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CERTAIN COEFFICIENT ESTIMATES FOR BI-UNIVALENT SAKAGUCHI TYPE FUNCTIONS

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ABSTRACT. Estimates on the initial coefficients are obtained for normalized analytic functions f in the open unit disk with f and its inverse $g = f^{-1}$ satisfying the conditions that $zf'(z)/f(z)$ and $zg'(z)/g(z)$ are both subordinate to a starlike univalent function whose range is symmetric with respect to the real axis. Several related classes of functions are also considered, and connections to earlier known results are made.

Key words and phrases: Univalent functions; Bi-univalent functions; Bi-starlike functions; Bi-convex functions; Subordination.

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1. INTRODUCTION

Let A be the class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The Koebe one-quarter theorem [8] ensures that the image of D under every univalent function $f \in A$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in D)$ and

$$f(f^{-1}(w)) = w, \ \ (|w| < r_0(f), r_0(f) \geq 1/4)$$

A function $f \in A$ is said to be bi-univalent in D if both f and f^{-1} are univalent in D. Let σ denote the class of bi-univalent functions defined in the unit disk D. A domain $D \subset \mathbb{C}$ is *convex* if the line segment joining any two points in D lies entirely in D, while a domain is *starlike* with respect to a point $w_0 \in D$ if the line segment joining any point of D to w_0 lies inside D. A function $f \in A$ is starlike if $f(D)$ is a starlike domain with respect to the origin, and convex if $f(D)$ is convex. Analytically, $f \in A$ is starlike if and only if $Re\{zf'(z)/f(z)\} > 0$, whereas $f \in A$ is convex if and only if $1 + Re\{zf''(z)/f'(z)\} > 0$. The classes consisting of starlike and convex functions are denoted by $\mathcal{ST}$ and $\mathcal{CV}$ respectively. The classes $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ of starlike and convex functions of order α, $0 \leq \alpha < 1$, are respectively characterized by $Re\{zf'(z)/f(z)\} > \alpha$ and $1 + Re\{zf''(z)/f'(z)\} > \alpha$. Various subclasses of starlike and convex functions are often investigated. These functions are typically characterized by the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying in a certain domain starlike with respect to 1 in the right-half plane. Subordination is useful to unify these subclasses.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on D with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [11] unified various subclasses of starlike and convex functions for which either of the quantity $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk D, $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps D onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in A$ satisfying the subordination $zf'(z)/f(z) \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f \in A$ satisfying the subordination $1 + zf''(z)/f'(z) \prec \phi(z)$. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{ST}_\sigma(\phi)$ and $\mathcal{CV}_\sigma(\phi)$.

Lewin [10] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [7, 12]). Brannan and Taha [6] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functins and bi-convex functions and obtained estimates on the initial coefficients. Recently, Srivastava et al. [14] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Bounds for the initial coefficients of several classes of functions were also investigated in [1, 2, 4, 5, 3, 13].

In this paper, estimates on the initial coefficients for bi-univalent Sakaguchi type functions are obtained. This class was motivated by Rosihan M. Ali et al. [3].

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk D, with $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(D)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0) \tag{1.2}$$

Definition 1.1. A function $f(z) \in A$ is said to be in the clas $M(\rho, \mu, \phi, t)$ if it satisfies

$$\left\{\frac{(1-t)[\rho\mu z^3 f'''(z) + (2\rho\mu + \rho - \mu)z^2 f''(z) + zf'(z)]}{\rho\mu z^2[f''(z) - t^2 f''(tz)] + (\rho - \mu)z[f'(z) - tf'(tz)] + (1 - \rho + \mu)[f(z) - f(tz)]}\right\} \prec \phi(z) \tag{1.3}$$

and,

$$\left\{\frac{(1-t)[\rho\mu w^3 g'''(w) + (2\rho\mu + \rho - \mu)w^2 g''(w) + wg'(w)]}{\rho\mu w^2[g''(w) - t^2 g''(tw)] + (\rho - \mu)w[g'(w) - tg'(tw)] + (1 - \rho + \mu)[g(w) - g(tw)]}\right\} \prec \phi(w) \tag{1.4}$$

$|t| \leq 1$, $t \neq 1$, $0 \leq \mu \leq \rho \leq 1$.
This class $M(\rho, \mu, \phi, t)$ was defined by Srutha Keerthi and Chinthamani [16].

For $\mu = 0$ in $M(\rho, \mu, \phi, t)$ we get the class $M(\rho, \phi, t)$ which was studied by Srutha Keerthi [15].

Definition 1.2. A function $f(z) \in A$ is in the class $M(\rho, \phi, t)$, if it satisfies

$$\left\{\frac{(1-t)[\rho z^2 f''(z) + zf'(z)]}{\rho z[f'(z) - tf'(tz)] + (1-\rho)[f(z) - f(tz)]}\right\} \prec \phi(z) \tag{1.5}$$

and,

$$\left\{\frac{(1-t)[\rho w^2 g''(w) + wg'(w)]}{\rho w[g'(w) - tg'(tw)] + (1-\rho)[g(w) - g(tw)]}\right\} \prec \phi(w) \tag{1.6}$$

$|t| \leq 1$, $t \neq 1$, $0 \leq \rho \leq 1$.
If $\rho = 0$ and $\rho = 1$ in $M(\rho, \phi, t)$, we get the classes $S^*(\phi, t)$ and $T(\phi, t)$ respectively, which were studied by Goyal and Pranay Goswami [9].

Definition 1.3. A function $f \in A$ is said to be in the class $M(\lambda, \phi, t)$ if it satisfies

$$\left\{\frac{(1-t)[\lambda z^2 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda[z^2 f''(z) - t^2 z^2 f''(tz)] + z[f'(z) - tf'(tz)]}\right\} \prec \phi(z) \tag{1.7}$$

and,

$$\left\{\frac{(1-t)[\lambda w^2 g'''(w) + (1+2\lambda)w^2 g''(w) + wg'(w)]}{\lambda[w^2 g''(w) - t^2 w^2 g''(tw)] + w[g'(w) - tg'(tw)]}\right\} \prec \phi(w) \tag{1.8}$$

$|t| \leq 1$, $t \neq 1$, $0 \leq \lambda \leq 1$.

For $\lambda = 0$, we get the class $T(\phi, t)$ which was studied by Goyal and Pranay Goswami [9].

2. COEFFICIENT ESTIMATES

For functions in the class $M(\rho, \mu, \phi, t)$, the following results are obtained.

Theorem 2.1. *If $f \in M(\rho, \mu, \phi, t)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1-t)|B_1^2[(1+2\rho-2\mu+6\rho\mu)(2+t) - (1+\rho-\mu+2\rho\mu)^2(1+t)] - (B_2 - B_1)(1-t)(1+\rho-\mu+2\rho\mu)^2|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{\frac{B_1}{2}[|2(1+2\rho-2\mu+6\rho\mu)(2+t) - (1+\rho-\mu+2\rho\mu)^2(1+t)| + |(1+\rho-\mu+2\rho\mu)^2(1+t)|] + |B_2 - B_1|(2+t)|1+2\rho-2\mu+6\rho\mu|}{(2+t)(1+t)|1+2\rho-2\mu+6\rho\mu||(2+t)(1+2\rho-2\mu+6\rho\mu) - (1+t)(1+\rho-\mu+2\rho\mu)^2|} \tag{2.2}$$

$|t| \leq 1, t \neq 1, 0 \leq \mu \leq \rho \leq 1$.

Proof. Let $f \in M(\rho, \mu, \phi, t)$ and $g = f^{-1}$. Then there are analytic functions $u, v : D \to D$, with $u(0) = v(0) = 0$, satisfying

$$\left\{ \frac{(1-t)[\rho\mu z^3 f'''(z) + (2\rho\mu + \rho - \mu)z^2 f''(z) + zf'(z)]}{\rho\mu z^2[f''(z) - t^2 f''(tz)] + (\rho - \mu)z[f'(z) - tf'(tz)] + (1 - \rho + \mu)[f(z) - f(tz)]} \right\} = \phi(u(z)) \tag{2.3}$$

and,

$$\left\{ \frac{(1-t)[\rho\mu w^3 g'''(w) + (2\rho\mu + \rho - \mu)w^2 g''(w) + wg'(w)]}{\rho\mu w^2[g''(w) - t^2 g''(tw)] + (\rho - \mu)w[g'(w) - tg'(tw)] + (1 - \rho + \mu)[g(w) - g(tw)]} \right\} = \phi(v(w)) \tag{2.4}$$

Define the functions p_1 and p_2 by

$$p_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad \text{and} \quad p_2(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

or, equivalently,

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right) \tag{2.5}$$

and

$$v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left(b_1 z + \left(b_2 - \frac{b_1^2}{2} \right) z^2 + \cdots \right) \tag{2.6}$$

Then p_1 and p_2 are analytic in D with $p_1(0) = 1 = p_2(0)$. Since $u, v : D \to D$, the functions p_1 and p_2 have positive real part in D, and $|b_i| \leq 2$ and $|c_i| \leq 2$. In view of (2.3), (2.4), (2.5) and (2.6), clearly

$$\left\{ \frac{(1-t)[\rho\mu z^3 f'''(z) + (2\rho\mu + \rho - \mu)z^2 f''(z) + zf'(z)]}{\rho\mu z^2[f''(z) - t^2 f''(tz)] + (\rho - \mu)z[f'(z) - tf'(tz)] + (1 - \rho + \mu)[f(z) - f(tz)]} \right\} = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1} \right)$$

$and,$

$$\left\{ \frac{(1-t)[\rho\mu w^3 g'''(w) + (2\rho\mu + \rho - \mu)w^2 g''(w) + wg'(w)]}{\rho\mu w^2[g''(w) - t^2 g''(tw)] + (\rho - \mu)w[g'(w) - tg'(tw)] + (1 - \rho + \mu)[g(w) - g(tw)]} \right\} = \phi\left(\frac{p_1(w) - 1}{p_1(w) + 1} \right) \tag{2.7}$$

Using (2.5) and (2.6) together with (1.2), it is evident that

$$\phi\left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2 c_1^2 \right) z^2 \cdots \tag{2.8}$$

and

$$\phi\left(\frac{p_2(w) - 1}{p_2(w) + 1} \right) = 1 + \frac{1}{2}B_1 b_1 w + \left(\frac{1}{2}B_1 \left(b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4}B_2 b_1^2 \right) w^2 \cdots \tag{2.9}$$

Since, $f \in M(\rho, \mu, \phi, t)$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \ldots \tag{2.10}$$

Since,

$$\left\{ \frac{(1-t)[\rho\mu z^3 f'''(z) + (2\rho\mu + \rho - \mu)z^2 f''(z) + zf'(z)]}{\rho\mu z^2[f''(z) - t^2 f''(tz)] + (\rho - \mu)z[f'(z) - tf'(tz)] + (1 - \rho + \mu)[f(z) - f(tz)]} \right\}$$
$$= 1 + a_2(1-t)(1 + \rho - \mu + 2\rho\mu)z + z^2(1-t)[a_3(2+t)(1 + 2\rho - 2\mu + 6\rho\mu) - a^2(1+t)(1 + \rho - \mu + 2\rho\mu)] + \cdots$$

and,

$$\left\{ \frac{(1-t)[\rho\mu w^3 g'''(w) + (2\rho\mu + \rho - \mu)w^2 g''(w) + wg'(w)]}{\rho\mu w^2[g''(w) - t^2 g''(tw)] + (\rho - \mu)w[g'(w) - tg'(tw)] + (1 - \rho + \mu)[g(w) - g(tw)]} \right\}$$
$$= 1 - a_2 w(1-t)(1 + \rho - \mu + 2\rho\mu)$$
$$+ w^2(1-t)\left\{ [a_2^2[2(1 + 2\rho - 2\mu + 6\rho\mu)(2+t) - (1 + \rho - \mu + 2\rho\mu)^2(1+t)] - a_3(1 + 2\rho - 2\mu + 6\rho\mu)(2+t)] \right\} + \cdots$$

It follows from (2.7), (2.8) and (2.9) that

$$a_2(1-t)(1+\rho-\mu+2\rho\mu)=\frac{1}{2}B_1c_1 \tag{2.11}$$

$$a_3(2+t)(1-t)(1+2\rho-2\mu+6\rho\mu)-a_2^2(1+t)(1-t)(1+\rho-\mu+2\rho\mu)^2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2 \tag{2.12}$$

$|t|\leq 1, t\neq 1, 0\leq\mu\leq\rho\leq 1.$

$$-a_2(1-t)(1+\rho-\mu+2\rho\mu)=\frac{B_1b_1}{2} \tag{2.13}$$

and

$$a_2^2(1-t)[2(1+2\rho-2\mu+6\rho\mu)(2+t)-(1+t)(1+\rho-\mu+2\rho\mu)^2]-a_3(1-t)(2+t)(1+2\rho-2\mu+6\rho\mu)=\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2 \tag{2.14}$$

From (2.11) and (2.13), it follows that

$$c_1=-b_1 \tag{2.15}$$

Now (2.12), (2.13), (2.14) and (2.15) yield

$$a_2^2=\frac{B_1^3(b_2+c_2)}{4(1-t)\{B_1^2[(1+2\rho-2\mu+6\rho\mu)(2+t)-(1+\rho-\mu+2\rho\mu)^2(1+t)]-(B_2-B_1)(1-t)(1+\rho-\mu+2\rho\mu)^2\}}$$

and

$$a_3=\frac{\begin{array}{r}B_1\{c_2[2(2+t)(1+2\rho-2\mu+6\rho\mu)-(1+t)(1+\rho-\mu+2\rho\mu)^2]+b_2(1+t)(1+\rho-\mu+2\rho\mu)^2\}\\+c_1^2(B_2-B_1)(2+t)(1+2\rho-2\mu+6\rho\mu)\end{array}}{4(1-t)(2+t)(1+2\rho-2\mu+6\rho\mu)[(2+t)(1+2\rho-2\mu+6\rho\mu)-(1+t)(1+\rho-\mu+2\rho\mu)^2]}$$

which in view of the well known inequalities $|b_i|\leq 2$ and $|c_i|\leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2^2|$ and $|a_3|$ as asserted in (2.2) and (2.3) respectively. ∎

Corollary 2.2. *If $f\in M(\rho,\phi,t)$, then the function has the coefficient inequalities as* $|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{(1-t)|B_1^2[(1+2\rho)(2+t)-(1+t)(1+\rho)^2]-(B_2-B_1)(1-t)(1+\rho)^2|}}$ *and* $|a_3|\leq\frac{\frac{B_1}{2}[|2(1+2\rho)(2+t)-(1+\rho)^2(1+t)|+|(1+\rho)^2(1+t)|+|B_2-B_1|(2+t)|1+2\rho|}{(2+t)(1+t)|1+2\rho||(2+t)(1+2\rho)-(1+t)(1+\rho)^2|}$.

By putting $\mu=0$ in (2.1) and (2.2), the above estimates can be derived.

Corollary 2.3. *If $f\in T(\phi,t)$, then the function gets the coefficient inequalities as* $|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{(1-t)|B_1^2(2-t)-4(1-t)(B_2-B_1)|}}$ *and* $|a_3|\leq\frac{B_1+|B_2-B_1|}{(1-t)(2-t)}$.

By putting $\rho=1$ in the above Corollary (2.2), these estimates can be obtained.

Corollary 2.4. *If $f\in S^*(\phi,t)$, then the function obtains the following coefficient inequalities as* $|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{(1-t)|B_1^2-(1-t)(B_2-B_1)|}}$ *and* $|a_3|\leq\frac{B_1+|B_2-B_1|}{(1-t)}$.

When we take $\rho=0$, the Corollary 2.3 yield these estimates.

Theorem 2.5. *If $f\in M(\lambda,\phi,t)$, then*

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{(1-t)|B_1^2[3(1+2\lambda)(2+t)-4(1+t)(1+\lambda)^2]-4(B_2-B_1)(1-t)(1+\lambda)^2|}} \tag{2.16}$$

and

$$|a_3|\leq\frac{B_1[|3(1+2\lambda)(2+t)-2(1+\lambda)^2(1+t)|+2(1+\lambda)^2(1+t)]+3|B_2-B_1|(2+t)|1+2\lambda|}{3(2+t)(1-t)|1+2\lambda||3(2+t)(1+2\lambda)-4(1+t)(1+\lambda)^2|} \tag{2.17}$$

Proof. Let $f \in M(\lambda, \phi, t)$ and $g = f^{-1}$. Then there are analytic functions $u, v : D \to D$ with $u(0) = v(0) = 0$, satisfying

$$\left\{\frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda[z^2 f''(z) - t^2 z^2 f''(tz)] + z[f'(z) - tf'(tz)]}\right\} = \phi(u(z))$$

$$and,$$

$$\left\{\frac{(1-t)[\lambda w^3 g'''(w) + (1+2\lambda)w^2 g''(w) + wg'(w)]}{\lambda[w^2 g''(w) - t^2 w^2 g''(tw)] + w[g'(w) - tg'(tw)]}\right\} = \phi(v(w)) \tag{2.18}$$

In view of (2.5), (2.6) and (2.18), clearly

$$\left\{\frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda[z^2 f''(z) - t^2 z^2 f''(tz)] + z[f'(z) - tf'(tz)]}\right\} = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)$$

$$and,$$

$$\left\{\frac{(1-t)[\lambda w^3 g'''(w) + (1+2\lambda)w^2 g''(w) + wg'(w)]}{\lambda[w^2 g''(w) - t^2 w^2 g''(tw)] + w[g'(w) - tg'(tw)]}\right\} = \phi\left(\frac{p_1(w)-1}{p_1(w)+1}\right)$$

Using (2.5) and (2.6) together with (1.2), we get (2.8) and (2.9).
Since $f \in M(\lambda, \phi, t)$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expression as (2.10).
Since

$$\left\{\frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda[z^2 f''(z) - t^2 z^2 f''(tz)] + z[f'(z) - tf'(tz)]}\right\}$$
$$= 1 + 2a_2(1+\lambda)(1-t)z + (1-t)[3a_3(1+2\lambda)(2+t) - 4a_2^2(1+\lambda)^2(1+t)]z^2 + \cdots$$

and

$$\left\{\frac{(1-t)[\lambda w^3 g'''(w) + (1+2\lambda)w^2 g''(w) + wg'(w)]}{\lambda[w^2 g''(w) - t^2 w^2 g''(tw)] + w[g'(w) - tg'(tw)]}\right\}$$
$$= 1 - 2a_2(1+\lambda)(1-t)w$$
$$+ (1-t)\{-3a_3(1+2\lambda)(2+t) + 2a_2^2[3(1+2\lambda)(2+t) - 2(1+\lambda)^2(1+t)]\}w^2 + \cdots$$

it follows from (2.8), (2.9) and (2.18) that

$$2a_2(1+\lambda)(1-t) = \frac{1}{2}B_1 c_1 \tag{2.19}$$

$$(1-t)[3a_3(1+2\lambda)(2+t) - 4a_2^2(1+\lambda)^2(1+t)] = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2 \tag{2.20}$$

$$-2a_2(1+\lambda)(1-t) = \frac{1}{2}B_1 b_1 \tag{2.21}$$

$$(1-t)\{-3a_3(1+2\lambda)(2+t) + 2a_2^2[3(1+2\lambda)(2+t) - 2(1+\lambda)^2(1+t)]\} = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2 \tag{2.22}$$

$|t| \le 1$, $t \ne 1$, $0 \le \lambda \le 1$
from (2.19) and (2.21), it follows (2.15).
Now (2.20), (2.21), (2.22) and (2.15) yield

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{4(1-t)\{B_1^2[3(1+2\lambda)(2+t) - 4(1+\lambda)^2(1+t)] - 4(1-t)(1+\lambda)^2(B_2 - B_1)\}}$$

and

$$a_3 = \frac{\frac{B_1}{2}\{c_2[3(1+2\lambda)(2+t) - 2(1+t)(1+\lambda)^2] + 2b_2(1+t)(1+\lambda)^2\} + \frac{B_2 - B_1}{4}3c_1^2(1+2\lambda)(2+t)}{3(1+2\lambda)(1-t)(2+t)[3(1+2\lambda)(2+t) - 4((1+\lambda)^2(1+t)]}$$

which in view of the well known inequalities $|b_i| \leq 2$ and $|c_i| \leq 2$ for functions with positive real part, gives us the required estimate on $|a_2^2|$ and $|a_3|$ as asserted in (2.16) and (2.17) respectively. ∎

Remark 2.1. For $\lambda = 0$, the function obtains coefficient estimates as in Corollary 2.3.

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