RENORMALIZED SOLUTIONS FOR NONLINEAR PARABOLIC EQUATION
WITH LOWER ORDER TERMS

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Received 18 December, 2012; accepted 5 July, 2013; published 30 December, 2013.

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ABSTRACT. In this paper, we study the existence of renormalized solutions for the nonlinear parabolic problem: \[ \frac{\partial b(u)}{\partial t} - \text{div}(a(x, t, u, \nabla u)) + \text{div}(\phi(x, t, u)) = f, \]
where the right side belongs to \( L^1(\Omega \times (0, T)) \) and \( b(u) \) is unbounded function of \( u \), the term \( -\text{div}(a(x, t, u, \nabla u)) \) is a Leray–Lions operator and the function \( \phi \) is a nonlinear lower order and satisfy only the growth condition.

Key words and phrases: Parabolic problems, Sobolev space, Renormalized solutions.

1. Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ ($N \geq 2$), $T$ is a positive real number, and $Q_T = \Omega \times (0, T)$. Let $b$ is a strictly increasing $C^1$-function, the data $f$ and $b(u_0)$ in $L^1(Q)$ and $L^1(\Omega)$ respectively, $-\text{div} \left( a(x, t, u, \nabla u) \right)$ is a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$ (see assumptions (2.2)-(2.4) of Section 2). The function $\phi(x, t, u)$ is a Carathéodory assumed to be continuous on $u$ (see assumptions (2.5)-(2.7)). We considere the following nonlinear parabolic problem:

$$
\begin{align*}
\frac{\partial b(u)}{\partial t} - \text{div}(a(x, t, u, \nabla u)) + \text{div}(\phi(x, t, u)) &= f \quad \text{in} \quad Q_T \\
u(x, t) &= 0 \quad \text{on} \quad \partial \Omega \times (0, T) \\
b(u(x, 0)) &= b(u_0(x)) \quad \text{in} \quad \Omega.
\end{align*}
$$

Under our assumptions, problem (1.1) does not admit, in general, a weak solution since the term $\phi(x, t, u)$ may not belong $(L^{1,\infty}(Q))^N$. In order to overcome this difficulty, we work with the framework of renormalized solutions (see definition (3.1)). The notion of renormalized solutions was introduced by R.-J. DiPerna and P.-L. Lions [14] for the study of the Boltzmann equation. It was then used by L. Boccardo and al (see [10]) when the right hand side is in $W^{-1,p}(\Omega)$ and by J.-M. Rakotoson (see [19]) when the right hand side is in $L^1(\Omega)$.

The existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [6] in the case where $a(x, t, s, \xi)$ is independent of $s$ and with $a(x, t, s, \xi)$ is independent of $s$ and $\gamma = 0$. For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch and al [11] in the case where $a$ is strictly monotone, $\phi = 0$ and $f \in L^p(0, T, W^{-1,p}(\Omega, \omega^*))$. See also the existence of renormalized solution proved by Y. Akdim and al [3] in the case where $a(x, t, s, \xi)$ is independent of $s$ and $\phi = 0$.

In the case where $b(u) = u$, the existence of renormalized solutions for (1.1) has been established by R.-Di Nardo (see [12]). For the degenerated parabolic equation with $b(u) = u$, $\text{div}(\phi(x, t, u)) = H(x, t, u, \nabla u)$ and $f \in L^1(Q)$, the existence of renormalized solution has been proved by Y. Akdim and al (see [4]).

The case where $b(u) = b(x, u)$, $\text{div}(\phi(x, t, u)) = H(x, t, u, \nabla u)$ and $f \in L^1(Q)$, the existence of renormalized solutions has been established by H. Redwane (see [17]) in the classical Sobolev space and by Y. Akdim and al (see [2]) in the degenerate Sobolev space.

It is our purpose, in this paper to generalize the result of ([3], [4], [12]) and we prove the existence of a renormalized solution of (1.1).

The plan of the paper is as follows: In Section 2 we give some preliminaries and basic assumptions. In Section 3 we give the definition of a renormalized solution of (1.1), and we establish (Theorem 3.1) the existence of such a solution.

2. Basic assumptions and preliminaries

2.1. Preliminaries. Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ ($N \geq 2$), $T$ is a positive real number, and $Q_T = \Omega \times (0, T)$. We need the Sobolev embeddings result

**Theorem 2.1.** (Gagliardo-Nirenberg) Let $v$ be a function in $W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$ with $q \geq 1$, $\rho \geq 1$. Then there exists a positive constant $C$, depending on $N$, $q$ and $\rho$, such that

$$
\| v \|_{L^\gamma(\Omega)} \leq C \| \nabla v \|_{L^\rho(\Omega)}^{\theta} \| v \|_{L^q(\Omega)}^{1-\theta}
$$

for every $\theta$ and $\gamma$ satisfying

$$
0 \leq \theta \leq 1, \quad 1 \leq \gamma \leq +\infty, \quad \frac{1}{\gamma} = \theta \left( \frac{1}{q} - \frac{1}{N} \right) + \frac{1-\theta}{\rho}.
$$
An immediate consequence of the previous result:

**Corollary 2.2.** Let \( v \in L^q((0, T), L^q(\Omega)) \) with \( q \geq 1, \rho \geq 1 \). Then \( v \in L^\sigma(\Omega) \) with \( \sigma = q\left(\frac{N+\rho}{N}\right) \)and

\[
\int_{Q_T} |v|^\sigma \, dx \, dt \leq C \int_{Q_T} |\nabla v|^\rho \, dx \, dt.
\]

**Lemma 2.3.** (see [12]) Assume that \( \Omega \) is an open set of \( \mathbb{R}^N \) of finite measure and \( 1 < p < +\infty \). Let \( u \) be a measurable function satisfying \( T_k(u) \in L^p((0, T), W_0^{1,p}(\Omega)) \cap L^\infty((0, T), L^2(\Omega)) \) for every \( k \) and such that:

\[
sup_{t \in (0, T)} |\nabla T_k(u)|^2 + \int_{Q_T} |\nabla T_k(u)|^p \leq M k, \quad \forall \quad k > 0
\]

where \( M \) is a positive constant. Then

\[
\||u|^{p-1}\|_{L^\frac{p(N+1)-N}{N(p+1)-N}(Q_T)} \leq CM\left(\frac{\gamma+1}{\gamma}\right)^{\frac{N}{N+p}}|Q_T|^\frac{1}{p} \leq CM^{\frac{(N+2)(p-1)}{p(N+1)-N}}
\]

where \( C \) is a constant depend only on \( N \) and \( p \).

### 2.2. Assumption(H)

Throughout this paper, we assume that the following assumptions hold true:

(2.1) \( b : \Omega \times \mathbb{R} \to \mathbb{R} \) is strictly increasing \( C^1 \)-function, such that \( b' > \beta > 0 \) and \( b(0) = 0 \).

and

(2.2) \( |a(x, t, s, \xi)| \leq \nu|h(x, t)| + |\xi|^{p-1}, \) with \( \nu > 0 \) and \( h(x, t) \in L^p(Q_T) \),

(2.3) \( a(x, t, s, \xi) \xi \geq \alpha|\xi|^p, \) with \( \alpha > 0 \),

(2.4) \( (a(x, t, s, \xi) - a(x, t, s, \eta))(\xi - \eta) > 0, \) with \( \xi \neq \eta \),

(2.5) \( |\phi(x, t, s)| \leq c(x, t)|s|^{\tau}, \)

(2.6) \( c(x, t) \in (L^\gamma(Q_T))^N, \quad \tau = \frac{N+p}{p-1} \),

(2.7) \( \gamma = \frac{N+2}{N+p}(p-1) \)

for almost every \( (x, t) \in Q_T \), for every \( s \in \mathbb{R} \) and every \( \xi, \eta \in \mathbb{R}^N \).

(2.8) \( f \in L^1(Q_T) \),

(2.9) \( u_0 \in L^1(\Omega) \) such that \( b(u_0) \in L^1(\Omega) \).

Throughout the paper, \( T_k \) denotes the truncation function at height \( k \geq 0 \):

\[
T_k(r) = \max(-k, \min(k, r))
\]
3. Main results

In this section, we study the existence of renormalized solutions to problem (1.1).

Definition 3.1. A measurable function \( u \) is a renormalized solution to problem (1.1), if

\[
\begin{align*}
(3.1) & \quad b(u) \in L^\infty((0, T), L^1(\Omega)). \\
(3.2) & \quad T_k(u) \in L^p((0, T), W^{1,p}_0(\Omega)), \text{ for any } k > 0, \\
(3.3) & \quad \lim_{n \to +\infty} \int_{\{ u \leq n \leq n+1 \}} a(x, t, u, \nabla u) \nabla u \, dx \, dt = 0,
\end{align*}
\]

and if for every function \( S \) in \( W^{2,\infty}(\mathbb{R}) \) which is piecewise \( C^1 \) and such that \( S' \) has a compact support

\[
\begin{align*}
(3.4) & \quad \frac{\partial B_S(u)}{\partial t} - \text{div} \left( a(x, t, u, \nabla u) S'(u) \right) + S''(u) a(x, t, u, \nabla u) \nabla u \\
& \quad + \text{div} \left( \phi(x, t, u) S'(u) \right) - S''(u) \phi(x, t, u) \nabla u = f S'(u) \quad \text{in} \quad D'(\Omega),
\end{align*}
\]

and

\[
(3.5) \quad B_S(u)(t = 0) = B_S(u_0) \quad \text{in} \quad \Omega,
\]

where

\[
B_S(z) = \int_0^z b'(s) S'(s) \, ds.
\]

Equation (3.4) is formally obtained through multiplication of (1.1) by \( S'(u) \). However while \( a(x, t, u, \nabla u) \) and \( \phi(x, t, u) \) does not in general make sense in (1.1), all the terms in (3.4) have a meaning in \( D'(Q_T) \). Indeed, if \( M \) is such that \( \text{supp} S' \subset [-M, M] \), the following identifications are made in (3.4):

- \( S'(u)a(x, t, u, \nabla u) \) identifies with \( S'(u) a(x, t, T_M(u), \nabla T_M(u)) \) a.e. in \( Q_T \).
- \( S''(u)a(x, t, u, \nabla u) \nabla u \) identifies with \( S''(u) a(x, t, T_M(u), \nabla T_M(u)) \nabla T_M(u) \) a.e. in \( Q_T \).
- \( S'(u)\phi(x, t, u) \) identifies with \( S'(u) \phi(x, t, T_M(u)) \) a.e. in \( Q_T \).
- \( S''(u)\phi(x, t, u) \nabla u \) identifies with \( S''(u) \phi(x, t, T_M(u)) \nabla T_M(u) \) a.e. in \( Q_T \).

The above consideration shows that equation (3.4) hold in \( D'(\Omega) \), \( \frac{\partial B_S(u)}{\partial t} \) belongs to \( L^1(Q) + L^{p'}(0, T, W^{-1,p'}(\Omega)) \) and \( B_S(u) \in L^p(0, T, W^{1,p}_0(\Omega)) \cap L^\infty(Q) \). It follows that \( B_S(u) \) belongs to \( C^0([0, T], L^1(\Omega)) \) so the initial condition (3.5) makes sense.

Theorem 3.1. Assume the assumption (H) hold, then problem (1.1) admits a renormalized solution \( u \) in the sense of Definition 3.1.

Proof. The proof is divided into six steps.

Step 1: Approximate problem and a priori estimates.

For each \( \epsilon > 0 \), we define the following approximations

\[
\begin{align*}
(3.6) & \quad b_\epsilon(r) = T_{\frac{1}{\epsilon}}(b(r)) + \epsilon r, \quad \forall \, r \in \mathbb{R}, \\
(3.7) & \quad a_\epsilon(x, t, s, \xi) = a(x, t, T_{\frac{1}{\epsilon}}(s), \xi) \text{ a.e. in } Q \quad \forall \, s \in \mathbb{R}, \forall \, \xi \in \mathbb{R}^N, \\
(3.8) & \quad \phi_\epsilon(x, t, r) = \phi(x, t, T_{\frac{1}{\epsilon}}(r)) \text{ a.e. in } Q_T, \forall \, r \in \mathbb{R}, \\
(3.9) & \quad f_\epsilon \in L^{p'}(Q_T) \text{ such that } f_\epsilon \to f \text{ strongly in } L^1(Q_T), \\
(3.10) & \quad u_{0\epsilon} \in D(\Omega) \text{ such that } b_\epsilon(u_{0\epsilon}) \to b(u_0) \text{ strongly in } L^1(\Omega),
\end{align*}
\]
Let us consider the approximate problem:

\[
\begin{align*}
\frac{\partial b_k(u_\epsilon)}{\partial t} - \text{div}(a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon)) + \text{div}(\phi_\epsilon(x, t, u_\epsilon)) &= f_\epsilon & \text{in} & & QT \\
& u_\epsilon(x, t) = 0 & \text{on} & & \partial \Omega \times (0, T) \\
& b_\epsilon(u_\epsilon(x, 0)) = b_\epsilon(u_{0\epsilon}(x)) & \text{in} & & \Omega.
\end{align*}
\]

(3.11)

As a consequence, proving existence of a weak solution \( u_\epsilon \in L^p((0, T), W^{1,p}_0(\Omega)) \) is an easy task (See [16]).

**Step 2:** The estimates derived in this step rely on standard techniques for problems of type (3.11).

Let \( \tau_1 \in (0, T) \) and \( t \) fixed in \( (0, \tau_1) \). Using in (3.11), \( T_k(u_\epsilon)\chi_{(0,t)} \) as test function, we integrate between \( (0, \tau_1) \), and by the condition (3.8) we have

\[
\int_{Q_\tau} B_k^e(u_\epsilon(t)) dx + \int a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) \nabla T_k(u_\epsilon) dx ds 
\]

(3.12)

\[
\leq \int_{Q_\tau} c(x, t) |u_\epsilon|^\gamma |\nabla T_k(u_\epsilon)| dx ds + \int_{Q_\tau} f_\epsilon T_k(u_\epsilon) dx ds + \int_{\Omega} B_k^e(u_{0\epsilon}) dx,
\]

where \( B_k^e(r) = \int_{0}^{r} T_k(s) b_\epsilon^e(s) ds \). Due to definition of \( B_k^e \) we have:

\[
0 \leq \int_{\Omega} B_k^e(u_{0\epsilon}) dx \leq k \int_{\Omega} |b_\epsilon(u_{0\epsilon})| dx \leq k ||b(u_0)||_{L^1(\Omega)} \quad \forall k > 0
\]

(3.13)

Using (3.12) and (2.3) we obtain:

\[
\int_{\Omega} B_k^e(u_\epsilon(t)) dx + \alpha \int_{Q_\tau} |\nabla T_k(u_\epsilon)|^p dx ds 
\]

(3.14)

\[
\leq \int_{Q_\tau} c(x, t) |u_\epsilon|^\gamma |\nabla T_k(u_\epsilon)| ds dx + k(||b(u_0)||_{L^1(\Omega)} + ||f_\epsilon||_{L^1(\Omega)})
\]

If we take the supremum for \( t \in (0, \tau_1) \) and we define \( M = \sup(||f_\epsilon||_{L^1(\Omega)} + ||b(u_0)||_{L^1(\Omega)}) \), we deduce from that above inequality (3.12) and (3.13),

\[
\frac{\beta}{2} \int_{\Omega} |T_k(u_\epsilon)|^2 dx + \alpha \int_{Q_\tau} |\nabla T_k(u_\epsilon)|^p dx ds \leq Mk + \int_{Q_\tau} c(x, t) |u_\epsilon|^\gamma |\nabla T_k(u_\epsilon)| dx ds
\]

(3.15)

By Gagliardo-Niremberg and Young inequalities we have:

\[
\int_{Q_{\tau_1}} c(x, t) |u_\epsilon|^\gamma |\nabla T_k(u_\epsilon)| dx ds \leq C \frac{\gamma}{N + 2} ||c(x, t)||_{L^p(Q_{\tau_1})} \sup_{t \in (0, \tau_1)} \int_{Q_{\tau_1}} |T_k(u_\epsilon)|^2 dx
\]

(3.16)

\[
+ C \frac{N + 2 + \gamma}{N + 2} ||c(x, t)||_{L^p(Q_{\tau_1})} \left( \int_{Q_{\tau_1}} |\nabla T_k(u_\epsilon)|^p dx ds \right)^\frac{1}{p} \left( \int_{Q_{\tau_1}} |T_k(u_\epsilon)|^2 dx \right)^\frac{N + 2}{p(N + 2 - \gamma)}.
\]

Since \( \gamma = \frac{(N + 2)(p - 1)}{N + p} \) and by using (3.15) and (3.16), we obtain

\[
\frac{\beta}{2} \int_{\Omega} |T_k(u_\epsilon)|^2 dx + \alpha \int_{Q_\tau} |\nabla T_k(u_\epsilon)|^p dx ds 
\]

\[
\leq Mk + C \frac{\gamma}{N + 2} ||c(x, t)||_{L^p(Q_{\tau_1})} \sup_{t \in (0, \tau_1)} \int_{\Omega} |T_k(u_\epsilon)|^2 dx
\]

(3.17)

\[
+ C \frac{N + 2 + \gamma}{N + 2} ||c(x, t)||_{L^p(Q_{\tau_1})} \int_{Q_{\tau_1}} |\nabla T_k(u_\epsilon)|^p dx ds
\]
Which is equivalent to
\[
\left( \frac{\beta}{2} - C \frac{\gamma}{N+2} \|c(x,t)\|_{L^\gamma(Q_{r_1})} \right) \sup_{t \in (0, \tau_1)} \int_{\Omega} |T_k(u_\epsilon)|^2 \, dx + \alpha \int_{Q_{r_1}} |\nabla T_k(u_\epsilon)|^p \, dx \, ds \\
- C \frac{N+2+\gamma}{N+2} \|c(x,t)\|_{L^\gamma(Q_{r_1})} \int_{Q_{r_1}} |\nabla T_k(u_\epsilon)|^p \, dx \, ds \leq Mk
\]
If we choose \( \tau_1 \) such that
\[
(3.17) \quad \left( \frac{\beta}{2} - C \frac{\gamma}{N+2} \|c(x,t)\|_{L^\gamma(Q_{r_1})} \right) \geq 0,
\]
and
\[
(3.18) \quad \left( \alpha - C \frac{N+2+\gamma}{N+2} \|c(x,t)\|_{L^\gamma(Q_{r_1})} \right) \geq 0,
\]
then, let us denote by \( C \) the minimum between (3.17) and (3.18), we obtain
\[
(3.20) \quad \sup_{t \in (0, \tau_1)} \int_{\Omega} |T_k(u_\epsilon)|^2 \, dx + \int_{Q_{r_1}} |\nabla T_k(u_\epsilon)|^p \, dx \, dt \leq CMk
\]
Then, by (3.19) and lemma 3.1, we conclude that \( T_k(u_\epsilon) \) is bounded in \( L^p(0, T, W_0^{1,p}(\Omega)) \) independently of \( \epsilon \) and for any \( k \geq 0 \), so there exists a subsequence still denoted by \( u_\epsilon \) such that
\[
(3.20) \quad T_k(u_\epsilon) \rightharpoonup \sigma_k \quad \text{in} \quad L^p(0, T, W_0^{1,p}(\Omega))
\]
We turn now to prove the almost every convergence of \( u_\epsilon \) and \( b_\epsilon(u_\epsilon) \). Let \( g_k \in C^2(\mathbb{R}) \) such that \( g_k(s) = s \) for \( |s| \leq \frac{k}{2} \) and \( g_k(s) = k \) for \( |s| \geq k \). Pointwise multiplication of the approximate equation (3.11) by \( g_k'(b_\epsilon(u_\epsilon)) \) leads to
\[
(3.21) \quad \frac{\partial g_k(b_\epsilon(u_\epsilon))}{\partial t} - div \left( a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) g_k'(b_\epsilon(u_\epsilon)) \right) + a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) g_k''(b_\epsilon(u_\epsilon)) b_\epsilon'(u_\epsilon) \nabla u_\epsilon + div \left( \phi_\epsilon(x, t, u_\epsilon) g_k'(b_\epsilon(u_\epsilon)) \right) - g_k''(b_\epsilon(u_\epsilon)) b_\epsilon'(u_\epsilon) \phi_\epsilon(x, t, u_\epsilon) \nabla u_\epsilon = f_{\epsilon} g_k(b_\epsilon(u_\epsilon)) \quad \text{in} \quad D'(\Omega)
\]
Now each term in (3.21) is taking into account because of (2.2), (3.7) and \( T_k(u_\epsilon) \) is bounded in \( L^p(0, T, W_0^{1,p}(\Omega)) \), we deduce that:
\[
- div \left( a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) g_k'(b_\epsilon(u_\epsilon)) \right) + a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) g_k''(b_\epsilon(u_\epsilon)) b_\epsilon'(u_\epsilon) \nabla u_\epsilon + f_{\epsilon} g_k'(b_\epsilon(u_\epsilon))
\]
is bounded in \( L^1(Q_{r_1}) + L^p(0, T, W^{-1,p}(\Omega)) \) independently of \( \epsilon \). Due to definition of \( b \) and \( b_\epsilon \), we have \( \{|b_\epsilon(u_\epsilon)| \leq k \} \subset \{|u_\epsilon| \leq k^* \} \) where \( k^* \) is a constant independent of \( \epsilon \). As a first consequence we have:
\[
Dg_k(b_\epsilon(u_\epsilon)) = g_k'(b_\epsilon(u_\epsilon)) b_\epsilon'(T_k^*(u_\epsilon)) DT^*_k(u_\epsilon) \quad \text{a.e in} \ Q
\]
as soon as \( k^* < \frac{1}{2} \). Secondly the following estimate hold true:
\[
\|g_k'(b_\epsilon(u_\epsilon)) b_\epsilon'(T_k^*(u_\epsilon))\|_{L^\infty(Q)} \leq \|g_k'\|_{L^\infty(Q)} \left( \max_{|r| \leq k^*} (b'(r) + 1) \right)
\]
As a consequence, we obtain:
\[
(3.22) \quad g_k(b_\epsilon(u_\epsilon)) \text{ is bounded in } L^p(0, T, W_0^{1,p}(\Omega)).
\]
Since \( \text{supp}(g_k'') \) and \( \text{supp}(g_k''') \) are both included in \([-k,k]\) by (3.8) it follows that for all \( \epsilon < \frac{1}{k} \) we have

\[
\left| \int_{Q_T} \phi_\epsilon(x, t, u_\epsilon) v' g_k''(b_\epsilon(u_\epsilon)) v' \ dx \ dt \right| \leq \int_{Q_T} c(x, t) v' |T^\frac{\alpha}{2}(u_\epsilon)| v' |g_k''(b_\epsilon(u_\epsilon))| v' \ dx \ dt \\
= \int_{\{|u_\epsilon| \leq k\}} c(x, t) v' |T_k(\epsilon)\epsilon(u_\epsilon)| v' |g_k''(b_\epsilon(u_\epsilon))| v' \ dx \ dt
\]

Furthermore, by Hölder and Gagliardo-Niremberg inequality, it results

\[
\int_{\{|u_\epsilon| \leq k\}} c(x, t) v' |T_k(\epsilon)\epsilon(u_\epsilon)| v' |g_k'(b_\epsilon(u_\epsilon))| v' \ dx \ dt \\
\leq \|g_k''\|_{L^\infty(\mathbb{R})} |c(x, t)| v'_{L^\infty(Q_T)} \left( \int_{\Omega} |T_k(\epsilon)\epsilon(u_\epsilon)|^2 \ dx \right)^{\frac{p}{2}} + \int_{Q_T} |\nabla T_k(\epsilon)\epsilon(u_\epsilon)|^p \ dx \ dt \leq c_k\epsilon
\]

where \( c_k \epsilon \) is a constant independently of \( \epsilon \) which will vary from line to line.

In the same by (3.8) we have:

(3.23)
\[
\left| \int_{Q_T} \phi_\epsilon(x, t, u_\epsilon) v' (g_k'''(b_\epsilon(u_\epsilon)) b_\epsilon'(u_\epsilon) \nabla u_\epsilon) v' \ dx \ dt \right| \\
\leq \int_{Q_T} (g_k'''(b_\epsilon(u_\epsilon)) b_\epsilon'(u_\epsilon) v' c(x, t) v' |T^\frac{\alpha}{2}(u_\epsilon)| v' |\nabla u_\epsilon| v' \ dx \ dt.
\]

Furthermore, by Hölder and Gagliardo-Niremberg inequality, we obtain for \( \epsilon < \frac{1}{k} \):

\[
\int_{Q_T} (g_k'''(b_\epsilon(u_\epsilon)) b_\epsilon'(u_\epsilon) v' c(x, t) v' |T_k(\epsilon)\epsilon(u_\epsilon)| v' |\nabla u_\epsilon| v' \ dx \ dt \\
= \int_{Q_T} (g_k'''(b_\epsilon(u_\epsilon)) b_\epsilon'(u_\epsilon) v' c(x, t) v' |T_k(\epsilon)\epsilon(u_\epsilon)| v' |\nabla T_k(\epsilon)\epsilon(u_\epsilon)| v' \ dx \ dt \\
\leq \|g_k''''\|_{L^\infty(\mathbb{R})} \sup_{|\gamma| \leq k'} \|b'(\gamma)| v' |c(x, t)| v' |T_k(\epsilon)\epsilon(u_\epsilon)| v' |\nabla T_k(\epsilon)\epsilon(u_\epsilon)| v' \ dx \ dt \leq c_k\epsilon
\]

We conclude by (3.24) that

(3.24)
\[
\frac{\partial g_k(b_\epsilon(u_\epsilon))}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0, T, W^{1,p'}(\Omega)).
\]

Arguing again as in [8], estimates (3.22) and (3.24) imply that, for a subsequence, still indexed by \( \epsilon \),

(3.25)
\[
u_\epsilon \to u \text{ a.e. } Q_T,
\]

where \( u \) is a measurable function defined on \( Q_T \).

Let us prove that \( \text{b}(\nu) \) belongs to \( L^\infty((0, T), L^1(\Omega)) \). We take \( T_k(b_\epsilon(u_\epsilon)) \) as test function in (3.11), by (3.8) we have

(3.26)
\[
\int_{\Omega} B_k(u_\epsilon) dx + \int_{Q_T} a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) \nabla T_k(b_\epsilon(u_\epsilon)) dx \ dt \\
\leq \int_{Q_T} |c(x, t)||T^\frac{\alpha}{2}(u_\epsilon)|^\gamma |\nabla T_k(b_\epsilon(u_\epsilon))| dx \ dt + k (||f_\epsilon||_{L^1(Q_T)} + ||b(u_0)||_{L^1(\Omega)}).
\]

with \( B_k(r) = \int_0^{b(r)} T_k(s) ds \). On the other hand, we have

(3.27)
\[
\int_{Q_T} a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) \nabla T_k(b_\epsilon(u_\epsilon)) dx \ ds
\]
we have

Using (3.13), (3.28) and (3.26) in (3.27), we have

Since

hence

∫

Proof. Using \( ψ_n(u_ε) \equiv T_{n+1}(u_ε) - T_n(u_ε) \) as a test function in (3.11), and by (3.8) we get

(3.29)

\[
\int_0^T \left( \frac{\partial b_ε(u_ε)}{\partial t}, ψ_n(u_ε) \right) dt + \int_{Q_T} a_ε(x, t, u_ε, \nabla u_ε) \nabla ψ_n(u_ε) dx dt \\
\leq \int_Ω c(x, t) |T_{1/2}(u_ε)| |\nabla ψ_n(u_ε)| dx dt + \int_{Q_T} f_ε ψ_n(u_ε) dx dt
\]

hence

(3.30)

\[
\int_Ω a_ε(x, t, u_ε, \nabla u_ε) \nabla ψ_n(u_ε) dx dt \leq \int_{Q_T} c(x, t) |T_{1/2}(u_ε)| |\nabla ψ_n(u_ε)| dx dt \\
+ \int_Ω B_n(u_0_ε) dx + \int_{Q_T} f_ε ψ_n(u_ε) dx dt.
\]
Proceeding as in ([6], [8]) it can be deduced from (3.30) that
\begin{equation}
\psi_n(u_\epsilon) \rightharpoonup \psi_n(u) \text{ weakly in } L^p(0, T, W^{1,p}_0(\Omega)).
\end{equation}

We have \( \nabla \psi_n(u_\epsilon) = \chi_{\{n \leq |u_\epsilon| \leq n+1\}} \nabla u_\epsilon \) a.e in \( Q_T \), by Young inequality and (2.3), we obtain
\begin{align}
\frac{\alpha}{p'} \int_{Q_T} |\nabla \psi_n(u_\epsilon)|^p \, dx \, dt & \leq \int_{Q_T} f^\ast \psi_n(u_\epsilon) \, dx \, dt + \int_{\Omega} B_n(u_{0_\epsilon}) \, dx \\
& + \frac{\alpha - p'}{p'} \int_{\{n \leq |u_\epsilon| \leq n+1\}} c(x, t) |T_{n+1}(u_\epsilon)| \gamma \, dx \, dt.
\end{align}

Using the weakly convergence of \( \psi_n(u_\epsilon) \), the pointwise convergence of \( u_\epsilon \) and the strongly convergence in \( L^1 \) of \( f^\ast \) and \( B_n(u_{0_\epsilon}) \), it follows that
\begin{align}
\frac{\alpha}{p'} \int_{Q_T} |\nabla \psi_n(u)|^p \, dx \, dt & \leq \int_{Q_T} f \psi_n(u) \, dx \, dt + \int_{\Omega} B_n(u_0) \, dx \\
& + \frac{\alpha - p'}{p'} \int_{\{n \leq |u| \leq n+1\}} c(x, t) |T_{n+1}(u)| \gamma \, dx \, dt.
\end{align}

The last inequality, together with the assumptions (2.8), (2.9), shows that \( \psi_n(u) \) is bounded in \( L^p(0, T, W^{1,p}_0(\Omega)) \) independently of \( n \). Thanks to the pointwise convergence of \( \psi_n(u) \) to 0 and weakly in \( L^p(0, T, W^{1,p}_0(\Omega)) \) as \( n \to +\infty \), we deduce that
\[ \lim_{n \to +\infty} \int_{Q_T} f \psi_n \, dx \, dt = 0, \]
and
\[ \lim_{n \to +\infty} \int_{\Omega} (c(x, t))^{p'} |u|^{p'} \, dx \, dt = 0. \]

Moreover
\[ |B_n(u_0)| \leq c \int_0^{u_0} \psi_n(s) \, ds \to 0 \text{ a.e as } n \to +\infty, \]
and \( |B_n(u_0)| \leq |b(u_0)| \) a.e. in \( \Omega \), since \( b(u_0) \in L^1(\Omega) \), by Lebesgue’s convergence theorem we obtain
\[ \lim_{n \to +\infty} \int_{\Omega} B_n(u_0) \, dx = 0. \]

Therefore
\[ \lim_{n \to +\infty} \int_{Q_T} |\nabla \psi_n(u)|^p \, dx \, dt = 0, \]
then
\begin{equation}
\psi_n(u) \to 0 \text{ strongly in } L^p(0, T, W^{1,p}_0(\Omega)).
\end{equation}

Finally, passing to the limit in (3.30) as \( n \to +\infty \), we get
\begin{equation}
\text{(3.34)}
\end{equation}
**Step 4:** In this step we introduce a time regularization of the $T_k(u)$ for $k > 0$ in order to perform the monotonicity method. This kind regularization has been introduced at the first time by R. Landes in [15]. Let $ν^μ_0$ be a sequence of function in $L^∞(Ω) \cap W^{1,p}_0(Ω)$ such that $||ν^μ_0||_L^∞(Ω) \leq k$ for all $μ > 0$ and $ν^μ_0$ converges to $T_k(u_0)$ a.e. in $Ω$ and $\frac{1}{p}||ν^μ_0||_{L^p(Ω)}$ converges to 0. For $k \geq 0$ and $μ > 0$, let us consider the unique solution $(T_k(u))_μ \in L^∞(Q) \cap L^p(0,T : W^{1,p}_0(Ω))$ of the monotone problem:

\[
\frac{∂(T_k(u))_μ}{∂t} + ω((T_k(u))_μ - T_k(u)) = 0 \text{ in } D'(Ω),
\]

\[
(T_k(u))_μ(t = 0) = ν^μ_0 \text{ in } Ω.
\]

Remark that $(T_k(u))_μ \to T_k(u)$ a.e. in $Q_T$, weakly-* in $L^∞(Q)$ and strongly in $L^p((0,T), W^{1,p}_0(Ω))$ as $μ \to +∞$

\[
|||T_k(u)|||_{L^∞(Q)} \leq \max(|||T_k(u)|||_{L^∞(Q)}, |||ν^μ_0|||_{L^∞(Ω)}) \leq k, \forall \, μ > 0 \, , \forall \, k > 0
\]

**Lemma 3.3.** (see H. Redwane [18]) Let $k \geq 0$ be fixed. Let $S$ be an increasing $C^∞(ℝ)$—function such that $S(r) = r$ for $|r| \leq k$, and $suppS′$ is compact. Then

\[
\liminf_{μ \to +∞} \liminf_{ε \to 0} \int_0^T \int_0^t \frac{∂b_k(u_ε)}{∂t}, S′(u_ε)(T_k(u_ε) - (T_k(u))_μ) ≥ 0.
\]

where $< , , >$ denotes the duality pairing between $L^1(Ω) + W^{-1,p′}(Ω)$ and $L^∞(Ω) \cap W^{1,p}(Ω)$.  

**Step 5:** We prove the following lemma which is the critical point in the development of the monotonic method.

**Lemma 3.4.** The subsequence of $u_ε$ satisfies for any $k ≥ 0$

\[
limsup_{ε \to 0} \int_0^T \int_0^t a(x,t,u_ε, \nabla T_k(u_ε)) \nabla T_k(u_ε) ≤ \int_0^T \int_0^t \int_0^T \int_0^t \int_0^T \int_0^t \sigma_k \nabla T_k(u_ε).
\]

**Proof.** Let $S_n$ be a sequence of increasing $C^∞$—function such that

\[
S_n(r) = r \text{ for } |r| ≤ n, \text{ supp}(S_n′) ⊂ (-(n + 1), (n + 1)) \text{ and } |||S_n′|||_{L^∞(ℝ)} ≤ 1 \text{ for any } n ≥ 1.
\]

We use the sequence $(T_k(u))_μ$ of approximation of $T_k(u)$, and plug the test function $S_n′(T_k(u_ε)) \nabla T_k(u_ε) - (T_k(u))_μ$ for $n > 0$ and $μ > 0$. For fixed $k ≥ 0$, let $W_ε^k = T_k(u_ε) - (T_k(u))_μ$

we obtain upon integration over $(0,t)$ and then over $(0,T)$:

\[
\int_0^T \int_0^t \frac{∂b_k(u_ε)}{∂t}, S_n′(u_ε) W_ε^k \, ds \, dt + \int_0^T \int_0^t \int_0^T \int_0^t a_ε(x,t,u_ε, \nabla u_ε) S_n′′(u_ε) \nabla W_ε^k \, dx \, ds \, dt
\]

\[
+ \int_0^T \int_0^t \int_0^T \int_0^t a_ε(x,t,u_ε, \nabla u_ε) S_n′′(u_ε) \nabla u_ε \nabla W_ε^k \, dx \, ds \, dt
\]

\[
- \int_0^T \int_0^t \int_0^T \int_0^t \phi_ε(x,t,u_ε) S_n′′(u_ε) \nabla W_ε^k \, dx \, ds \, dt
\]

\[
- \int_0^T \int_0^t \int_0^T \int_0^t \phi_ε(x,t,u_ε) \nabla u_ε \nabla W_ε^k \, dx \, ds \, dt = \int_0^T \int_0^t \int_0^T \int_0^t \int_0^T \int_0^t f_ε S_n′′(u_ε) W_ε^k \, dx \, ds \, dt.
\]

Now we pass to the limit in (3.36) as $ε \to 0$, $μ \to +∞$ and then $n \to +∞$ for $k$ real number fixed. In order to perform this task we prove below the following results for any fixed $k ≥ 0$

\[
\liminf_{μ \to +∞} \liminf_{ε \to 0} \int_0^T \int_0^t \frac{∂b_k(u_ε)}{∂t}, W_ε^k > ds \, dt \geq 0 \quad \text{for any } n ≥ k,
\]
\[(3.40)\quad \lim_{\mu \to +\infty} \lim_{\epsilon \to 0} \int_0^T \int_0^t \phi_\epsilon(x, t, u_\epsilon) S'_n(u_\epsilon) \nabla W^\epsilon_\mu dx \, ds \, dt = 0 \quad \text{for any } n \geq 1,\]

\[(3.41)\quad \lim_{\mu \to +\infty} \lim_{\epsilon \to 0} \int_0^T \int_0^t f_\epsilon(x, t, u_\epsilon) W^\epsilon_\mu dx \, ds \, dt = 0.\]

**Proof of (3.37):** The function \(S_n\) belongs \(C^\infty(\mathbb{R})\) and is increasing. We have \(n \geq k\), \(S_n(r) = r\) for \(|r| \leq k\) while \(\text{supp} S'_n\) is compact. In view of the definition of \(W^\epsilon_\mu\) and lemma 3.3 applies with \(S = S_n\) for fixed \(n \geq k\). As a consequence \(3.37\) holds true.

**Proof of (3.38):** Let us recall the main properties of \(W^\epsilon_\mu\). For fixed \(\mu > 0\): \(W^\epsilon_\mu\) converges to \(T_k(u) - (T_k(u))_\mu\) weakly in \(L^p(0, T, W^{1,p}_0(\Omega))\) as \(\epsilon \to 0\). Remark that

\[(3.42)\quad ||W^\epsilon_\mu||_{L^\infty(Q_T)} \leq 2k \quad \text{for any } \epsilon > 0, \mu > 0,\]

then we deduce that

\[(3.43)\quad W^\epsilon_\mu \to T_k(u) - (T_k(u))_\mu \quad \text{a.e. in } Q_T \text{ and } L^\infty(Q_T)\]

weakly-\(^*\) when \(\epsilon \to 0\). One had \(\text{supp} S'_n \subset \left[-(n+1), -n\right] \cup \left[n, n+1\right]\) for any fixed \(n \geq 1\) and \(0 < \epsilon < \frac{1}{n+1}\).

\[\phi_\epsilon(x, t, u_\epsilon) S'_n(u_\epsilon) \nabla W^\epsilon_\mu = \phi_\epsilon(x, t, T_{n+1}(u_\epsilon)) S'_n(u_\epsilon) \nabla W^\epsilon_\mu \quad \text{a.e. in } Q_T\]

since \(\text{supp} S' \subset \left[-(n+1), n+1\right]\), on the other hand

\[\phi_\epsilon(x, t, T_{n+1}(u_\epsilon)) S'_n(u_\epsilon) \to \phi(x, t, T_{n+1}(u)) S'_n(u) \quad \text{a.e. in } Q_T\]

and

\[|\phi_\epsilon(x, t, T_{n+1}(u_\epsilon)) S'_n(u_\epsilon)| \leq c(x, t)(n+1)^\gamma \quad \text{for any } n \geq 1\]

by \(3.43\) and strongly convergence of \(T_k(u_\epsilon)_\mu\) in \(L^p(0, T, W^{1,p}_0(\Omega))\) we obtain \(3.38\).

**Proof of (3.39):** For any fixed \(n \geq 1\) and \(0 < \epsilon < \frac{1}{n+1}\):

\[\phi_\epsilon(x, t, u_\epsilon) S''_n(u_\epsilon) \nabla u_\epsilon W^\epsilon_\mu = \phi_\epsilon(x, t, T_{n+1}(u_\epsilon)) S''_n(u_\epsilon) \nabla T_{n+1}(u_\epsilon) W^\epsilon_\mu \quad \text{a.e. in } Q_T\]

as in the previous step it is possible to pass to the limit for \(\epsilon \to 0\) since by \(3.42\) and \(3.43\)

\[\phi_\epsilon(x, t, T_{n+1}(u_\epsilon)) S''_n(u_\epsilon) W^\epsilon_\mu \to \phi(x, t, T_{n+1}(u)) S''_n(u_\epsilon) W^\epsilon_\mu \quad \text{a.e. in } Q_T.\]

Since \(|\phi(x, t, T_{n+1}(u_\epsilon)) S''_n(u_\epsilon) W^\epsilon_\mu| \leq 2k|c(x, t)|(n+1)^\gamma \quad \text{a.e. in } Q_T\) and \((T_k(u))_\mu\) converges to 0 in \(L^p(0, T; W^{1,p}_0(\Omega))\), we obtain \(3.39\).

**Proof of (3.40):** In view of the definition of \(S_n\) we have \(\text{supp} S' \subset \left[-(n+1), -n\right] \cup \left[n, n+1\right]\) for any \(n \geq 1\), as a consequence

\[\left| \int_0^T \int_0^t \int_\Omega a(x, t, u_\epsilon, \nabla u_\epsilon) S''_n(u_\epsilon) W^\epsilon_\mu dx \, ds \, dt \right|\]

\[\leq T ||S''_n(u_\epsilon)||_{L^\infty(\mathbb{R})} ||W^\epsilon_\mu||_{L^\infty(Q_T)} \int_{\max \{|u_\epsilon| | u_\epsilon| \leq n+1\}} a(x, t, u_\epsilon, \nabla u_\epsilon) \nabla u_\epsilon dx \, ds \, dt\]
for any \( n \geq 1 \), any \( 0 < \epsilon < \frac{1}{n+1} \) and any \( \mu > 0 \). By (3.35) it is possible to establish (3.40).

**Proof of (3.41):** By (3.9), the pointwise convergence of \( u_\epsilon \) and \( W_\mu^\epsilon \) and its boundedness it is possible to pass the limit for \( \epsilon \to 0 \) for any \( \mu > 0 \) and any \( n \geq 1 \)

\[
\lim_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega f_n^\epsilon(u_\epsilon)(T_k(u) - (T_k(u))) dx \, ds \, dt = \int_0^T \int_0^t \int_\Omega f_n^\epsilon(u_\epsilon)(T_k(u) - (T_k(u))) dx \, ds \, dt.
\]

Now for fixed \( n \geq 1 \), using that \( \| (T_k(u))_\mu \|_{L^\infty(Q)} \leq \max\{\| (T_k(u))_\mu \|_{L^\infty(Q)}, \| \nu_\mu^\epsilon \|_{L^\infty(Q)} \} \leq k, \forall \mu > 0, \forall k > 0 \) (see [18]), it possible to pass to the limit as \( \mu \) tends to +\( \infty \) in the above inequality.

Now we turn back to the proof of lemma 3.4. Due to (3.37)-(3.41) we can to pass to the limit-sup when \( \mu \) tends to +\( \infty \) and to the limit as \( n \) tends to +\( \infty \) in (3.36), using the definition of \( W_\mu^\epsilon \) we deduce that for any \( k \geq 0 \)

\[
\lim_{n \to +\infty} \limsup_{\mu \to +\infty} \limsup_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega S_n^\epsilon(u_\epsilon) a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) (\nabla T_k(u_\epsilon) - \nabla (T_k(u)_\mu)) dx \, ds \, dt \leq 0.
\]

Since \( S_n^\epsilon(u_\epsilon) a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) \nabla T_k(u_\epsilon) = a(x, t, u_\epsilon, \nabla u_\epsilon) \nabla T_k(u_\epsilon) \) for \( k \leq \frac{1}{\epsilon} \) and \( k \leq n \), using the properties of \( S_n^\epsilon \) the above inequality implies that for \( k \leq n \)

\[
\text{(3.44)} \quad \lim_{n \to +\infty} \limsup_{\mu \to +\infty} \limsup_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) (\nabla T_k(u_\epsilon)) dx \, ds \, dt
\]

On the other hand, for \( \epsilon < \frac{1}{n+1} \)

\[
S_n^\epsilon(u_\epsilon) a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) = S_n^\epsilon(u_\epsilon) a(x, t, T_{n+1}(u_\epsilon), \nabla T_{n+1}(u_\epsilon)) \quad \text{a.e. in } Q_T.
\]

Furthermore we have

\[
\text{(3.45)} \quad a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) \rightharpoonup \sigma_k \quad \text{weakly in } (L^p(Q_T))^N
\]

it follows that for a fixed \( n \geq 1 \)

\[
S_n^\epsilon(u_\epsilon) a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) \rightharpoonup S_n^\epsilon(u_\epsilon) \sigma_{n+1} \quad \text{weakly in } L^p(Q_T)
\]

when \( \epsilon \) tends to 0. Finally, using the strong convergence of \( (T_k(u)_\mu) \) to \( T_k(u) \) in \( L^p(0, T, W_0^{1,p}(\Omega)) \) as \( \mu \) tends to +\( \infty \), we get

\[
\text{(3.46)} \quad \lim_{\epsilon \to 0} \lim_{\mu \to +\infty} \int_0^T \int_0^t \int_\Omega S_n^\epsilon(u_\epsilon) a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) \nabla (T_k(u_\epsilon)) dx \, ds \, dt
\]

as soon as \( k \leq n \). Now for \( k \leq n \) we have

\[
a(x, t, T_{n+1}(u_\epsilon), \nabla T_{n+1}(u_\epsilon)) \chi_{\{|u_\epsilon| \leq k\}} = a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \chi_{\{|u_\epsilon| \leq k\}} \quad \text{a.e. in } Q_T
\]

which implies that, by (3.25), (3.45), and by passing to the limit when \( \epsilon \) tends to 0,

\[
\text{(3.47)} \quad \sigma_{n+1} \chi_{\{|u| \leq k\}} = \sigma_k \chi_{\{|u| \leq k\}} \quad \text{a.e. in } Q_T - \{|u| = k\} \quad \text{for } k \leq n
\]

Finally, by (3.47) and (3.45) we have for \( k \leq n : \sigma_{n+1} \nabla T_k(u) = \sigma_k \nabla T_k(u) \quad \text{a.e. in } Q_T.
\]

Recalling (3.44), (3.46) the proof of the lemma is complete.

**Step 6:** In this step we prove that the weak limit \( \sigma_k \) of \( a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \) can be identified with \( a(x, t, T_k(u), \nabla T_k(u)) \). In order to prove this result we recall the following lemma:
Lemma 3.5. the subsequence of $u_\epsilon$ defined in Step 1 satisfies for any $k \geq 0$

\[(3.48)\]
\[
\lim_{\epsilon \to 0} \int_t^T \int_0^1 \int_{\Omega} \left( a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) - a(x, t, T_k(u), \nabla T_k(u)) \right) \left( \nabla T_k(u_\epsilon) - \nabla T_k(u) \right) = 0
\]

\[\text{Proof.}\] Using (2.3) we have

\[(3.49)\]
\[
\lim_{\epsilon \to 0} \int_t^T \int_0^1 \int_{\Omega} \left( a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) - a(x, t, T_k(u), \nabla T_k(u)) \right) \left( \nabla T_k(u_\epsilon) - \nabla T_k(u) \right) \geq 0.
\]

Furthermore, by (2.2), (3.48) we have

\[
a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \to a(x, t, T_k(u), \nabla T_k(u)) \quad \text{a.e. in } Q_T,
\]

and

\[
|a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon))| \leq \nu |h(x, t) + |\nabla T_k(u_\epsilon)|^{p-1}| \quad \text{a.e. in } Q_T,
\]

uniformly with respect to $\epsilon$. As a consequence

\[(3.50)\]
\[
a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \to a(x, t, T_k(u), \nabla T_k(u)) \quad \text{strongly in } (L^p(Q_T))^N.
\]

Finally, using (3.25), (3.45) and (3.50) make it possible to pass to the limit $- \sup$ as $\epsilon$ tends to 0 in (3.49) and we have (3.48). $\blacksquare$

Lemma 3.6. For fixed $k \geq 0$, we have

\[(3.51)\]
\[
\sigma_k = a(x, t, T_k(u), \nabla T_k(u)) \quad \text{a.e. in } Q_T,
\]

and as $\epsilon$ tends to 0

\[(3.52)\]
\[
a(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \nabla T_k(u_\epsilon) \to a(x, t, T_k(u), \nabla T_k(u)) \nabla T_k(u)
\]

weakly in $L^1(Q_T)$.

\[\text{Proof.}\] We observe that for any $k > 0$, any $0 < \epsilon < \frac{1}{k}$ and any $\xi \in \mathbb{R}^N$:

\[
a_\epsilon(x, t, T_k(u_\epsilon), \xi) = a(x, t, T_k(u_\epsilon), \xi) = a_\frac{\epsilon}{k}(x, t, T_k(u_\epsilon), \xi) \quad \text{a.e. in } Q_T.
\]

Since

\[(3.53)\]
\[
T_k(u_\epsilon) \to T_k(u) \quad \text{weakly in } L^p((0, T), W^p_0(\Omega)),
\]

and by (3.48) we obtain

\[(3.54)\]
\[
\lim_{\epsilon \to 0} \int_t^T \int_0^1 \int_{\Omega} a_\frac{\epsilon}{k}(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \nabla T_k(u_\epsilon) \ dx \, ds \, dt = \int_t^T \int_0^1 \int_{\Omega} \sigma_k \nabla T_k(u) \ dx \, ds \, dt.
\]

Since, for fixed $k > 0$, the function $a_\frac{\epsilon}{k}(x, t, s, \xi)$ is continuous and bounded with respect to $s$, the usual Minty’s argument applies in view of (3.53), (3.45) and (3.54). It follows that (3.51) holds true. In order to prove (3.54), by (2.3), (3.48) and proceeding as in [5, 8] it’s easy to show (3.52). $\blacksquare$

Taking the limit as $\epsilon$ tends to 0 in (3.35) and using (3.52) show that $u$ satisfies (3.3). Our aim is to prove that $u$ satisfies (3.4) and (3.5). Now we want to prove that $u$ satisfies the equation (3.4).

Let $S$ be a function in $W^{2,\infty}(\mathbb{R})$ such that $supp S' \subset [-k, k]$ where $k$ is a real positive number. Pointwise multiplication of the approximate equation (3.11) by $S'(u_\epsilon)$ leads to

\[(3.55)\]
\[
\frac{\partial B_k(u_\epsilon)}{\partial t} - div \left( a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon) S'(u_\epsilon) \right) + S''(u_\epsilon) a(x, t, u_\epsilon, \nabla u_\epsilon) \nabla u_\epsilon
\]
Using the weakly convergence of $T\phi S(u_\epsilon)$ implies that $B_S(u_\epsilon)$ converge to $B_S(u)$ a.e. in $Q_T$ and $L^\infty(Q_T)$ weak-*. Then $\frac{\partial B_S(u_\epsilon)}{\partial t}$ converges to $\frac{\partial B_S(u)}{\partial t}$ in $D'(\Omega)$. We observe that the term $a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon)$ can be identified with $a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))S'(u_\epsilon)$ for $\epsilon \leq \frac{1}{k}$, so using the pointwise convergence of $u_\epsilon \to u$ in $Q_T$, the weakly convergence of $T_k(u_\epsilon) \to T_k(u)$ in $L^p((0,T),W^p_0(\Omega))$, we get

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon) \to a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))S'(u) \quad \text{in} \quad L^p(Q_T),$$

and

$$S''(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \to S''(u)a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\nabla T_k(u) \quad \text{in} \quad L^1(Q_T).$$

Furthermore, since $\phi(x,t,u_\epsilon)S'(u_\epsilon) = \phi(x,t,T_k(u_\epsilon))S'(u_\epsilon)$ a.e. in $Q_T$. By (3.8) we obtain $|\phi(x,t,T_k(u_\epsilon))S'(u_\epsilon)| \leq |c(x,t)|k^{\gamma}$, it follows that

$$\phi(x,t,T_k(u_\epsilon))S'(u_\epsilon) \to \phi(x,t,T_k(u))S'(u) \quad \text{strongly in} \quad L^p(Q_T).$$

In a similar way, it results

$$S''(u_\epsilon)\phi(x,t,u_\epsilon)\nabla u_\epsilon = S''(T_k(u_\epsilon))\phi(x,t,T_k(u_\epsilon))\nabla T_k(u_\epsilon) \quad \text{a.e. in} \quad Q_T.$$

Using the weakly convergence of $T_k(u_\epsilon)$ in $L^p((0,T);W^p_0(\Omega))$ it is possible to prove that

$$S''(u_\epsilon)\phi(x,t,u_\epsilon)\nabla u_\epsilon \to S''(u)\phi(x,t,u)\nabla u \quad \text{in} \quad L^1(Q_T).$$

Finally by (3.9) we deduce that $f_\epsilon S'(u_\epsilon)$ converges to $f S'(u)$ in $L^1(Q_T)$.

It remains to prove that $B_S(u)$ satisfies the initial condition $B_S(t = 0) = B_S(u_0)$ in $\Omega$. To this end, firstly remark that $S$ being bounded, $B_S(u_\epsilon)$ is bounded in $L^\infty(Q)$. Secondly the above considerations of the behavior of the terms of this equation show that $\frac{\partial B_S(u_\epsilon)}{\partial t}$ is bounded in $L^1(Q_T) + L^p((0,T);W^{-1,p'}(\Omega))$. As a consequence, an Aubin’s type lemma (See e.g [20]) implies that $B_S(u_\epsilon)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. On the other hand, the smoothness of of $S$ implies that $B_S(t = 0) = B_S(u_0)$ in $\Omega$. [1]

**References**


