The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 10, Issue 1, Article 13, pp. 1-3, 2013

NEW SOLUTIONS TO NON-SMOOTH PDES

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Received 13 May, 2013; accepted 29 July, 2013; published 4 November, 2013.

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ABSTRACT. We provide strong solutions to partial differential equations when the function is non-differentiable.

Key words and phrases: HJB PDEs, Strong solution, Viscosity solutions.

2000 *Mathematics Subject Classification.* 35-xx.

ISSN (electronic): 1449-5910

This note overcomes a major obstacle in the area of (stochastic) partial differntial equations and their applications. In so doing, it provides strong solutions to partial differential equations when the function is non-differentiable. It is established that if the function is non-differentiable, the existing methods adopt viscosity and minimax weak solutions (see, for example, Crandall and Lions (1983), among many others). Below is a description of the method.

We express the function $H(x)$ as $H(x+\epsilon)$, where ϵ is a shift parameter with an initial value equal to zero (see Alghalith (2008), among others). If $H(x)$ is differentiable with respect to $x,$

we have

$$H_x = H_\epsilon; H_{xx} = H_{\epsilon\epsilon},$$

where the subscript denotes a partial derivative. Therefore we can substitute H_δ for H_x even if H is not differentiable with respect to $x.$

Consider this function $H(x,y)$; it can be expressed as $H(x+\epsilon, \varphi y)$, where φ is a shift parameter with an initial value equal to one (see Alghalith (2008), among others). We define $f \equiv x+\epsilon, g \equiv \varphi y$; differentiating $H(f,g)$ with respect to x and ϵ, respectively, yields

$$H_x = H_f = H_\epsilon; H_{xx} = H_{ff} = H_{\epsilon\epsilon.} \tag{1.1}$$

Similarly, differentiating $H(f,g)$ with respect to φ and y, respectively, yields

$$H_\varphi = H_g y; H_y = H_g \varphi.$$

Thus

$$\frac{H_y}{H_\varphi} = \frac{\varphi}{y} \Rightarrow H_y = \frac{\varphi H_\varphi}{y}. \tag{1.2}$$

It is also clear that the second derivatives of $H(g,y)$ with respect to φ and y, respectively, are

$$H_{\varphi\varphi} = H_{gg} y^2; H_{yy} = \varphi^2 H_{gg}. \tag{1.3}$$

Therefore

$$\frac{H_{yy}}{H_{\varphi\varphi}} = \frac{y^2}{\varphi^2} \Rightarrow H_{yy} = \frac{\varphi^2 H_{\varphi\varphi}}{y^2}. \tag{1.4}$$

Using (1.2), we obtain

$$H_{yx} = \frac{\varphi H_{\varphi f}}{y} = \frac{\varphi H_{\varphi x}}{y} = \frac{\varphi H_{\varphi\epsilon}}{y}. \tag{1.5}$$

For example, we consider the following known Hamilton-Jacobi-Bellman PDE (however our approach is virtually applicable to any form of PDEs)

$$H_x + +a(.)\, H_y + b(.)\, H_{yy} + e(.)\, H_{xy} = 0. \tag{1.6}$$

Substituting $(1.1) - (1.5)$ into (1.6) yields

$$H_\epsilon + a(.)\, \frac{\varphi H_\varphi}{y} + b(.)\, \varphi^2 H_{gg} + e(.)\, \frac{\varphi H_{\varphi\epsilon}}{y} = 0.$$

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