max_{|z|=k} |P(z)| \), then by Rouche’s Theorem, the polynomial \( F(z) = P(z) - \mu M \) does not vanish in \( |z| < k \) for every \( \mu \in \mathbb{C} \) with \( |\mu| > 1 \). Applying Lemma 2.3 to polynomial \( F(z) \), we get for \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( |z| \geq 1 \),

\[
\left| F(Rz) - \alpha F(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} F(rz) \right|
\]
\[\leq k^n \left| H(Rz/k^2) - \alpha H(rz/k^2) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} H(rz/k^2) \right|
\]

where \( H(z) = z^n F(1/z) = z^n P(1/z) - \mu M z^n \). Replacing \( F(z) \) by \( P(z) - \mu M \) and \( H(z) \) by \( Q(z) - \mu M z^n \), we have for \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( |z| \geq 1 \),

\[
\left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz) \right.
\]
\[\left. - \mu \left\{ 1 - \alpha + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} \right\} M \right|
\]
\[\leq k^n \left| Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q(rz/k^2) \right|
\]
\[\leq k^n \left| \frac{R^n}{k^n} - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right| M z^n,
\]

(2.10)

where \( Q(z) = z^n P(1/z) \). Choosing argument of \( \lambda \) in the right hand side of inequality (2.10) such that

\[
k^n \left| Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q(rz/k^2) \right.
\]
\[\left. - \frac{\mu}{k^{2n}} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right| M z^n \right|
\]
\[= \frac{|\mu| z^n |M|}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right|
\]
\[\leq k^n \left| Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q(rz/k^2) \right|
\]

(2.11)
which is possible by applying Theorem 1.3 to the polynomial $Q(z/k^2)$ and using the fact that $\max_{|z|=k} |Q(z/k^2)| = M/k^n$, we get for $|\alpha| \leq 1$, $|\beta| \leq 1$ and $|z| \geq 1$,

$$P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz)$$

$$\leq |\mu| \left\{ 1 - \alpha + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right\} M$$

$$\leq \frac{|\mu|^n |M|}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right|$$

$$\leq k^n \left( Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} Q(rz/k^2) \right).$$

Equivalently for $|\alpha| \leq 1$, $|\beta| \leq 1$ and $|z| \geq 1$,

$$P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz)$$

$$+ k^n \left( Q(Rz/k^2) - \alpha Q(z/k^2) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} Q(z/k^2) \right)$$

$$\leq |\mu| \left\{ \frac{|z|^n}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \right\} M.$$

Letting $|\mu| \to 1$, we get the conclusion of Lemma 2.4 and this completes proof of Lemma 2.4.

3. PROOFS OF THE THEOREMS

**Proof of Theorem 1.3** Let $M = \max_{|z|=k} |P(z)|$. Then the polynomial $F(z) = Mz^n/k^n$ has all its zeros in $|z| \leq k$ where $k \leq 1$ and,

$$|P(z)| \leq |F(z)| \text{ for } |z| = k,$$

therefore by Lemma 2.2 for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz)$$

$$\leq |F(Rz) - \alpha F(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} F(rz)|.$$

Replacing $F(z)$ by $Mz^n/k^n$, we get the conclusion of Theorem 1.3.

**Proof of Theorem 1.7** Let $m = \min_{|z|=k} |P(z)|$. If $P(z)$ has zeros on $|z| = k$, $k \leq 1$, then the result is trivially true. Assume that all the zeros of $P(z)$ lie in $|z| < k$, $k \leq 1$, so that $m > 0$. An application of Rouche’s Theorem shows that the polynomial $f(z) = k^n P(z) - \lambda mz^n$ has all its zeros in $|z| < k$, $k \leq 1$, for every $\lambda$ with $|\lambda| < 1$. Applying Lemma 2.1 to $f(z)$, we have

$$|f(Rz)| \geq \left( \frac{R+k}{k+r} \right)^n |f(rz)| \text{ for } |z| = 1, \ R > r \geq 1,$$
which implies,

$$|f(Rz)| > |f(rz)| \quad \text{for } R > r \geq 1 \quad \text{and} \quad |z| = 1.$$  

Thus by Rouche’s Theorem for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, all the zeros of $F(z) = f(Rz) - \alpha f(rz)$ lie in $|z| < 1$ and we have

$$|f(Rz) - \alpha f(rz)| \geq |f(Rz)| - |\alpha f(rz)|$$

$$\geq \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} |f(rz)|$$

for $|z| = 1$ and $R > r \geq 1$. Again by Rouche’s Theorem, it follows that all the zeros of the polynomial

$$g(z) = f(Rz) - \alpha f(rz) + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) f(rz)$$

$$= k^n \left[ P(Rz) - \alpha P(rz) + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) P(rz) \right]$$

$$- \lambda m z^n \left[ R^n - \alpha r^n + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) r^n \right]$$

lie in $|z| < 1$ for $\beta \in \mathbb{C}$ with $|\beta| < 1$. This implies for every $\lambda$ with $|\lambda| < 1$,

$$k^n \left| P(Rz) - \alpha P(rz) + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) P(rz) \right|$$

$$\geq |z|^n \left| R^n - \alpha r^n + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) r^n \right| m$$  

(3.1)

for $|z| \geq 1$. If inequality (3.1) is not true, then there exists a point $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$ such that

$$k^n \left| P(Rz_0) - \alpha P(rz_0) + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) P(rz_0) \right|$$

$$< |z_0|^n \left| R^n - \alpha r^n + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) r^n \right| m$$

We take

$$\lambda = \frac{k^n \left[ P(Rz_0) - \alpha P(rz_0) + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) P(rz_0) \right]}{m |z_0|^n \left[ R^n - \alpha r^n + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) r^n \right]}$$

then $|\lambda| < 1$ and with this choice of $\lambda$, we have $g(z_0) = 0$ with $|z_0| \geq 1$, which is a contradiction, since all the zeros of $g(z)$ lie in $|z| < 1$. Hence for $R > r \geq 1$, $|\alpha| \leq 1$, $|\beta| < 1$ and $|z| \geq 1$,

$$k^n \left| P(Rz) - \alpha P(rz) + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) P(rz) \right|$$

$$\geq |z|^n \left| R^n - \alpha r^n + \beta \left( \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right) r^n \right| \min_{|z|=k} |P(z)|.$$  

(3.2)

For $\beta$ with $|\beta| = 1$, (3.2) follows from continuity. Hence inequality (3.2) immediately leads to inequality (1.16) and this completes the proof of Theorem 1.7.
Proof of Theorem 1.10 Since \( P(z) \) does not vanish in \( |z| < k, k \leq 1 \), by Lemma 2.3 we have for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1 \) and \( |z| \geq 1 \),

\[
P(Rz) - \alpha P(rz) + \beta \left\{ \frac{R + k}{k + r} \right\} P(rz)
\]

(3.3)

\[
\leq k^n \left| Q \left( Rz/k^2 \right) - \alpha Q \left( rz/k^2 \right) + \beta \left\{ \frac{R + k}{k + r} \right\} Q \left( rz/k^2 \right) \right|
\]

where \( Q(z) = z^n \overline{P(1/z)} \). Inequality (3.3) in conjunction with Lemma 2.4 gives for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1 \) and \( |z| \geq 1 \),

\[
2 \left| P(Rz) - \alpha P(rz) + \beta \left\{ \frac{R + k}{k + r} \right\} P(rz) \right|
\]

\[
\leq \left| P(Rz) - \alpha P(rz) + \beta \left\{ \frac{R + k}{k + r} \right\} P(rz) \right|
\]

\[
+ k^n \left| Q \left( Rz/k^2 \right) - \alpha Q \left( rz/k^2 \right) + \beta \left\{ \frac{R + k}{k + r} \right\} Q \left( rz/k^2 \right) \right|
\]

\[
\leq \left[ \frac{|z|^n}{k^n} R^m - |\alpha|^n + \beta \left\{ \frac{R + k}{k + r} \right\} r^n \right] \max_{|z|=k} |P(z)|
\]

this completes the proof of Theorem 1.10.

Proof of Theorem 1.11 Let \( m = \min_{|z|=k} |P(z)| \). If \( P(z) \) has a zero on \( |z| = k \), then the result follows from Theorem 1.10. We assume that \( P(z) \) has all its zeros in \( |z| < k \) where \( k \leq 1 \) so that \( m > 0 \). Now for every \( \lambda \) with \( |\lambda| < 1 \), it follows by Rouche’s Theorem \( h(z) = P(z) - \lambda mz \) does not vanish in \( |z| < k \). Applying Lemma 2.2 to the polynomial \( h(z) \), we get for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1 \) and \( |z| \geq 1 \),

\[
h(Rz) - \alpha h(rz) + \beta \left\{ \frac{R + k}{k + r} \right\} h(rz)
\]

\[
\leq k^n \left| Q_1(Rz/k^2) - \alpha Q_1(rz/k^2) + \beta \left\{ \frac{R + k}{k + r} \right\} Q_1(rz/k^2) \right|
\]

where \( Q_1(z) = z^n \overline{h(1/z)} = z^n \overline{P(1/z)} - \overline{\lambda} m n \). Equivalently,

\[
\left| P(Rz) - \alpha P(rz) + \beta \left\{ \frac{R + k}{k + r} \right\} P(rz)
\]

\[
- \lambda \left\{ 1 - \alpha + \beta \left\{ \frac{R + k}{k + r} \right\} \right\} m \right| \]

\[
\leq k^n \left| Q \left( Rz/k^2 \right) - \alpha Q \left( rz/k^2 \right) + \beta \left\{ \frac{R + k}{k + r} \right\} Q \left( rz/k^2 \right) \right|
\]

(3.4)

\[
- \overline{\lambda} \left\{ R^m - |\alpha|^n + \beta \left\{ \frac{R + k}{k + r} \right\} r^n \right\} \frac{m}{k^{2n}} \right| \text{ for } |z| = 1.
\]
where \(Q(z) = z^n P(1/z)\). Since all the zeros of \(Q(z/k^2)\) lie in \(|z| \leq k, k \leq 1\), by Theorem 1.7 applied to \(Q(z/k^2)\), we have for \(R > r \geq 1\) and \(|z| = 1\),

\[
\left| Q \left( \frac{Rz}{k^2} \right) - \alpha Q \left( \frac{rz}{k^2} \right) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q \left( \frac{rz}{k^2} \right) \right| \\
\geq \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right| \min_\{|z|=k| Q(z/k^2) | \\
= \frac{1}{k^{2n}} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right| \min_\{|z|=k| P(z) | \\
= \frac{1}{k^{2n}} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right| m. \\
(3.5)
\]

Now, choosing the argument of \(\lambda\) on the right hand side of inequality (3.4) such that

\[
k^n \left| Q \left( \frac{Rz}{k^2} \right) - \alpha Q \left( \frac{rz}{k^2} \right) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q \left( \frac{rz}{k^2} \right) \right| \\
- \lambda \left\{ R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right\} \frac{m}{k^{2n}} \\
= k^n \left| Q \left( \frac{Rz}{k^2} \right) - \alpha Q \left( \frac{rz}{k^2} \right) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q \left( \frac{rz}{k^2} \right) \right| \\
- \lambda \left\{ R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right\} \frac{m}{k^{2n}} \\
\text{for } |z| = 1, \text{ which is possible by inequality (3.5). We get for } |z| = 1, \\
\left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz) \right| \\
- |\lambda| \left\{ 1 - \alpha + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} \right\} m \\
\leq k^n \left| Q \left( \frac{Rz}{k^2} \right) - \alpha Q \left( \frac{rz}{k^2} \right) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} Q \left( \frac{rz}{k^2} \right) \right| \\
- \frac{|\lambda|}{k^n} \left\{ R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right\} \frac{m}{k^{2n}}.
\]

Equivalently for \(|z| = 1, R > r \geq 1\), we have

\[
\left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} P(rz) \right| \\
- \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} r^n \right| m. \\
(3.6)
\]
Letting $|\lambda| \to 1$ in inequality (3.6), we obtain for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| = 1$,

$$\left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right|$$

$$- k^n \left| Q \left( \frac{Rz/k^2}{r/k^2} \right) - \alpha Q \left( \frac{rz/k^2}{r/k^2} \right) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} Q \left( \frac{rz/k^2}{r/k^2} \right) \right|$$

$$\leq \left| 1 - \alpha + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right|$$

(3.7)

$$- \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right|$$

Inequality (3.7) in conjunction with Lemma 2.3 gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| = 1$,

$$2 \left| P(Rz) - \alpha P(rz) + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right|$$

$$\leq \left\{ \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \right.$$  

$$+ \left| 1 - \alpha + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right|_{|z|=k} \max |P(z)|$$

$$- \left( \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \right.$$

$$- \left| 1 - \alpha + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right|_{|z|=k} \min |P(z)| \right\} \right],$$

which is equivalent to inequality (1.19) and thus completes the proof of Theorem 1.11.

**REFERENCES**


