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UNIFORM CONTINUITY AND K-CONVEXITY

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ABSTRACT. A closed arcwise-connected subset A of R^n is called k-convex if for each positive number a and for all elements x and y in A there is a positive number b such that if the norm of $x - y$ is less than or equal to b then the length of the shortest curve $l(x, y)$ in A is less than k times the norm of $x - y$ plus a. We show that a union of two non disjoint closed finite convex subsets need not be $k-$convex. Let $f(x)$ be a uniformly continuous functions on a finite number of closed subsets $A_1, ..., A_n$ of R^n such that the union of $A_j, ..., A_n, j = 1, ..., n - 1$is k-convex. We show that f is uniformly continuous on the union of the sets $A_i, i = 1, ..., n$. We give counter examples if this condition is not satisfied. As a corollary we show that if $f(x)$ is uniformly continuous on each of two closed convex sets A, B then $f(x)$ is uniformly continuous on the union of A and B.

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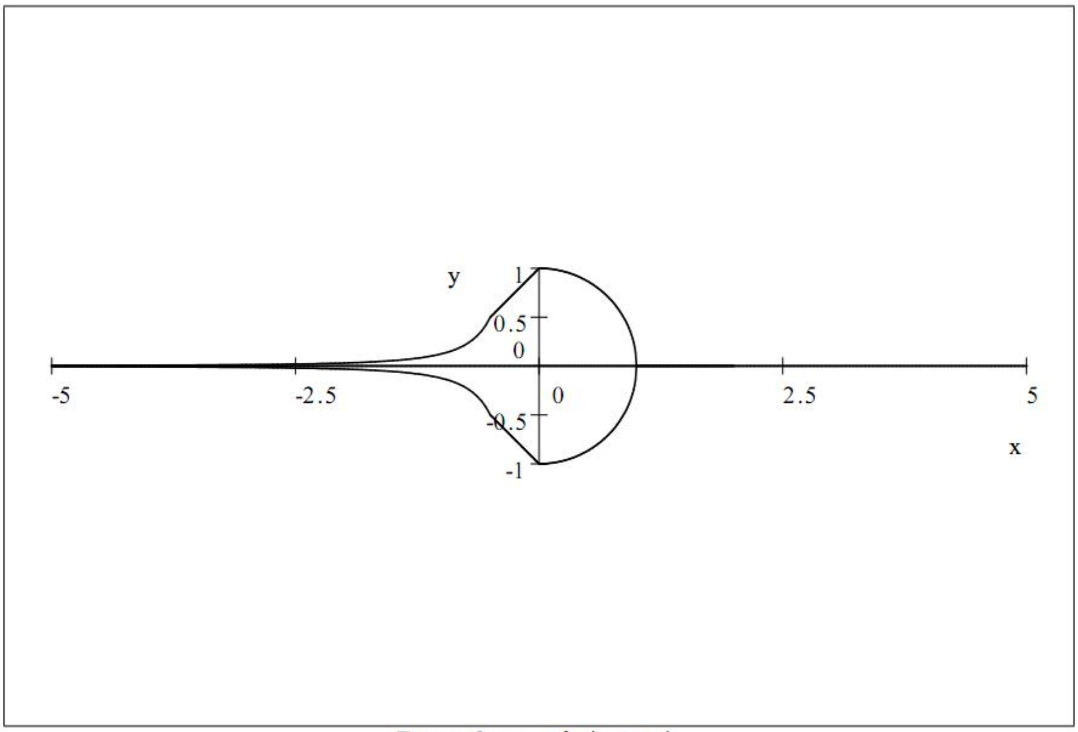
1. INTRODUCTION

The main purpose of this article is to lay down certain sufficient conditions to ensure that a uniformly continuous function on each member of a finite family of subsets in R^n is uniformly continuous on the union. As usual we define a function f defined on a subset A of a metric space (X, d) to be uniformly continuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that $x, y \in A, d(x, y) < \delta$ implies that $d(f(x), f(y)) < \varepsilon$. Alternatively, f is not uniformly continuous if there is $\varepsilon > 0$, two sequences x_n, y_n in A and a sequence of positive numbers a_n in R that converges to 0 such that for all n it holds that $d(x_n, y_n) < a_n$, and $d(f(x_n), f(y_n)) > \varepsilon$. It is well-known (See[**?**], p78) that if a real-valued function f is uniformly continuous on a subset A of a metric space X then it can extended to a uniformly continuous function on the closure of A in X. So we will assume that a uniformly continuous function is always defined on a closed subset A of a metric space X. We first give a counter example to show that a uniformly continuous function on two subsets of R^n (with the usual topology) need not be uniformly continuous on the union of these subsets.

Example 1.1. *We define two functions $f(x), -f(x)$:*

$$f(x) = \begin{cases} 1 + x & if \quad -1/2 \leq x \leq 0 \\ 1/(8x^2) & if \quad x \leq -1/2 \end{cases}, -f(x) = \begin{cases} -1 - x & if \quad -1/2 \leq x \leq 0 \\ -1/(8x^2) & if \quad x \leq -1/2 \end{cases}$$



$Boundary\ of\ A_1 \cup A_2$

Let C be the closed right-half plane with the right half of the open unit disc removed. Let A_1 be the set defined by the union of the graph of $f(x)$, the region above the graph of $f(x)$ and the subset C. Let A_2 be the reflection in the x-axis of the subset A_1consisting of the union of the graph of $-f(x)$, the region below the graph of $-f(x)$ and the subset C. We notice that

$$dist((0, 0), A_1) = dist((0, 0), A_2)) = \parallel (-1/2, 1/2) \parallel = \sqrt{2}/2. \tag{1.1}$$

Let $F(x, y) = F(z) = Arg(z) = \tan^{-1}(y/x)$, where $Arg(z)$ is the principal argument of z lying in the interval $(-\pi, \pi)$. We notice that $\parallel \nabla F(x, y) \parallel = \parallel (-y/(x^2 + y^2), x/(x^2 + y^2)) \parallel = 1/(x^2 + y^2) \leq 1/2$. Let $z_1, z_2 \in A_i, i = 1, 2$. Then there is (x, y) on the segment joining z_1, z_2

such that $| F(z_1) - F(z_2) |=\| \nabla F(x,y).(z_1 - z_2) \| \leq (1/2) \| z_1 - z_2 \|$. *Thus* F *is uniformly continuous on* $A_i, i = 1,2$. *However,* $F(z)$ *is not uniformly continuous on* $A_1 \cup A_2$. *To see this we take the sequence*

$z_n = -n + (i/n)$ *and the sequence* $z'_n = -n - (i/n), n = 1,2,\ldots$. *Then* $| z_n - z'_n | \to 0$ *as* $n \to \infty$. *But* $| F(z_n) - F(z'_n) | \to 2\pi$. *Thus* $F(z)$ *is not uniformly continuous on* $A_1 \cup A_2$.

So the question arises: Under what conditions will a uniformly continuous function f on two subsets A and B in R^n be uniformly continuous on the union? To answer this question we need some definitions and lemmas.

Let $x,y \in R^n$. We define as usual the *segment* $[x,y]$ to be the image of the function $f : [0,1] \to R^n$ defined by $f(t) = (1-t)x + tz$. Let $x_1,\ldots,x_n$ be distinct points in R^n. By an $(n-1)$-*chain* $x_1 \to \ldots \to x_n$ joining $x_1$to x_n we mean the set $[x_1,x_2] \cup [x_2,x_3] \cup \ldots \cup [x_{n-1},x_n]$. Let $A \subseteq R^n$ be closed (and arc-wise connected). Then for all $x,y \in A$,there is a curve $\gamma(x,y)$ joining x,y in A with length being the smallest possible all curves joining x,y in A . We denote such a length by $l(x,y)$. Let n be a positive integer. We say that A is n-*convex* if for all $\varepsilon > 0$ there is $\delta > 0$ such that $x,y \in A, \| x-y \| \leq \delta$ implies that $l(x,y) \leq n \| x-y \| + \varepsilon$. In such a case we assume that we can partition any curve $\gamma(x,y)$ whose length is $l(x,y)$ through a partition $\{x, x_2, x_3, \ldots, x_n, y\}$ such that the resulting n-chain has segments each of length $a \leq \| x-y \| + (\varepsilon/n)$. A convex set is 1-convex. *A U-shaped* set is an example of a 2-convex which is not 1-convex and an *S-shaped* set is an example of a 3-convex which is not 2-convex. We call a subset a *finite-convex* if it is m-convex for some positive integer m. A set which is not finitely convex is *0-convex*. The complement sets A'_1, A'_2 of A_1, A_2, respectively, in Example 1.1 are finite convex sets whose union $A_1 \cup A_2$ is not finite convex, see Proposition 1.6 below.

Remark 1.1. Let A be 0-convex. Then A is not m-convex for every positive integer m. Thus since A is not 2-convex there are $\varepsilon_2 > 0, x_1, y_1 \in A$ such that $\| x_1 - y_1 \| \leq 1/2, l(x_1,y_1) - 2. \| x_1 - y_1 \| > 2\varepsilon_2$. Since A is not 3-convex there are $\varepsilon_3 > \varepsilon_2 > 0$ and two points $x_2, y_2 \in A$ such that $\| x_2 - y_2 \| < 1/3$ and $l(x_2,y_2) - 3. \| x_2 - y_2 \| > \varepsilon_3 > \varepsilon_2$. Otherwise A is 3-convex. In general for positive integer $n > 2$ and since A is not n-convex there are $\varepsilon_n > \varepsilon_{n-1} > \varepsilon_2 > 0, x_n, y_n \in A$ such that

$$\| x_n - y_n \| < 1/(n+1), \tag{1.2}$$

$$l(x_n,y_n) - (n+1). \| x_n - y_n \| > \varepsilon_{n-1} > \varepsilon_0, n = 2,\ldots. \tag{1.3}$$

The union of two *non disjoint* closed $2-$convex sets may not be finite-convex as the following example shows.

Example 1.2. *Let* A_1, A_2*, and* C *be as in Example 1.1. Although it needs some computations, each of the subsets* A_1, A_2 *is a closed 2-convex. Also* $A_1 \cap A_2 = C$. *Let* $B = A_1 \cup A_2$. *Then* B *is not* $m-$*convex for every positive integer* m. *To see this we take two sequences*

$x_n = -n + i/12n^2, y_n = -n - i/12n^2, n = 1,2,\ldots$. *We can verify the property 1.2.The verification is trivial.*

In the Example 1.2 the two subsets were closed and 2-convex. But the union may be not finite-convex even if the two subsets are convex. In the following example we give two non closed convex sets(not just 2-convex as in Example 1.2 but with dropping the closure condition), with connected non empty intersection whose union is not finite-convex.

Example 1.3. *Let* A *be the upper half plane with the negative* x*-axis removed and let* B *be the lower-half plane with the negative real axis removed. Then* A, B *are convex non closed sets with intersection being the positive real axis. The union* $C = A \cup B$ *is not finite-convex. For we can take the sequence* $x_n = -n + i/n, n = 1,2,\ldots$ *and the sequence* $y_n = -n - i/n, n = 1,2,\ldots$.

Then $\| x_n - y_n \| \to 0$ *as* $n \to \infty$. *But for each* n *every route between* x_n, y_n *in* C *has length* $l \geq 2\sqrt{(n^4+1)/n^2} \to \infty$. *Thus* C *is not finite-convex.*

We will see (Proposition 1.7) that the union of two closed convex sets with non empty intersection is finite-convex.

Lemma 1.1. *Let* $A, B \subset R^n$ *be closed subsets and let* $x, y \in A \cup B$. *Let* $[x, y] \subset A \cup B$. *Then there is* $z \in [x, y]$ *such that each of the subsets* $\{x, z\}, \{z, y\}$ *is contained in either* A *or* B.

Proof. Without loss of generality we assume $x \in A \backslash B, y \in B \backslash A$. If $(x, y) \cap A = \phi$ then $(x, y) \subset B$ and it follows that $x, y \in B$ since B is closed. Since this is not the case it follows that $(x, y) \cap A \neq \phi$. We choose the farthest point z from x in $A \cap [x, y]$. Then $x \neq z \neq y, z \in A \cap B$. Thus $\{x, z\} \subset A, \{z, y\} \subset B$ as required. ∎

Lemma 1.2. *Let* A, B, C *be three closed subsets of* R^n *such that* $B \cup C$ *is* k*-convex. Let* $[x, y]$ *be a segment in* $A \cup B \cup C$ *of length* ε. *Then there are at most* $1 + 2k$ *pairs of ends of segments such that each pair is contained in one of* A, B, C *and such that the segment corresponding to any of these pairs of ends has length at most* ε.

Proof. We can assume that $x \in A \backslash (B \cup C), y \in (B \cup C) \backslash A$. Otherwise $A \subset B \cup C$ and we are back to previous Lemma. Then there is z such that $\{x, z\} \subset A, \{z, y\} \subset B \cup C$. Since $B \cup C$ is $k-$polygonally connected there are k segments in $B \cup C$ connecting z and y. By Lemma 1.1 each of these segments can be replaced by two ends that are contained in B or C. Thus we have in total at most $1 + 2k$ pairs of ends such that each pair is contained in one of A, B, C. ∎

Proposition 1.3. *Let* A *and* B *be closed subsets in* R^n *such that* $A \cup B$ *is* k*-convex. Let* $x, y \in A \cup B$ *and let* ε *be the distance between* x *and* y. *Then there are at most* $2k$ *segments joining* x *and* y *such that the ends of each segment are in* A *or in* B *and the length of each new segment is* $l \leq \varepsilon$.

Proof. Let us call a segment whose two ends are in one subset A_i to be "good". Let $x_1 = x, x_{k+1} = y$. Since $A \cup B$ is $k-$convex there is a sequence $\{x_1, ..., x_{k+1}\} \subset A \cup B$. We take x_1 and assume without loss of generality that $x_1 \in A$. If $x_2 \in A$ we stop. Otherwise $x_2 \in B \backslash A$. If the open segment $(x_1, x_2) \cap A = \phi$ then $(x_1, x_2) \subset B$. Since B is closed $x_1 \in B$. In this case we stop. On the other hand if $(x_1, x_2) \cap A \neq \phi$ we pick the farthest element $z_1 \neq x_1$ on the segment (x_1, x_2) from x_1; it exists since $[x_1, x_2] \cap A$ is non empty and compact. Then $z_1 \in A \cap B$. If $z_1 = x_2$ we stop. Otherwise we have $[x_1, x_2] = [x_1, z_1] \cup [z_1, x_2]$ such that $\{x_1, z_1\} \subset A, \{z_1, x_2\} \subset B$ and the length of each new segment is $\leq \varepsilon$.Thus we have tackled the segment $[x_1, x_2]$ by replacing it by one or two "good" segments.

Next we notice that $[x_2, x_3] \subset A \cup B$ of length ε and we tackle the segment $[x_2, x_3]$ as we have tackled the segment $[x_1, x_2]$ by writing if necessary $[x_2, x_3] = [x_2, z_2] \cup [z_2, x_3]$, such that the ends of each new segment are in one of A or B and the length of each new segment is does not exceed ε. We proceed to tackle the remaining finite number of segments in the same way and replace them by "good" segments. ∎

Lemma 1.4. *Let* $A_1, ...A_n$ *be n closed subsets of* $R^m, n \geq 2$. *Let* $B_i = \cup_{j=i}^n A_i, j = 1, ..., n$. *Let* $B_2, ...B_{n-1}$ *be* $k-$*convex. Let* $[x, y] \subset B_1 = \cup_{i=1}^n A_i$ *and let* $\varepsilon = \| x - y \|$. *Then there is a finite number* K *of pairs of segment ends in* B_1 *joining* x, y *such that each segment has length not exceeding* ε *and each pair of ends is contained in one* A_i.

Proof. We use induction on n. The assertion is true for $n = 2$ without the assumption of $k-$convexity. The assertion is true for $n = 3$. So assume the assertion to be true for $n - 1$. Then $[x, y] \subset A_1 \cup B_2$. By Lemma 1.1 there is z such that each of $\{x, z\}, \{z, y\}$ is contained in A_1 or

B_2. If, for example, $\{y, z\} \subset B_2$ then since B_2 is $k-$convex there are k segments in B_2 joining y and z. By induction hypothesis each of these segments can be replaced by a finite number of segments whose end points are contained in a single A_i. Thus the assertion follows. ∎

Proposition 1.5. *Let $A_1, ...A_n$ be n closed subsets of $R^m, n \geq 2$. Let $B_i = \cup_{j=i}^n A_j, i = 1, ..., n$. Let $B_1, ...B_{n-1}$ be $k-$convex. Let $x, y \in B_1, \| x - y \| = \varepsilon > 0$. Then there is a finite number p (depending only the sets A_i) of points $x_1, ...x_p$ in B_1 with $x_1 = x, x_p = y, \| x_i - x_{i+1} \| \leq \varepsilon, i = 2, ..., p-1$ and such that each of the subsets $\{x_i, x_{i+1}\}, i = 1, ..., p-1$ is contained in one $A_i, i = 1, ..., n$.*

Proof. Since B_1 is $k-$convex there are k segments joining x, y in B_1. Using the previous Lemma we can replace each segment by a finite number of "good" pairs. The assertion follows. ∎

Remark 1.2. We call the number p corresponding to the closed sets in the previous Proposition the *index* of these sets.

Proposition 1.6. *Let f be a uniformly continuous real function on each of the closed subsets $A_1, ..., A_n$ of R^m Let $B_i = \cup_{j=i}^n A_j, i = 1, ..., n$. Assume $B_1, ...B_{n-1}$ be $k-$convex. Then f is uniformly continuous on the union $B_1 = \cup_{j=1}^n A_j$. In particular if $f(x)$ is uniformly continuous on two closed subsets whose union is finite convex then $f(x)$ is uniformly continuous on the union of these subsets.*

Proof. Let p be the index of the sets $A_i, i = 1, ..., n$. Let $\varepsilon > 0$ be given. Then there are $\delta_i > 0$ such that for all $x, y \in A_i, \| x - y \| < \delta_i$ implies that $| f(x) - f(y) | < \varepsilon/p, i = 1, ..., n$.Let δ be the minimum of the $\delta_i, i = 1, ..., n$. Then there are p points $x_1, ..., x_p$, such that each pair $\{x_i, x_{i+1}\}$ is contained in one $A_i, i = 1, ..., p$. Thus $| f(x) - f(y) | \leq \sum_{i=1}^p | f(x_i) - f(x_{i+1}) | < p.\varepsilon/p = \varepsilon$. The assertion follows. ∎

Remark 1.3. We notice that the two sets in Example 1.1, namely A_1', A_2' are finite convex while their union is not. Otherwise the function $f(z)$ defined there would be uniformly continuous and this was not the case.

Finally we prove the following.

Proposition 1.7. *Let A, B be two convex closed sets in R^n such that $D = A \cap B \neq \phi$ and let $C = A \cup B$. Then C is finite-convex. Thus if $f(x)$ is uniformly continuous on A and B then $f(x)$ is uniformly continuous on $C = A \cup B$.*

Proof. By Proposition 1.7 it suffices to show that C is finite convex. Assume that C is not finite-convex. Then there are two sequences $x_n, y_n, n = 1, 2, ...$of points in C such that $\| x_n - y_n \| \to 0$ and a positive number $c > 0$ such that for each $n = 1, 2, ...$ a smallest route between x_n, y_n has length $l_n \geq c$. We can take the x_n to be boundary points in A and the y_n to be boundary points in B and outside D. For each $n = 1, 2, ...$the chosen route between x_n, y_n, aside from x_n, y_n , is outside D. Since A, B, C, D are closed, there is z such that $x_n, y_n \to z$. Then $z \in D$. It follows there is a non trivial "hole" in D and this contradicts the fact that D is convex. The last part of the proposition statement is a direct consequence of Proposition 1.6 . ∎

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