NEW SOLUTIONS TO NON-SMOOTH PDES

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ABSTRACT. We provide strong solutions to partial differential equations when the function is non-differentiable.

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This note overcomes a major obstacle in the area of (stochastic) partial differential equations and their applications. In so doing, it provides strong solutions to partial differential equations when the function is non-differentiable. It is established that if the function is non-differentiable, the existing methods adopt viscosity and minimax weak solutions (see, for example, Crandall and Lions (1983), among many others). Below is a description of the method.

We express the function \( H(x) \) as \( H(x + \epsilon) \), where \( \epsilon \) is a shift parameter with an initial value equal to zero (see Alghalith (2008), among others). If \( H(x) \) is differentiable with respect to \( x \), we have

\[
H_x = H_x^\epsilon; \quad H_{xx} = H_{x\epsilon},
\]

where the subscript denotes a partial derivative. Therefore we can substitute \( H_\delta \) for \( H_x \) even if \( H \) is not differentiable with respect to \( x \).

Consider this function \( H(x, y) \); it can be expressed as \( H(x + \epsilon, \varphi y) \), where \( \varphi \) is a shift parameter with an initial value equal to one (see Alghalith (2008), among others). We define \( f \equiv x + \epsilon, g \equiv \varphi y \); differentiating \( H(f, g) \) with respect to \( x \) and \( \epsilon \), respectively, yields

\[
(1.1) \quad H_x = H_f = H_\epsilon; \quad H_{xx} = H_{ff} = H_{x\epsilon}.
\]

Similarly, differentiating \( H(f, g) \) with respect to \( \varphi \) and \( y \), respectively, yields

\[
H_\varphi = H_g y; \quad H_y = H_g \varphi.
\]

Thus

\[
(1.2) \quad \frac{H_y}{H_\varphi} = \frac{\varphi}{y} \Rightarrow H_y = \frac{\varphi H_\varphi}{y}.
\]

It is also clear that the second derivatives of \( H(g, y) \) with respect to \( \varphi \) and \( y \), respectively, are

\[
(1.3) \quad H_{\varphi\varphi} = H_{gg} y^2; \quad H_{yy} = \varphi^2 H_{gg}.
\]

Therefore

\[
(1.4) \quad \frac{H_{yy}}{H_{\varphi\varphi}} = \frac{y^2}{\varphi^2} \Rightarrow H_{yy} = \frac{\varphi^2 H_{\varphi\varphi}}{y^2}.
\]

Using (1.2), we obtain

\[
(1.5) \quad H_{yx} = \frac{\varphi H_{xf}}{y} = \frac{\varphi H_{\varphi x}}{y} = \frac{\varphi H_{x\epsilon}}{y}.
\]

For example, we consider the following known Hamilton-Jacobi-Bellman PDE (however our approach is virtually applicable to any form of PDEs)

\[
(1.6) \quad H_x + a(.) H_y + b(.) H_{yy} + e(.) H_{xy} = 0.
\]

Substituting (1.1) – (1.5) into (1.6) yields

\[
H_\epsilon + a(.) \frac{\varphi H_\varphi}{y} + b(.) \varphi^2 H_{gg} + e(.) \frac{\varphi H_{x\epsilon}}{y} = 0.
\]
REFERENCES
