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SCHWARZ METHOD FOR VARIATIONAL INEQUALITIES RELATED TO ERGODIC CONTROL PROBLEMS

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ABSTRACT. In this paper, we study variational inequalities related to ergodic control problems studied by M. Boulbrachène and H. Sissaoui [11, 1996], where the "discount factor" (i.e., the zero order term) is set to 0, we use an overlapping Schwarz method on nomatching grid which consists in decomposing the domain in two subdomains. For $\alpha \in]0.1[$ we provide the discretization on each subdomain converges in L^∞-norm.

Key words and phrases: Variational Inequalities, Ergodic control, Schwarz method, Finite element method.

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1. INTRODUCTION

The ergodic control problems my be solved by considering the solution u_α of the following variational inequalities:

$$\left\{\begin{array}{l} a\left(u_\alpha, v-u_\alpha\right)+\alpha\left(u_\alpha, v-u_\alpha\right) \geq\left(f, v-u_\alpha\right), \\ u_\alpha \in H^1(\Omega), u_\alpha \leq \psi\,;\; v \in H^1(\Omega), v \leq \psi, \end{array}\right. \tag{1.1}$$

as α tends to 0; $\alpha > 0$.

Let Ω is a given bounded smooth open set in $\mathbb{R}^N$, f is a given positive function in $L^\infty(\Omega)$, ψ a positive obstacle of $W^{2,\infty}(\Omega)$,

$$a(u, v)=\int_\Omega \nabla(u) . \nabla(v)\, dx. \tag{1.2}$$

$(.,.)$ denotes the L^2-scalar product, $\|.\|_\infty$ the L^∞-norm.

It is proved under suitable assumptions that u_α converges to u_0, the unique solution of:

$$\left\{\begin{array}{l} a\left(u_0, v-u_0\right) \geq\left(f, v-u_0\right), \\ u_0 \in H^1(\Omega), u_0 \leq \psi, v \in H^1(\Omega), v \leq \psi. \end{array}\right. \tag{1.3}$$

The method characterizes the solution of the continuous problem (respectively, the discret problem) as the upper of the set continuous subsolutions (respectively, the discrete subsolutions). [3]

Let $\alpha \in]0,1[$, by easy transformation, u_α is also a solution of the following problem:

$$\left\{\begin{array}{l} b\left(u_\alpha, v-u_\alpha\right) \geq\left(f+\lambda u_\alpha, v-u_\alpha\right), \\ u_\alpha \in H^1(\Omega), u_\alpha \leq \psi, v \in H^1(\Omega), v \leq \psi, \end{array}\right. \tag{1.4}$$

where

$$\lambda = 1-\alpha,$$

$$b(u, v)=a(u, v)+(u, v). \tag{1.5}$$

Proposition 1.1. (see [2, 5]) *Under assumptions* (1.1)-(1.3)*, u_α is uniformly bounded in $W^{p,\infty}(\Omega)$, $p < +\infty$. Moreover u_α converges uniformly on Ω and strongly in $H^1(\Omega)$ to u_0 the unique solution of* (1.3).

Notation . We denote by $w=\sigma(g,\psi)$ the solution of the following (VI):

$$\begin{array}{l} b(w, v-w) \geq (g, v-w) \\ v \leq \psi, w \leq \psi. \end{array} \tag{1.6}$$

Then

$$u_\alpha=\sigma\left(f+\lambda u_\alpha ; \psi\right) \tag{1.7}$$

1.1. Lipschitz Continuous Dependence.

Let $g, \tilde{g} \in L^\infty(\Omega)$ and $u=\sigma(g, \psi)\,;\, \tilde{u}=\sigma(\tilde{g}, \psi)$ the associated solutions.

Proposition 1.2. *Under the above notations, we have*

$$\|\sigma(g, \psi)-\sigma(\tilde{g}, \psi)\|_\infty \leq\|g-\tilde{g}\|_\infty . \tag{1.8}$$

The following property is crucial in this paper.

1.2. **The concept of subsolution.** Let X be the set of subsolutions of problem (1.4),that is, $w \in X$ if and only if

$$b(w,v) \leq (f+\lambda w, v), \quad \forall v \in H^1(\Omega), \quad v \geq 0, w \leq \psi. \tag{1.9}$$

Lemma 1.3. *The solution u_α of* (1.4) *is the upper bound of X.*

The proof is standard, we adapt [3]. Clearly $u_\alpha \in X$ and if $w \in X$, an iterative schema starting from $w \leq u_\alpha$ leads to an increasing sequence $w_k, w_\infty = \lim w_k = u_\alpha$.

2. THE SCHWARZ METHOD FOR THE OBSTACLE PROBLEM 1

2.1. **The continuous Schwarz sequences.** we consider the following problem:

$$\begin{cases} \text{Find } u_\alpha \in H_0^1(\Omega) \text{ the solution of:} \\ b(u_\alpha, v-u_\alpha) \geq (f+\lambda u_\alpha, v-u_\alpha) \quad \forall v \leq \psi, \forall u_\alpha \leq \psi. \end{cases} \tag{2.1}$$

We decompose Ω into two overlapping polygonal subdomains $\Omega_1$and Ω_2, such that

$$\Omega = \Omega_1 \cup \Omega_2, \tag{2.2}$$

and u_α satisfies the local regularity condition

$$u_\alpha \mid_{\Omega_i} \in W^{2,p}(\Omega_i), 2 \leq p \leq \infty.$$

We denote by $\partial\Omega_i$ the boundary condition of Ω_i, and $\Gamma_i = \partial\Omega_i \cap \Omega_j$. The intersection of $\bar{\Gamma}_i$ and $\bar{\Gamma}_j$ is assumed to be empty.

Choosing $u_\alpha^0 = u_0$,such that u_0 the unique solution of:

$$a(u_0, v-u_0) \geq (f, v-u_0),$$

we respectively define the alternating Schwarz sequences $(u_{\alpha 1}^{n+1})$ on Ω_1 such that:

$$\begin{cases} b_1(u_{\alpha 1}^{n+1}, v-u_{\alpha 1}^{n+1}) \geq (f_1+\lambda u_{\alpha 1}, v-u_{\alpha 1}^{n+1}) \quad \forall v \leq \psi, v = u_{\alpha 2}^n \text{ on } \Gamma_1 \\ \\ u_{\alpha 1}^{n+1} = u_{\alpha 2}^n \text{ on } \Gamma_1, \end{cases} \tag{2.3}$$

and

$$\begin{cases} b_2(u_{\alpha 2}^{n+1}, v-u_{\alpha 2}^{n+1}) \geq (f_2+\lambda u_{\alpha 2}, v-u_{\alpha 2}^{n+1}) \quad \forall v \leq \psi, v = u_{\alpha 1}^{n+1} \quad \text{on } \Gamma_2 \\ u_{\alpha 2}^{n+1} = u_{\alpha 1}^{n+1} \text{ on } \Gamma_2, \end{cases} \tag{2.4}$$

where

$$f_i = f\backslash_{\Omega_i}, \tag{2.5}$$

and

$$b_i(u_\alpha, v) = a_i(u_\alpha, v) + \int_{\Omega_i} \lambda u_\alpha v dx \quad i = 1, 2. \tag{2.6}$$

The following geometrical convergence is du to Lions [10].

2.2. **Geometrical convergence.**

Theorem 2.1. [12] *The sequences* $\left(u_{\alpha 1}^{n+1}\right)$, $\left(u_{\alpha 2}^{n+1}\right)$, $n \geq 0$, *produced by the Schwarz alternating method converge geometrically to the solution* u_α *of obstacle problem* (2.1). *More precisely, there exist two constants* $k_1, k_2 \in]0,1[$ *such that for all* $n \geq 0$,

$$\left\| u_{\alpha i} - u_{\alpha i}^{n+1} \right\|_{L^\infty(\Omega_i)} \leq \left(k_1^n k_2^n\right) \left\| u_\alpha - u_\alpha^0 \right\|_{L^\infty(\Gamma_i)}, i = 1, 2 \tag{2.7}$$

Moreover, $\alpha \to 0$,

$$\left\| u_{0i} - u_{0i}^{n+1} \right\|_{L^\infty(\Omega_i)} \leq \left(k_1^{n+1} k_2^n\right) \left\| u_0 - u_0^0 \right\|_{L^\infty(\Gamma_i)}, i = 1, 2. \tag{2.8}$$

3. The discret problem 2

We suppose for simplicity that Ω is polyhedral. Let r_h be a regular, quasi uniform triangulation of Ω into n-simplexes of diameter less than h.

We denote by V_h the standard piecewise linear finite element espace, we consider the discrete variational inequality:

$$\begin{cases} a\left(u_{\alpha h}, v_h - u_{\alpha h}\right) + \alpha\left(u_{\alpha h}, v_h - u_{\alpha h}\right) \geq \left(f, v_h - u_{\alpha h}\right), \\ u_{\alpha h} \in V_h\,, \quad u_{\alpha h} \leq r_h \psi, v_h \in V_h, v_h \leq r_h \psi. \end{cases} \tag{3.1}$$

Or:

$$\begin{cases} b\left(u_{\alpha h}, v_h - u_{\alpha h}\right) \geq \left(f + \lambda u_{\alpha h}, v_h - u_{\alpha h}\right), \\ u_{\alpha h} \in V_h, u_{\alpha h} \leq r_h \psi, v_h \in V_h,\ v \leq r_h \psi. \end{cases} \tag{3.2}$$

4. The discrete maximum principle 3[1]

We assume that the matrix A with generic coefficient

$$a\left(\varphi_i, \varphi_j\right) \tag{4.1}$$

is a $M-$matrix.

As for the continuous problem, it is easy to prove that $u_{\alpha h}$ converges to u_{0h}, the solution of the following (VI):

$$\begin{cases} b\left(u_{0h}, v_h - u_{0h}\right) \geq \left(f, v_h - u_{0h}\right), \\ u_{0h} \in V_h, u_{0h} \leq r_h \psi, v_h \in V_h,\ v_h \leq r_h \psi. \end{cases} \tag{4.2}$$

Proposition 4.1. [5] *Under the assumption* (2.3)*, then* $u_{\alpha h}$ *converges uniformly on* Ω *and strongly in* $H^1(\Omega)$ *to* u_{0h}*, the unique solution of* (2.4).

Notation . We note $w_h = \sigma_h(g, \psi)$ the solution of the following discrete (VI):

$$b\left(w_h, v_h - w_h\right) \geq \left(g, v_h - w_h\right), v_h \leq r_h \psi,\ w_h \leq r_h \psi.$$

Then

$$u_{\alpha h} = \left(\sigma_h f + \lambda u_{\alpha h}; \psi\right).$$

As for the continuous case, we establish a Lipschitz discrete property.

Proposition 4.2. *Under the assumption* (2.3). *Then*

$$\left\| \sigma_h(g, \psi) - \sigma_h(\tilde{g}, \psi) \right\|_\infty \leq \left\| g - \tilde{g} \right\|_\infty. \tag{4.3}$$

The proof is similar to that of proposition 4.1.

We now define X_h to be the set of discret subsolutions of problem (2.2), that is $w_h \in X_h$ if and only if

$$b(w_h, \varphi_i) \leq (f + \lambda w_h, \varphi_i), \forall i \in \{1, ...m(h)\}, w_h \leq r_h \psi \tag{4.4}$$

Lemma 4.3. *Under the assumption* (4.2)*, then the solution $u_{\alpha h}$ of (2.2) is the upper bound of X_h.*

The proof is similar to that of Lemma 1.3.

Results. Let $\bar{u}_h$ be the solution of discrete VI :

$$\begin{cases} b(\bar{u}_h, v_h - \bar{u}_h) \geq (f + \lambda u_\alpha, v_h - \bar{u}_h), \\ v_h \leq r_h \psi, \bar{u}_h \leq r_h \psi, \end{cases} \tag{4.5}$$

u_α being the solution of (1.4)

Let $u^{(h)}$ be the solution of the continuous VI:

$$\begin{cases} b\left(u^{(h)}, v - u^{(h)}\right) \geq \left(f + \lambda u_{\alpha h}, v - u^{(h)}\right), \\ v \leq \psi, u^{(h)} \leq \psi, \end{cases} \tag{4.6}$$

$u_{\alpha h}$ being the solution of (2.2).

Lemma 4.4.

$$\|u_\alpha - \bar{u}_h\|_\infty \leq Ch^2 |\log h|^2, \tag{4.7}$$

and

$$\left\|u^{(h)} - u_{\alpha h}\right\|_\infty \leq Ch^2 |\log h|^2. \tag{4.8}$$

Proof. We adapt [4]. ∎

Theorem 4.5. [11]

$$\|u_\alpha - u_{\alpha h}\|_\infty \leq Ch^2 |\log h|^2, \tag{4.9}$$

and

$$\|u_0 - u_{0h}\|_\infty \leq Ch^2 |\log h|^2. \tag{4.10}$$

Remark 4.1. In the sequel, the constant C is always considered independent of both α and h.

4.1. **The discrete Schwarz sequences.** Let $V_{h_i} = V_{h_i}(\Omega_i)$ be the space of continuous piecewise linear function on τ_{hi} which vanish on $\partial\Omega \cap \partial\Omega_i$.

For $w \in C(\Gamma_i)$, we define

$$V_{h_i}^{(w)} = \{v \in V_h / v = 0 \text{ on } \partial\Omega \cap \Omega_i,\ v = \pi_{hi}(w) \text{ on } \Gamma_i\}$$

Where π_{hi} denotes the interpolation operator on Γ_i.

For $i = 1, 2$. Let τ_{hi} be a standard regular finite element triangulation in Ω_i , h_i being the mesh size. We suppose that the two triangulation are mutually independent on $\Omega_1 \cup \Omega_2$ a triangle belonging to one triangulation does not necessarily belong to the other.

Choosing $u_{\alpha h}^0 = u_{0h}$, such that u_{0h} is a solution of the following inequation:

$$a(u_{0h}, v - u_{0h}) \geq (f, v - u_{0h}),$$

We define the discrete sequence of Schwarz $(u_{\alpha 1h}^{n+1})$ on Ω_1:

$$\begin{cases} b_1(u_{\alpha 1h}^{n+1}, v - u_{\alpha 1h}^{n+1}) \geq (f_1 + \lambda u_{\alpha 1h}^n, v - u_{1\alpha h}^{n+1}) \quad \forall v \leq \psi, v = u_{\alpha 2h}^n \text{ on } \Gamma_1 \\ u_{\alpha 1h}^{n+1} = u_{\alpha 2h}^n \text{ on } \Gamma_1, \end{cases} \tag{4.11}$$

and $(u_{\alpha 2h}^{n+1})$ on Ω_2 such that

$$b_2(u_{\alpha 2h}^{n+1}, v - u_{\alpha 2h}^{n+1}) \geq (f_2 + \lambda u_{\alpha 2h}^{n}, v - u_{\alpha 2h}^{n+1}) \quad \forall v \leq \psi,\ v = u_{\alpha 1h}^{n+1} \text{ on } \Gamma_2$$
$$u_{\alpha 2h}^{n+1} = u_{\alpha 1h}^{n+1} \text{ on } \Gamma_2.$$

5. L^{∞}-ERROR ANALYSIS

This section is devoted to the proof of the present paper to that end we begin by introducing two discrete auxiliary sequences and prove a fundamental Lemma.

5.1. **Definition of two auxiliary sequences.** For $w_{\alpha ih}^{0} = u_{\alpha ih}^{0} = u_{0h}$; $i = 1, 2$, we défine the sequence $w_{\alpha 1h}^{n+1}$on Ω_1:

$$\text{(5.1)} \qquad \begin{cases} b_1(w_{\alpha 1h}^{n+1}, v - w_{\alpha 1h}^{n+1}) \geq (f_1 + \lambda w_{\alpha 1h}^{n}, v - w_{\alpha 1h}^{n+1}) \quad \forall v \leq \psi, v = u_{\alpha 2h}^{n} \text{ on } \Gamma_1 \\ w_{\alpha 1h}^{n+1} = u_{\alpha 2h}^{n} \text{ on } \Gamma_1 \end{cases}$$

and $w_{\alpha 2h}^{n+1}$ on Ω_2 such that:

$$\text{(5.2)} \qquad \begin{cases} b_2(w_{\alpha 2h}^{n+1}, v - w_{\alpha 2h}^{n+1}) \geq (f_2 + \lambda w_{\alpha 2h}^{n}, v - w_{\alpha 2h}^{n+1}) \quad \forall v \leq \psi,\ v = u_{\alpha 1h}^{n+1} \text{ on } \Gamma_2 \\ w_{\alpha 2h}^{n+1} = u_{\alpha 1h}^{n+1} \text{ on } \Gamma_2 \end{cases}$$

Note that $(w_{\alpha ih}^{n+1}\ )$ is the finite element approximation of $\left(u_{\alpha i}^{n+1}\right)$ definite in (2.13), (1.14).

Notation . We will adapt the following notations:

$$\text{(5.3)} \qquad \begin{aligned} |.|_i &= \|.\|_{L^{\infty}(\Gamma_i)}, i = 1.2 \\ \|.\|_i &= \|.\|_{L^{\infty}(\Omega_i)}, i = 1.2 \\ r_{hi} &= r_h, i = 1.2 \end{aligned}$$

The following Lemma will play a key role in proving the main result of this paper.

Lemma 5.1. *We have*

$$\text{(5.4)} \qquad \left\|u_{1\alpha}^{n+1} - u_{1\alpha h}^{n+1}\right\|_1 \leq \sum_{p=1}^{n+1} \|u_{1\alpha}^{p} - w_{\alpha 1h}^{p}\|_1 + \sum_{p=0}^{n} \|u_{2\alpha}^{p} - w_{\alpha 2h}^{p}\|_2$$

$$\text{(5.5)} \qquad \leq \sum_{p=1}^{n+1} \|u_{1\alpha}^{p} - w_{\alpha 1h}^{p}\|_1 + \left\|u_{2\alpha}^{0} - w_{\alpha 2h}^{0}\right\|_2 + \sum_{p=1}^{n} \|u_{2\alpha}^{p} - w_{\alpha 2h}^{p}\|_2$$

Where

$$\left\|u_{2\alpha}^{0} - w_{\alpha 2h}^{0}\right\|_2 = \|u_0 - u_{0h}\|_{\infty} \leq Ch^2 \left|\log h\right|^2$$

and

$$\text{(5.6)} \qquad \left\|u_{2\alpha}^{n+1} - u_{2\alpha h}^{n+1}\right\|_2 \leq \sum_{p=0}^{n+1} \|u_{2\alpha}^{p} - w_{\alpha 2h}^{p}\|_1 + \sum_{p=0}^{n+1} \|u_{1\alpha}^{p} - w_{\alpha 1h}^{p}\|_1$$

$$\text{(5.7)} \qquad \leq \sum_{p=0}^{n+1} \|u_{2\alpha}^{p} - w_{\alpha 2h}^{p}\|_1 + + \left\|u_{2\alpha}^{0} - w_{\alpha 2h}^{0}\right\|_1 + \sum_{p=1}^{n+1} \|u_{1\alpha}^{p} - w_{\alpha 1h}^{p}\|_1$$

In the same way

$$\left\|u_{2\alpha}^{0} - w_{\alpha 2h}^{0}\right\|_1 = \|u_0 - u_{0h}\|_{\infty} \leq Ch^2 \left|\log h\right|^2$$

Proof. We will take

$$h_1 = h_2 = h$$

Indeed, for $n = 0$, using the discrete form of Proposition 4.1, we get

$$\left\|u_{1\alpha}^1 - u_{1\alpha h}^1\right\|_1 \leq \sum_{p=1}^{1} \|u_{1\alpha}^p - w_{\alpha 1h}^p\|_1 + \sum_{p=0}^{0} \|u_{2\alpha}^p - w_{\alpha 2h}^p\|_2$$

$$\left\|u_{2\alpha}^1 - u_{2\alpha h}^1\right\|_2 \leq \sum_{p=0}^{1} \|u_{2\alpha}^p - w_{\alpha 2h}^p\|_1 + \sum_{p=0}^{1} \|u_{1\alpha}^p - w_{\alpha 1h}^p\|_1$$

For subdomain Ω_1 :

$$\begin{aligned} \left\|u_{1\alpha}^1 - u_{1\alpha h}^1\right\|_1 &\leq \left\|u_{1\alpha}^1 - w_{\alpha 1h}^1\right\|_1 + \left\|w_{\alpha 1h}^1 - u_{1\alpha h}^1\right\|_1 \\ &\leq \left\|u_{1\alpha}^1 - w_{\alpha 1h}^1\right\|_1 + \left|\pi_h u_{\alpha 2}^0 - \pi_h u_{\alpha 2h}^0\right|_1 \\ &\leq \left\|u_{1\alpha}^1 - w_{\alpha 1h}^1\right\|_1 + \left\|u_{\alpha 2}^0 - w_{\alpha 2h}^0\right\|_2 \end{aligned}$$

and subdomain Ω_2 :

$$\begin{aligned} \left\|u_{2\alpha}^1 - u_{2\alpha h}^1\right\|_2 &\leq \left\|u_{2\alpha}^1 - w_{\alpha 2h}^1\right\|_2 + \left\|w_{\alpha 2h}^1 - u_{2\alpha h}^1\right\|_2 \\ &\leq \left\|u_{2\alpha}^1 - w_{\alpha 2h}^1\right\|_2 + \left|\pi_h u_{\alpha 1}^1 - \pi_h u_{\alpha 1h}^1\right|_2 \\ &\leq \left\|u_{2\alpha}^1 - w_{\alpha 2h}^1\right\|_2 + \left\|u_{\alpha 1}^1 - u_{\alpha 1h}^1\right\|_1 \\ &\leq \left\|u_{2\alpha}^1 - w_{\alpha 2h}^1\right\|_2 + \left\|u_{1\alpha}^1 - w_{\alpha 1h}^1\right\|_1 + \left\|u_{\alpha 2}^0 - w_{\alpha 2h}^0\right\|_2 \end{aligned}$$

Let as now suppose that

$$\|u_{2\alpha}^n - w_{\alpha 2h}^n\|_2 \leq \sum_{p=0}^{n} \|u_{2\alpha}^p - w_{\alpha 2h}^p\|_2 + \sum_{p=1}^{n} \|u_{1\alpha}^p - w_{\alpha 1h}^p\|_1$$

Then

$$\begin{aligned} \left\|u_{1\alpha}^{n+1} - u_{1\alpha h}^{n+1}\right\|_1 &\leq \left\|u_{1\alpha}^{n+1} - w_{\alpha 1h}^{n+1}\right\|_1 + \left\|w_{\alpha 1h}^{n+1} - u_{1\alpha h}^{n+1}\right\|_1 \\ &\leq \left\|u_{1\alpha}^{n+1} - w_{\alpha 1h}^{n+1}\right\|_1 + |\pi_h u_{\alpha 2}^n - \pi_h u_{\alpha 2h}^n|_1 \\ &\leq \left\|u_{1\alpha}^{n+1} - w_{\alpha 1h}^{n+1}\right\|_1 + \|u_{\alpha 2}^n - u_{\alpha 2h}^n\|_2 \\ &\leq \left\|u_{1\alpha}^{n+1} - w_{\alpha 1h}^{n+1}\right\|_1 + \sum_{p=0}^{n} \|u_{2\alpha}^p - w_{\alpha 2h}^p\|_2 + \sum_{p=1}^{n} \|u_{1\alpha}^p - w_{\alpha 1h}^p\|_1 \end{aligned}$$

And consequently,

$$\left\|u_{1\alpha}^{n+1} - u_{1\alpha h}^{n+1}\right\|_1 \leq \sum_{p=1}^{n+1} \|u_{1\alpha}^p - w_{\alpha 1h}^p\|_2 + \sum_{p=0}^{n} \|u_{2\alpha}^p - w_{\alpha 2h}^p\|_2$$

Likewise, using the above estimate, we get

$$\begin{aligned} \left\|u_{2\alpha}^{n+1} - u_{2\alpha h}^{n+1}\right\|_2 &\leq \left\|u_{2\alpha}^{n+1} - w_{\alpha 2h}^{n+1}\right\|_2 + \left\|w_{\alpha 2h}^{n+1} - u_{2\alpha h}^{n+1}\right\|_2 \\ &\leq \left\|u_{2\alpha}^{n+1} - w_{\alpha 2h}^{n+1}\right\|_2 + \left|\pi_h u_{\alpha 1}^{n+1} - \pi_h u_{\alpha 1h}^{n+1}\right|_2 \\ &\leq \left\|u_{2\alpha}^{n+1} - w_{\alpha 2h}^{n+1}\right\|_2 + \left\|u_{\alpha 1}^{n+1} - u_{\alpha 1h}^{n+1}\right\|_1 \\ &\leq \sum_{p=0}^{n+1} \|u_{2\alpha}^p - w_{\alpha 2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_{1\alpha}^p - w_{\alpha 1h}^p\|_1 \end{aligned}$$

For $n=1$:

$$\left\|u_{1\alpha}^2-u_{1\alpha h}^{n+1}\right\|_1 \leq \sum_{p=1}^{2}\|u_{1\alpha}^p-w_{\alpha 1h}^p\|_2+\sum_{p=0}^{1}\|u_{2\alpha}^p-w_{\alpha 2h}^p\|_2$$

and

$$\left\|u_{2\alpha}^2-u_{2\alpha h}^{2}\right\|_2 \leq \sum_{p=1}^{2}\|u_{1\alpha}^p-w_{\alpha 1h}^p\|_1+\sum_{p=0}^{2}\|u_{2\alpha}^p-w_{\alpha 2h}^p\|_2$$

∎

Theorem 5.2. *Let $h=\max(h_1,h_2)$. Then, there exists two constants C and k, $0<k<1$ independent of both h and n such that*

$$n+1 \leq \frac{\log h}{\log k}. \tag{5.8}$$

We have

$$\left\|u_{\alpha i}-u_{\alpha ih}^{n+1}\right\|_{L^\infty(\Omega_i)} \leq Ch^2\left|\log h\right|^3,\ i=1,2 \tag{5.9}$$

and

$$\left\|u_{0i}-u_{0ih}^{n+1}\right\|_{L^\infty(\Omega_i)} \leq Ch^2\left|\log h\right|^3,\ i=1,2 \tag{5.10}$$

Proof. Let us give the proof for $i=1$, The case $i=2$ is similar.

$$\left\|u_{\alpha 1}-u_{\alpha 1h}^{n+1}\right\|_1 \leq \left\|u_{\alpha 1}-u_{\alpha 1}^{n+1}\right\|_1+\left\|u_{\alpha 1}^{n+1}-u_{\alpha 1h}^{n+1}\right\|_1$$

$$\leq (k)^{n+1}\left\|u_{\alpha 1}-u_{\alpha 1}^{0}\right\|_1+(n+1)C_2h^2\left|\log h\right|^2+nC_3h^2\left|\log h\right|^2$$

$$\leq (k)^{n+1}\left\|u_{\alpha 1}-u_{01}\right\|_1+(n+1)C_2h^2\left|\log h\right|^2+nC_3h^2\left|\log h\right|^2$$

$$\leq (k)^{n+1}\lim_{\alpha\to 0}\left\|u_{\alpha 1}-u_{01}\right\|_1+(n+1)C_1h^2\left|\log h\right|^2+nC_2h^2\left|\log h\right|^2.$$

We obtain

$$\left\|u_{\alpha 1}-u_{\alpha 1h}^{n+1}\right\|_1 \leq Ch^2\left|\log h\right|^3$$

Which is the desired error estimate.

For the case $\alpha=0$, we adapt [14]. ∎

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